- Ordinary representations

Deligne: $f\in S_k(\Gamma_0(N),\varepsilon)$, $p\nmid N \Rightarrow \exists\, \rho_{f,p}: G_{\mathbb{Q}}\to GL_2(\mathcal{O}_{f,p})$ s.t

a) $\rho_{f,p}$ unramified at $\ell\nmid Np$, i.e. $\rho_{f,p}(I_\ell)=\{Id\}$.

b) $\forall \ell\nmid Np$, $tr(\rho_{f,p}(Frob_\ell))=a_\ell(f)$

$$\det(\rho_{f,p}(Frob_\ell))=\varepsilon(\ell)\ell^{k-1}=\varepsilon(\ell)\chi_p^{k-1}(\ell).$$

In particular, $\det(\rho_{f,p})=\varepsilon.\chi_p^{k-1}$.

Mazur-Wiles: Conditions as before, if $a_p(f)\in\mathcal{O}_{f,p}^*$, we say that f is p-ordinary. In that case $\Rightarrow \rho_{f,p}|_{G_{\mathbb{Q}_p}}\simeq\begin{pmatrix}\varepsilon_1 & * \\ 0 & \varepsilon_2\end{pmatrix}$ with ε_2 unramified

$$\Rightarrow \rho_{f,p}|_{I_p}\simeq\begin{pmatrix}\chi_p^{k-1}.\varepsilon & \gamma \\ 0 & 1\end{pmatrix} \Rightarrow \underset{\substack{\uparrow\\ \text{contragradient}}}{\rho^T_{f,p}}|_{I_p}=\begin{pmatrix}\chi_p^{k-1}\varepsilon & 0 \\ \gamma & 1\end{pmatrix}.$$

Fontaine: $a_p(f)\notin\mathcal{O}_{f,p}^*$

$\Rightarrow \rho_{f,p}|_{I_p}\simeq\begin{pmatrix}\omega_2^{k-1} & 0 \\ 0 & \omega_2^{p(k-1)}\end{pmatrix}$

$\omega_2: I_t\to\mathbb{F}_{p^2}^*$

$\quad\ \, \text{''}$

$Gal(\mathbb{Q}_{p,t}|\mathbb{Q}_p^{ur})$.

In this case: $M:=\mathcal{O}_{f,p}^2$

$$M^{I_p}=\left\{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\in\mathcal{O}_{f,p}^2 \;\middle|\; \forall\sigma\in I_p,\ \rho^T(\sigma)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\chi_p^{k-1}(\sigma^{-1}) & 0\\ \gamma(\sigma^{-1}) & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\chi_p^{k-1}(\sigma^{-1})\alpha\\ \gamma_p(\sigma^{-1})\alpha+\beta\end{pmatrix}=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right\}$$

$$\underset{\substack{\uparrow\\ \chi_p|_{I_p}\neq 1\Rightarrow\alpha=0}}{=}\left\{\begin{pmatrix}0\\ \beta\end{pmatrix} \;\middle|\; \beta\in\mathcal{O}_{f,p}\right\}$$ i.e. $\mathcal{O}_{f,p}$-mod of rank 1.

Def: ~~$\mathcal{O}_L$ DVR finite/$\mathbb{Z}_p$~~ $R\in\mathcal{C}$, Π profinite and $\rho:\Pi\to GL_m(R)$ def of $\overline{\rho}$. Denote $M=R^2$. Let $I\leq\Pi$ closed. We say that ρ is I-ordinary if $M^I\subseteq M$ is free of rank 1 and a direct summand of M.

Notice that if $M^I = M$, ρ it's not I-ordinary (call them I-unramified), think e.g. of $I = I_p$.

<u>Thm 6.5</u> Suppose $\bar{\rho}$ is I-ordinary $\Rightarrow$ being I-ordinary is a def. condition.

<u>Proof</u>: check that if $\rho: \Pi \to GL_2(A)$ is I-ord. and $\pi: A \to A'$ hom of coeff Λ-algebras $\Rightarrow \pi_* \rho: \Pi \to GL_2(A')$ is I-ord.

After base change, can assume that $\forall x \in I$, $\rho(x) = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ (since $(A \times A)^I$ is free, after base change, say, $\langle (1,0) \rangle$) $\Rightarrow$ under π, it has the same form.

Second condition: given $R_0, R_1, R_2 \in \mathcal{C}_0$, $\phi_1: R_1 \to R_0$, $\phi_2: R_2 \to R_0$

$R_3 := R_1 \times_{R_0} R_2$, given $\rho: \Pi \to GL_2(R_3)$, this is I-ordinary $\Leftrightarrow$ $\phi_{1*}\rho$ and $\phi_{2*}\rho$ are so. ($\Rightarrow$ trivial)

$\Leftarrow$: self study. #

Given $\bar{\rho}: \Pi \to GL_2(K)$ I-ordinary, $D_{\bar{\rho}, I}(R) = \{\rho: \Pi \to GL_2(R)$, def of $\bar{\rho}$, I-ordinary$\}$

this is relatively rep.

<u>Cor</u>: Supp $\bar{\rho}$ I-ordinary, $C(\bar{\rho}) = K \Rightarrow \exists\ \rho_{univ, I}$ universal I-ordinary deformation of $\bar{\rho}$. In particular, $\exists\ R_I = R_{\bar{\rho}}/Q$, $\rho_{univ, I}: \Pi \to GL_2(R_I)$ s.t $\forall \rho: \Pi \to GL_2(R)$ I-ordinary, $\exists!\ \pi: R_I \to R$ s.t $\rho = \pi \circ \rho_{univ}$. #

Obs: given a set of closed $I_1, \dots, I_n \leq \Pi$, $\bar{\rho}$ I_k-ordinary $\Rightarrow \exists\, \rho_{univ,I}$ universal def. I_k-ordinary $\forall\, 1 \leq k \leq n$. Check it!

Def: $\rho: \Pi \to GL_2(A)$ co-I-ordinary if $M = A^2$ has M_1 submodule of rank 1 s.t it's a direct summand of M and M/M_1 is I-invariant.

- Deformation conditions for global Galois reps

$\Pi = G_{\mathbb{Q},S}$, $S \ni \{p, \infty\}$.

e.g.: $\bar{\rho} = \rho_{E,p}$. Assume $\bar{\rho}$ abs irr (so that $\exists\, \rho_{univ}$)

S = primes of bad reduction $\cup \{p\} \cup \{\infty\}$.

$\rho_{E,p}: G_{\mathbb{Q},S} \to GL_2(\mathbb{Z}_p)$ = Galois action on $T_p(E)$.

Fact: $N(\rho_{E,p}) \equiv$ Artin conductor of $\rho_{E,p}$ = prime to p part of N_E

(Hodge-Serre) weights of $\rho_{E,p} = \{0, 1\}$.

$\det(\rho_{E,p}) = \chi_p$ (as we can prove using Weil-pairing).

Local conditions at $\ell \in S$ like "finite-flat".

Def: Let $\bar{\rho}: G_{\mathbb{Q},S} \to GL_n(k)$. A Global Galois deformation problem Q is the problem of representing a subfunctor of $D_{\bar{\rho}}$ defined by giving, for each $\ell \in S$, a deformation condition D_ℓ for $\bar{\rho}|G_{\mathbb{Q}_\ell}$.

Lemma: This is a def condition for reps of $G_{\mathbb{Q},S}$.

· The tangent space:

$H^1_{\mathcal{Q}}(G_{\mathbb{Q},S}, Ad(\bar{\rho}))$ = subspace of $H^1(G_{\mathbb{Q},S}, Ad(\bar{\rho}))$ corresponding to $D_{\mathcal{Q}}(k[\varepsilon]) \subseteq D_{\bar{\rho}}(k[\varepsilon])$.

$H^1_{\mathcal{Q}_\ell}(G_{\mathbb{Q}_\ell}, Ad(\bar{\rho}))$ = subspace of $H^1(G_{\mathbb{Q}_\ell}, Ad(\bar{\rho}))$ corresponding to $D_{\mathcal{Q}_\ell}(k[\varepsilon])$.

Thm (6.Γ)

$$\begin{tikzcd}
H^1_{Q}(G_{\mathbb{Q},S}, Ad(\bar{\rho})) \arrow[r, hook] \arrow[d, "Res"'] \arrow[dr, phantom, "\circlearrowleft"] & H^1(G_{\mathbb{Q},S}, Ad(\bar{\rho})) \arrow[d, "Res"] \\
\bigoplus_{\ell \in S} H^1_{\mathcal{Q}_\ell}(G_{\mathbb{Q}_\ell}, Ad(\bar{\rho})) \arrow[r, hook] & \bigoplus_{\ell \in S} H^1(G_{\mathbb{Q}_\ell}, Ad\bar{\rho})
\end{tikzcd}$$

is Cartesian.

· Representations that are ordinary at p

$\Pi = G_{\mathbb{Q},S}$. Choose 1 local def cond: ρ I_p-ordinary.

$D^0(R) = \{$defs of $\bar{\rho}$ to R which are I_p-ordinary$\}$.

$\Rightarrow$ Representable by $R^0(\bar{\rho})$ = universal ordinary def ring.

Wiles: $\bar{\rho}$ modular $\Rightarrow$ any ordinary $\bar{\rho}$-def is so.

Prop (Mazur, Martin). Let $S = \{p, \infty\}$, $k = \mathbb{F}_p$, $\bar{\rho}: G_{\mathbb{Q},S} \to GL_2(\mathbb{F}_p)$ p-ordinary.

Let $\omega: G_{\mathbb{Q},S} \to \mathbb{F}_p^*$ Teichmüller character. Supp $\det(\bar{\rho}) \neq 1, \omega, \omega^{-1}, \omega^{\frac{p-1}{2}}$ or $\bar{\rho}$ tamely ramified

$\Rightarrow \operatorname{Ker}[R(\bar{\rho}) \to R^0(\bar{\rho})] = \langle \alpha, \beta \rangle$.

Cor: Krulldim $(R^0(\bar{\rho})/pR^0(\bar{\rho})) \geq 1$. If $\dim t_{R^0(\bar{\rho})} \leq 1 \Rightarrow R^0(\bar{\rho}) \simeq \mathbb{Z}_p[[T]]$.