Advanced Microeconomics 1: Problem set 1 Solutions

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Problem 1.a)

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A decision maker makes choices in a domain consisting of five alternatives: $X = \{a, b, c, d, e\}$. A choice function assigns to any non-empty subset A of X a choice $c(A) \in A$.

(a) How many different choice functions could the decision maker have?

- X has $\binom{5}{5} = 1$ subsets of cardinality 5, $\binom{5}{4} = 5$ subsets of cardinality 4, etc.
- In each set of cardinality m, there are m different ways to pick the choice.
- Therefore, there are

$$5^{\binom{5}{5}} 4^{\binom{5}{4}} 3^{\binom{5}{3}} 2^{\binom{5}{2}} 1^{\binom{5}{1}} = 309,586,821,120$$

different choice functions.

Problem 1.b)

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A decision maker makes choices in a domain consisting of five alternatives: $X = \{a, b, c, d, e\}$. A choice function assigns to any non-empty subset A of X a choice $c(A) \in A$.

(b) How many different coherent choice functions satisfying the Independence of Irrelevant Alternatives can she have?

- There are as many coherent choice functions as there are complete and transitive strict preferences over X, that is, $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Problem 2.a)

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Determine for the following binary relations if they are reflexive, symmetric or transitive:

(a) xRy on natural numbers if x divides y.

- Reflexivity: True. (Any $x \in \mathbb{N}$ is divisible by itself.)
- Symmetry: Not true. (E.g., 1 divides 2 but 2 doesn't divide 1.)
- Transitivity: True. ($y/x = m$ and $z/y = n$ imply $z/x = mn$.)

Problem 2.b)

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Determine for the following binary relations if they are reflexive, symmetric or transitive:

(b) xRy on natural numbers if $x + y$ is even.

- Reflexivity: True. ($x + x = 2x$ even for all $x \in \mathbb{N}$.)
- Symmetry: True. ($x + y = y + x$ for any x, y.)
- Transitivity: True. ($x + y$ is even iff x and y are both even or both odd. So, xRy, yRz implies that x, y, z are all even or all odd, and $x + z$ is even.)

Problem 2.c)

Problem 2.c)

Determine for the following binary relations if they are reflexive, symmetric or transitive:

(c) xRy on natural numbers if xy is even.

- Reflexivity: Not true. (x^2 is not even if x is odd.)
- Symmetry: True. ($xy = yx$ for any x, y.)
- Transitivity: Not true. (E.g., for $x, z = 1$, $y = 2$, xy and yz are even but xz not.)

Problem 3

Problem 3

Show that if $\succ$ is asymmetric and negatively transitive, then $\succeq$ derived from $\succ$ by setting $x \succeq y \iff \neg(y \succ x)$, is a rational preference relation.

To show **completeness** of $\succeq$, show that $\neg(x \succeq y)$ implies $y \succeq x$:

$$\neg(x \succeq y) \implies y \succ x \implies \neg(x \succ y) \implies y \succeq x$$

where the first and third implication follow by definition of $\succeq$ and the second one follows from asymmetry of $\succ$.

To show **transitivity** of $\succeq$, show that $x \succeq y \wedge y \succeq z$ implies $x \succeq z$:

$$x \succeq y \wedge y \succeq z \implies \neg(z \succ y) \wedge \neg(y \succ x) \implies \neg(z \succ x) \implies x \succeq z$$

where the first and third implication follow by definition of $\succeq$ and the second one follows from negative transitivity of $\succ$.

Problem 4

Problem 4

Start with a strict preference relation $\succ$ that is asymmetric and transitive, (but not negatively transitive). Define weak preference $\succeq$ by $x \succeq y \iff \neg(y \succ x)$, and indifference $\sim$ by $x \sim y \iff \neg(x \succ y)$ and $\neg(y \succ x)$. Give an example of such a strict preference where the derived $\succeq$ is not a rational preference relation.

Consider $\succ$ over alternatives x, y, z such that

$$
\begin{aligned}
&\neg(x \succ y), \neg(y \succ x) \quad (\text{i.e. } x \sim y)\\
&\neg(y \succ z), \neg(z \succ y) \quad (\text{i.e. } y \sim z)\\
&\neg(x \succ z),\ z \succ x
\end{aligned}
$$

$\succ$ is asymmetric and transitive.

By the definition of $\succeq$, we have $x \succeq y$, $y \succeq z$, and $\neg(x \succeq z)$. So $\succeq$ is not transitive.

Problem 5

Problem 5

Two decision makers $i \in \{1, 2\}$ have complete and transitive preferences $\succeq_i$ on X. They agree to a common preference $\succeq$ satisfying for all $x, y \in X$,

$$x \succeq y \iff x \succeq_i y \text{ for some } i \in \{1, 2\}$$

(a) Is $\succeq$ complete?
(b) Is it transitive?

(a) $\succeq$ is complete, since for all $x, y \in X$ there is some i such that $x \succeq_i y$ or $y \succeq_i x$ by completeness of $\succeq_i$.

(b) $\succeq$ is not transitive in general. Suppose 1 and 2 have complete and transitive strict preferences over $\{x, y, z\}$ s.t. $z \succ_1 x \succ_1 y$ and $y \succ_2 z \succ_2 x$. The corresponding $\succeq_1$ and $\succeq_2$ are rational, but $\succeq$ is not transitive as $x \succeq y$, $y \succeq z$ and $\neg(x \succeq z)$.

Problem 6

Problem 6

Explain if the following procedures lead to a coherent choice rule (i.e. a choice rule satisfying Sen's α and Sen's β):

(a) (Choosing the second cheapest wine) The decision maker has a strict rational preference relation in mind and chooses the second cheapest alternative from any set with two or more alternatives (and the only choice if faced with a single alternative).

(a) does NOT lead to a coherent choice rule: when the cheapest wine is deleted from the choice set[1], the chosen wine changes, violating Sen's α.

[1] And there are at least 3 wines in the original choice set, and all the wines have different prices

Problem 6

Problem 6

Explain if the following procedures lead to a coherent choice rule (i.e. a choice rule satisfying Sen's α and Sen's β):

(b) Two committee members $i \in \{1, 2\}$ with a strict ordering on a finite set rank alternatives from best (smallest score) to worst (highest score) and choose the alternative with the lowest sum of scores.

(b) does NOT lead to a coherent choice rule. E.g., suppose there are 4 alternatives a, b, c, d. Committee member 1 has ordering $a \succ_1 b \succ_1 c \succ_1 d$, and member 2 has ordering $c \succ_2 d \succ_2 a \succ_2 b$. Then $c(\{a, b, c, d\}) = \{a, c\}$, $c(\{a, b, c\}) = \{a\}$, $c(\{a, c\}) = \{a, c\}$. Both Sen's α and β are violated.

Problem 6

Problem 6

Explain if the following procedures lead to a coherent choice rule (i.e. a choice rule satisfying Sen's α and Sen's β):

(c) Continue with the assumptions of the previous part except that now committee member $i = 2$ gets to make the choice. She maximizes the suffering of member $i = 1$, i.e. $c(A)$ is the worst alternative in $A \subset X$ for $i = 1$.

(c) leads to a coherent choice rule. The derived choice rule is a well-defined function as member 1 has a strict ordering $\succ_1$ on a finite set. It's coherent since if $c(A) \in B \subset A$, then $a \succ_1 c(A)$ for all $a \in A \backslash c(A)$, and consequently $b \succ_1 c(A)$ for all $b \in B \backslash c(A)$, so $c(B) = c(A)$.

Problem 7

Problem 7

You are an HR manager at a firm in charge of hiring new workers. You conduct job interviews in the following manner. For any set of job applicants A, you invite them for the interview in alphabetical order. You score their interview on a scale from 0 to 100 and the first applicant that gets a score at or above 90 is hired. If none of them gets such a grade, you hire the last applicant in alphabetical order. Is this procedure rational? Does it satisfy Independence of Irrelevant Alternatives?

- The procedure satisfies IIA. If $c(A) = a$, then a is either the alphabetically first applicant in A to score 90 or more, or a is the alphabetically last one in A and no-one in A scores 90 or more. In the first case, if $a \in B \subset A$, then a is also the alphabetically first to score at least 90 in B. In the second case, no candidate scores 90 in B and a is the last in alphabetical order in B. So in either case, $c(B) = a$.
- The revealed preference relation ranks those with scores of at least 90 first and the ranking among them is alphabetical. Those with lower scores come after and are ranked in reversed alphabetical order.

Problem 8 (Bonus)

Problem 8 (Bonus)

Read 'A Condition for the Completeness of Partial Preference Relations' by David Schmeidler in Econometrica (1971) pages 403–404, and explain the logic of the proof in your own words. Can you find an example of a continuous but incomplete preference relation on $\mathbb{R}_+$?

- The theorem says that on connected X, transitive and nontrivial $\succeq$ is complete if for all $x \in X$, "weak" upper and lower contour sets are closed and "strict" upper and lower contour sets are open.
  - It's first shown that if $x \succ y$, every element in X belongs to the strict upper contour set at y or strict lower contour set at x.
  - Then it's supposed by contradiction that for some $v, w \in X$, $\neg(v \succeq w)$ and $\neg(w \succeq v)$. Using the previous part, this is shown to imply that an intersection of strict contour sets at v and w is equal to an intersection of weak contour sets at v and w—a contradiction, since the first set (non-trivial subset of connected X) is open but the second one closed.
- Example of a continuous but incomplete preference relation on $\mathbb{R}^2_+$: $(x, y) \succeq (x', y')$ iff $x \geq x'$ and $y \geq y'$ OR $x = x'$ OR $y = y'$.

Problem 9 (Bonus)

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Suppose that X is a finite set and the decision maker has in mind two one-to-one numerical functions u and v on X. Interpret the following choice procedures in words, and determine whether they lead to choice rules satisfying the Independence of Irrelevant Alternatives.

(a) Fix a number v^*. For any choice problem $A \subset X$, let a^* be the maximizer of u over A and b^* be the maximizer of v over A. The decision maker chooses a^* if $v(a^*) \geq v^*$ and she chooses b^* otherwise.

- The procedure in (a) uses a u-based rule to pick the choice if the resulting selection satisfies a v-based threshold criterion. Otherwise a v-based rule is used to pick the choice.
- IIA is not satisfied: if $v(a^*) < v^*$ so that b^* is chosen, then after deleting a^* from the choice set, the maximizer of u over the new choice set may satisfy the threshold criterion and be therefore chosen.

Problem 9 (Bonus)

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Suppose that X is a finite set and the decision maker has in mind two one-to-one numerical functions u and v on X. Interpret the following choice procedures in words, and determine whether they lead to choice rules satisfying the Independence of Irrelevant Alternatives.

(b) Fix u^*. For any choice problem A, the DM chooses a^* maximizing u if $u(a^*) \geq u^*$, otherwise she chooses b^* maximizing v.

- The procedure in (b) uses a u-based rule to pick the choice if the resulting selection satisfies a u-based threshold criterion. Otherwise a v-based rule is used to pick the choice.
- IIA is satisfied. To prove this, suppose $c(A) = a$.
  - If $u(a) \geq u^*$, then $a = \text{argmax}_{x \in A} u(x)$. Clearly for any B s.t. $a \in B \subset A$, $a = \text{argmax}_{x \in B} u(x)$ and $u(a) \geq u^*$, so $c(B) = a$.
  - If $u(a) < u^*$, then $a = \text{argmax}_{x \in A} v(x)$ and $u(x) < u^* \ \forall x \in A$. Clearly for any B s.t. $a \in B \subset A$, $a = \text{argmax}_{x \in B} v(x)$ and $u(x) < u^* \ \forall x \in B$, so $c(B) = a$.

Problem 10

Problem 10 (Bonus)

We say that a collection $\mathcal{P}$ of subsets of X is a nested family of subsets if for any $P, Q \in \mathcal{P}$, either $P \subset Q$ or $Q \subset P$. We say that $x \succ y$ if there is an set $P \subset \mathcal{P}$ such that $x \in P$ such that $y \notin P$. Write $x \succeq y$ if $\neg(x \succ y)$.

(a) Show that $\succeq$ is complete and transitive.

- Completeness: We must show that for any $x, y \in X$, $\neg(x \succ y)$ or $\neg(y \succ x)$, i.e., $x \in P \implies y \in P$ for all $P \subset \mathcal{P}$ OR $y \in P \implies x \in P$ for all $P \subset \mathcal{P}$. If this were not true, we would have some $P \in \mathcal{P}$ s.t. $x \in P, y \notin P$ and some $P' \in \mathcal{P}$ s.t. $x \notin P', y \in P'$, but then we would have $P \not\subset P'$ and $P' \not\subset P$—a contradiction.
- Transitivity:
  1. Take any $x, y, z \in X$ s.t. $x \succeq y, y \succeq z$. Then $\neg(y \succ x), \neg(z \succ y)$.
  2. So for any $P \in \mathcal{P}$ s.t. $y \in P$, we have $x \in P$. And for any $P \in \mathcal{P}$ s.t. $z \in P$, $y \in P$.
  3. So for any $P \in \mathcal{P}$ s.t. $z \in P$, we have $x \in P$, implying $\neg(z \succ x)$ and $x \succeq z$.

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We say that a collection $\mathcal{P}$ of subsets of X is a nested family of subsets if for any $P, Q \in \mathcal{P}$, either $P \subset Q$ or $Q \subset P$. We say that $x \succ y$ if there is an set $P \subset \mathcal{P}$ such that $x \in P$ such that $y \notin P$. Write $x \succeq y$ if $\neg(x \succ y)$.

(b) For any complete and transitive relation on X, show that there exists a family of nested subsets corresponding to the order as above.

- Given complete and rational relation $\succeq$ on X, define for every $x \in X$ set $P_x := \{z \in X : z \succeq x\}$.
- To show that $\mathcal{P} := \{P_x : x \in X\}$ satisfies the nestedness property, take any $P_x, P_{x'} \in \mathcal{P}$ and $y \in P_x, y' \in P_{x'}$. We'll show $y \notin P_{x'} \implies y' \in P_x$.
  1. From $y \notin P_{x'}$ it follows that $\neg(y \succeq x')$. By completeness, $x' \succeq y$.
  2. Since $y \in P_x, y' \in P_{x'}$, we have $y' \succeq x' \succeq y \succeq x$.
  3. By transitivity, $y' \succeq x$, so $y' \in P_x$.

Problem 10

We can show that the rational preference we can retrieve from the built nested family in the manner described in the problem is exactly the same that was used for building the nested structure.

- If there is set $P_z \in \mathcal{P}$ s.t. $x \in P_z$, $y \notin P_z$, then $x \succeq z$, $\neg(y \succeq z)$. Then in order to not violate transitivity, we must have $\neg(y \succeq x)$, that is, $x \succ y$.
- Conversely, if there is no set $P_z \in \mathcal{P}$ s.t. $x \in P_z$, $y \notin P_z$, then since $x \in P_x$, we have $y \in P_x$, so $y \succeq x$, so $\neg(x \succ y)$.

Problem 10 (Bonus)

(c) Give an economic interpretation to the nested family of sets in the previous part.

P_x is an upper contour set at x.