Advanced Microeconomics 1: Problem set 2 Solutions

Mikael Mäkimattila (Aalto University)
mikael.makimattila@aalto.fi

September 2024

Problem 1

Problem 1

Show that the following demand observations satisfy WARP, but not SARP (or GARP)

$$\mathbf{p}^1 = (1,1,2),\ \mathbf{p}^2 = (2,1,1),\ \mathbf{p}^3 = (1,2,1+\epsilon),$$

$$\mathbf{x}(\mathbf{p}^1,1) = (1,0,0),\ \mathbf{x}(\mathbf{p}^2,1) = (0,1,0),\ \mathbf{x}(\mathbf{p}^3,1+\epsilon) = (0,0,1).$$

$$\mathrm{x}(\mathrm{p}^1,1) = (1,0,0) \in B(\mathbf{p}^1,1) \quad \mathrm{x}(\mathrm{p}^1,1) = (1,0,0) \notin B(\mathbf{p}^2,1) \quad \mathrm{x}(\mathrm{p}^1,1) = (1,0,0) \in B(\mathbf{p}^3,1+\epsilon)$$

$$\mathrm{x}(\mathrm{p}^2,1) = (0,1,0) \in B(\mathbf{p}^1,1) \quad \mathrm{x}(\mathrm{p}^2,1) = (0,1,0) \in B(\mathbf{p}^2,1) \quad \mathrm{x}(\mathrm{p}^2,1) = (0,1,0) \notin B(\mathbf{p}^3,1+\epsilon)$$

$$\mathrm{x}(\mathrm{p}^3,1+\epsilon) = (0,0,1) \notin B(\mathbf{p}^1,1) \quad \mathrm{x}(\mathrm{p}^3,1+\epsilon) = (0,0,1) \in B(\mathbf{p}^2,1) \quad \mathrm{x}(\mathrm{p}^3,1+\epsilon) = (0,0,1) \in B(\mathbf{p}^3,1+\epsilon)$$

We do not find a violation of WARP since in our $\mathbf{x}$-observations, whenever we have $\mathbf{x}(\mathbf{p},w) \in B(\mathbf{p}',w')$ and $\mathbf{x}(\mathbf{p},w) \neq \mathbf{x}(\mathbf{p}',w')$, then $\mathbf{x}(\mathbf{p}',w') \notin B(\mathbf{p},w)$.

SARP (and GARP) violated, since $(1,0,0) \succeq_{R^0} (0,1,0) \succeq_{R^0} (0,0,1)$, so $(1,0,0) \succeq_{R} (0,0,1)$, but $(0,0,1) \succeq_{R^0} (1,0,0)$.

Problem 2

Problem 2

(a) Are all utility functions representing continuous preferences on $\mathbb{R}^n_+$ continuous?

No. For example, utility function

$$u(\mathbf{x}) = \begin{cases} \sum_{i=1}^n x_i & \text{if } \sum_{i=1}^n x_i \leq 1 \\ \sum_{i=1}^n x_i + 1 & \text{if } \sum_{i=1}^n x_i > 1 \end{cases}$$

represents continuous preferences but is not continuous.

Problem 2

Problem 2

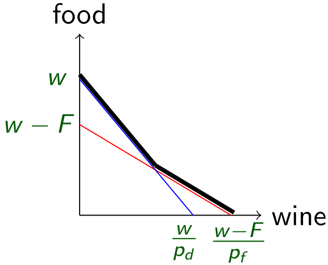
(b) A decision maker evaluates the available options in a finite set X according to three separate criteria $i \in \{1, 2, 3\}$. She assigns to each option $x \in X$ numerical rating $u_i(x)$ for each $i \in \{1, 2, 3\}$. To aggregate over the criteria, she computes an aggregate score $g(u_1(x), u_2(x), u_3(x))$ using a function $g : \mathbb{R}^3 \to \mathbb{R}$ and chooses the alternative with the highest score in any feasible set $A \subset X$. Under what assumptions on g is the decision maker's choice behavior consistent (in the sense of satisfying Sen's α and β)?

Any function g assigns a numerical rating to each option and therefore gives consistent choice. WARP is satisfied: if $a, b \in A \cap B$ and $a \in c(A)$, then it must be that $g(u_1(a), u_2(a), u_3(a)) \geq g(u_1(b), u_2(b), u_3(b))$. So if $b \in c(B)$, then necessarily $a \in c(B)$.

Problem 2

Problem 2

(c) Sketch the budget set of a consumer that divides her income w between food and wine when price of food is normalized to 1 and wine (assumed to be of homogenous quality) can be purchased either domestically at price p_d or at a cheaper foreign rate $p_f < p_d$. The catch is that in order to get the cheaper rate for wine, you have to pay a fixed cost of $F > 0$. What can you say about optimal quantities demanded if the consumer has a strictly monotone, strictly convex and continuously differentiable utility over wine and food?



If the utility function is strictly increasing and strictly convex in wine and food, then the consumer spends all her money on food or all on wine (indifference curves in the LHS graph are concave to origin).

Problem 3

Problem 3

Affordable living for low-income households is a goal for many governments. Three schemes may be tried to help them: i) Lump-sum income subsidy of size $I > 0$. ii) Rent subsidy: a fraction r of a household's rent is paid by the government. iii) Voucher: a fixed amount $s > 0$ is paid from the household's rent if rent exceeds R. The consumer has a Cobb-Douglas utility function for housing services x and other consumption y given by:

$$u(x, y) = \frac{1}{5} \ln x + \frac{4}{5} \ln y$$

and her monthly income is $w = 500$. Assume that the units of consumption and housing are chosen so that the price of each is 1 per unit. Assume also that the low-income population is small enough so that the schemes' effect on market prices can be ignored (i.e. they stay fixed at unity).

Problem 3

Problem 3

(a) Without any subsidies, how much will the household spend on housing?

- The utility maximization problem is

$$\max_{x,y} \frac{1}{5}\ln x + \frac{4}{5}\ln y$$
$$\text{s.t. } x + y \leq 500.$$

- The budget constraint binds, i.e. $x + y = 500$, since we have strictly increasing preferences. The FOCs also give us $y = 4x$. So $x = 100$, i.e., the household spends 100 on housing.

Problem 3

Problem 3

(b) Suppose that the government wants the household to live in a unit producing 150 worth services. What is the cost to the government for achieving this target under each of the schemes above?

(i) With lump-sum income subsidy $I > 0$, the household's UMP is otherwise the same but the budget constraint is $x + y \leq 500 + I$. Again, at the optimum $y = 4x$. So $x = 150$ requires $I = 250$.

(ii) With rent subsidy, the housing price faced by the household decreases to $1 - r$. The budget constraint is then $(1 - r)x + y \leq 500$. Now the FOCs give $y = 4x(1 - r)$. So $x = 150$ requires $r = 1/3$, and the government's cost is $(1/3) \cdot 150 = 50$.

Problem 3

(iii) With voucher, the government must choose size s and eligibility threshold R. If the threshold doesn't bind, there is no difference to a lump-sum income subsidy. If it binds, then the government achieves the target by setting $R = 150$. Solve the minimal voucher size s s.t. the household wants to choose voucher-eligible amount of housing rather than the bundle in (a). The budget constraint binds so $y = 350 + s$. The utility from the voucher-eligible bundle is $(1/5)\ln 150 + (4/5)\ln(350 + s)$. Set equal to the utility in (a), $(1/5)\ln 100 + (4/5)\ln 400$, so the needed voucher is $s \approx 11.4$.

Problem 4

Problem 4

Suppose that a non-satiated consumer has a continuously differentiable demand function $\mathbf{x}(p, w)$ that satisfies WARP.

(a) Show that the Slutsky matrix $S(\mathbf{p}, w)$ does not have full rank by showing that $\mathbf{p}^T S(\mathbf{p}, w) = 0$ and also $S(\mathbf{p}, w)\mathbf{p} = 0$.

Non-satiated rational preferences imply Walras' law,

$$\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, w) = w. \tag{1}$$

By differentiating (1) w.r.t. w and p_k, we get

$$\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial w} = 1, \qquad x_k + \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial p_k} = 0. \tag{2}$$

Problem 4

$$\mathbf{p}^T S(\mathbf{p}, w) = (x_1 \sum_{i=1}^n p_i \frac{\partial x_i}{\partial w} + \sum_{i=1}^n p_i \frac{\partial x_i}{\partial p_1}, \ldots, x_n \sum_{i=1}^n p_i \frac{\partial x_i}{\partial w} + \sum_{i=1}^n p_i \frac{\partial x_i}{\partial p_n}) = (0, \ldots, 0)$$

where the second equality follows from (2).

$$\begin{aligned} S(\mathbf{p}, w)\mathbf{p} &= (\frac{\partial x_1}{\partial w} \sum_{i=1}^n p_i x_i + \sum_{i=1}^n \frac{\partial x_1}{\partial p_i} p_i, \ldots, \frac{\partial x_n}{\partial w} \sum_{i=1}^n p_i x_i + \sum_{i=1}^n \frac{\partial x_n}{\partial p_i} p_i)^T \\ &= (\frac{\partial x_1}{\partial w} w + \sum_{i=1}^n \frac{\partial x_1}{\partial p_i} p_i, \ldots, \frac{\partial x_n}{\partial w} w + \sum_{i=1}^n \frac{\partial x_n}{\partial p_i} p_i)^T \\ &= (0, \ldots, 0)^T \end{aligned}$$

where the second equality follows from Walras' law (1), and the third equality follows from Walrasian demand being homogeneous of degree 0.

Problem 4

Problem 4

(b) In the case with two goods, use the above result to show that the Slutsky matrix is symmetric. Comment.

- For a two-good case, and denoting the $(l, k)^{th}$ entry of Slutsky matrix by s_{lk}, we showed in part (a) that $p_1 s_{11} + p_2 s_{12} = 0 = p_1 s_{11} + p_2 s_{21}$ for any $p_1, p_2 > 0$. Clearly $s_{12} = s_{21}$, that is, the Slutsky matrix is symmetric.
- The Slutsky matrix is the Hessian of the expenditure function (therefore symmetry follows from continuity of 2nd partials by Young's theorem), and the Jacobian of Hicksian demand. So symmetry tells us that the change of compensated demand h_k due to a change in price p_l equals the change of compensated demand h_l due to a change in price p_k.

Problem 5

Problem 5

In many instances, a consumer chooses an indivisible item from a set of differentiated alternatives with different prices. For concreteness, let's think of a growing city located on a lake and the choice of housing. Assume that housing units are identical except in their proximity to the lake denoted by $t \in [0, 1]$. All consumers prefer locations closer to the lake to more distant ones. The consumers vary by their available income $w \in [\underline{w}, \overline{w}]$ and they split their income between choosing a location t at price $p(t)$ and other consumption y.

(a) Formulate the problem for choosing the optimal housing for a consumer with income w assuming that her utility is strictly increasing in y and strictly decreasing in t.

The problem is to choose the most preferred pair (t, y) within the budget set. If utility function $u(t, y)$ represents the preferences of the consumer, and prices are normalized so that the unit price of other consumption y is 1, then the problem is

$$\max_{t,y} u(t, y)$$
$$\text{s.t. } p(t) + y \leq w, \quad t \in [0, 1], \quad y \geq 0$$

Problem 5

Problem 5

(b) Assume that utility is additively separable and strictly concave in t and y (it can be written as $u(t,y) = v_t(t) + v_y(y)$ for some functions v_t and v_y). Assume also that the price function $p(t)$ is differentiable. Write the first-order conditions for the problem and argue that any point satisfying them is an optimum.

- Additively separable preferences admit representation $u(t,y) = v_t(t) + v_y(y)$.
- Assume that utility is differentiable. The KKT conditions are

$$v_t'(t) - \lambda p'(t) - \mu_1 + \mu_0 = 0, \quad v_y'(y) - \lambda + \gamma = 0 \qquad \text{(FOCs)}$$

$$\lambda(p(t) + y - w) = 0, \quad \mu_0 t = 0, \quad \mu_1(t-1) = 0, \quad \gamma y = 0 \qquad \text{(Compl. slackness)}$$

$$p(t) + y - w \leq 0, \quad t - 1 \leq 0, \quad -t \leq 0, \quad -y \leq 0 \qquad \text{(Feasibility)}$$

$$\lambda, \mu_0, \mu_1, \gamma \geq 0 \qquad \text{(Multiplier positivity)}$$

- Strict concavity of objective $u(t,y)$ and quasiconvexity of constraint $p(t) + y - w$ in (t,y) imply sufficiency of KKT conditions.

Problem 5

Problem 5

(c) Use the implicit function theorem to derive comparative statics of t in w.

Since utility is strictly increasing in y, the budget constraint always binds in the optimum. Assuming an interior solution, our FOCs are

$$v_t'(t) - \lambda p'(t) = 0,$$
$$v_y'(y) - \lambda = 0,$$
$$p(t) + y - w = 0$$

and we can use the implicit function theorem:

$$\begin{pmatrix} v_t'' - \lambda p'' & 0 & -p' \\ 0 & v_y'' & -1 \\ p' & 1 & 0 \end{pmatrix} \begin{pmatrix} dt \\ dy \\ d\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dw$$

Problem 5

By Cramer's rule,

$$dt = \frac{\det\begin{pmatrix} 0 & 0 & -p' \\ 0 & v_y'' & -1 \\ dw & 1 & 0 \end{pmatrix}}{\det\begin{pmatrix} v_t'' - \lambda p'' & 0 & -p' \\ 0 & v_y'' & -1 \\ p' & 1 & 0 \end{pmatrix}} = \frac{dw \cdot p' \cdot v_y''}{v_t'' + (p')^2 \cdot v_y'' - \lambda p'}$$

So,

$$\frac{dt}{dw} = \frac{p'(t^*)v_y''(y^*)}{v_t''(t^*) + p'(t^*)^2 v_y''(y^*) - \lambda^* p''(t^*)}$$

which is negative when $v_y'', v_t'', p' < 0$, $p'' > 0$.

Problem 5

Problem 5

(d) What if only locations $\{t_1, t_2, t_3\} \in [0, 1]$ are available. Can you use calculus for solving the problem? Can you determine the comparative statics of t in w for this case by other methods?

- The budget constraint will bind in the optimum, since v_y is strictly increasing in y. The consumer's problem can therefore be written simply $\max_{t \in \{t_1, t_2, t_3\}} v_t(t) + v_y(w - p(t))$.
- By Milgrom-Shannon theorem, optimizer t^* is decreasing in w if $v_t(t) + v_y(w - p(t))$ has the following kind of single-crossing-differences property: for all $t' > t''$,

$$\Big(v_t(t'') + v_y(w - p(t''))\Big) - \Big(v_t(t') + v_y(w - p(t'))\Big)$$

is single-crossing. A sufficient condition is that $p(t)$ is decreasing in t and v_y is concave.

Problem 5

Problem 5

(e) Draw the feasible set for general utility function $u(t, y)$. What assumptions on u do you need to have the same comparative statics results (you can assume the differentiable case)?

- Assume more general $u(t, y)$ which is however still strictly increasing in y and twice differentiable.
- To get the same comparative statics result $dt^*/dw \leq 0$, we need the same kind of single-crossing differences property, i.e., for all $t' > t''$,

$$u(t'', w - p(t'')) - u(t', w - p(t')).$$

whould be single-crossing. This is satisfied if u satisfies

$$\frac{d^2}{dtdw}u(t, w - p(t)) \leq 0 \iff \frac{\partial^2 u}{\partial t \partial y} - \frac{\partial^2 u}{\partial y^2}p'(t) \leq 0$$

So in addition to what we previously assumed to get $dt^*/dw \leq 0$, we should now make complementarity assumption $\frac{\partial^2 u}{\partial t \partial y} \leq 0$.

Problem 6

Problem 6

You need to evaluate the welfare effects of a price change from $\mathbf{p}^0$ to $\mathbf{p}^1 \neq \mathbf{p}^0$ for a given household whose income does not change. Unfortunately you only have data on the realized demand at the current prices, i.e. you know the demand vector $\mathbf{x}(\mathbf{p}^0, w)$ and you know that the household has strictly monotonic preferences.

(a) Show that a sufficient condition for the household to be strictly better off at $\mathbf{p}^1$ is that

$$(\mathbf{p}^1 - \mathbf{p}^0) \cdot \mathbf{x}(\mathbf{p}^0, w) < 0.$$

$(\mathbf{p}^1 - \mathbf{p}^0) \cdot \mathbf{x}(\mathbf{p}^0, w) < 0$ and $\mathbf{p}^0 \cdot \mathbf{x}(\mathbf{p}^0, w) \leq w$ imply $\mathbf{p}^1 \cdot \mathbf{x}(\mathbf{p}^0, w) < w$. So there is some positive vector ϵ such that $\mathbf{p}^1 \cdot (\mathbf{x}(\mathbf{p}^0, w) + \epsilon) < w$, that is, $\mathbf{x}(\mathbf{p}^0, w) + \epsilon$ is in the new budget set. It is strictly preferred to $\mathbf{x}(\mathbf{p}^0, w)$ by strict monotonicity of preferences.

Problem 6

Problem 6

(b) Can the household be better off at $\mathbf{p}^1$ if the reverse strict inequality holds?

Yes, because of substitution.

Problem 6

(c) Give a sufficient condition for the household preferences so that

$$(\mathbf{p}^1 - \mathbf{p}^0) \cdot \mathbf{x}(\mathbf{p}^0, w) \leq 0$$

implies that the household is strictly better off at $\mathbf{p}^1$.

Sufficient condition: (i) preferences can be represented by a differentiable and strictly increasing utility function u, and (ii) $x_i(\mathbf{p}^0, w) > 0$ for some i such that $p_i^0 \neq p_i^1$.

Problem 6

Show the result for $(\mathbf{p}^1 - \mathbf{p}^0) \cdot \mathbf{x}(\mathbf{p}^0, w) = 0$ (the rest was shown in a):

- There must be goods a, b s.t. $x_a(\mathbf{p}^0, w), x_b(\mathbf{p}^0, w) > 0$ and $p_a^0 < p_a^1$, $p_b^0 > p_b^1$.
- Then, and by (i) and by optimality of $\mathbf{x}(\mathbf{p}^0, w)$ at $(\mathbf{p}^0, w)$: $\frac{\partial u(\mathbf{x}(\mathbf{p}^0,w))/\partial x_a}{\partial u(\mathbf{x}(\mathbf{p}^0,w))/\partial x_b} = \frac{p_a^0}{p_b^0}$.
- Now take $\mathbf{x}(\mathbf{p}^0, w)$ and consider a marginal decrease in consumption of good a and a (p_a^1/p_b^1) times as large increase in consumption of good b. This change is affordable with income w and prices $\mathbf{p}^1$ since
  $\mathbf{p}^1 \cdot \mathbf{x}(\mathbf{p}^0, w) - p_a^1 + \frac{p_a^1}{p_b^1} p_b^1 = \mathbf{p}^1 \cdot \mathbf{x}(\mathbf{p}^0, w)$, and it's strictly preferred as

$$-\frac{\partial u(\mathbf{x}(\mathbf{p}^0, w))}{\partial x_a} + \frac{p_a^1}{p_b^1} \frac{\partial u(\mathbf{x}(\mathbf{p}^0, w))}{\partial x_b} = \frac{\partial u(\mathbf{x}(\mathbf{p}^0, w))}{\partial x_b} \left(\frac{p_a^1}{p_b^1} - \frac{p_a^0}{p_b^0} \right) > 0$$

Problem 7

Problem 7 (Bonus)

Read Giving According to Garp by Andreoni and Miller in Econometrica (2002).

(a) How do the authors assess the consistency or rationality of their subjects in the experiment? Is giving to others irrational?

Consistency/rationality is assessed in a modified dictator game by checking whether the subjects' choices (about their own and another subject's payment given a budget set) satisfy WARP, GARP, and SARP. Giving to others is not considered irrational.

(b) Could you find a subject whose choice behavior is consistent with WARP but not SARP?

No (see p. 742).

(c) How do the authors measure the severity of violations to consistent behavior?

With Afriat's (1972) Critical Cost Efficiency Index (CCEI) that measures how much budget constraints should be relaxed to avoid violations of revealed preference axioms.

Problem 8

Problem 8 (Bonus)

Read The Money Pump as a Measure of Revealed Preference Violations by Echenique, Lee, and Shum, in the Journal of Political Economy, 2011.

(a) Aggregation of purchases over time. Why not use individual purchases?

Aggregation reduces stockpiling issues (a consumer is not likely to check the butter shelf today if she bought butter yesterday).

(b) Observation of prices from consumer data. What if prices vary daily and consumers visit the store on different days?

This might be problematic: even if a good is available to some purchasers at a given price, it might not be effectively available to all others at the same price.

(c) Separability of food purchases from other consumption: why might this be problematic?

If food not separable from other consumption in preferences, other consumption choices affect preferences over food also beyond income effects.

Problem 9

Problem 9 (Bonus)

Le Chatelier's Principle states that if one factor (say capital) is in fixed supply in short run, then the variable factor (say labor) reacts to factor price changes gradually. More concretely, following a change in the price of labor, one sees first a small adjustment in labor, then following a long-run adjustment in capital, a further adjustment in the same direction.

(a) Consider the following production problem. A firm produces one unit of output at price $p = 10$ by one of two technologies: i) Two units of l or ii) One unit of both l and k. Let the initial labor price be $w = 2$ and capital price $r = 3$. What is the optimal choice of technology? What happens in the short run (i.e. holding k fixed) if labor price changes to $w = 6$. What happens after capital adjusts? What about Le Chatelier's Principle?

- (i) is initially the cheapest technology, costing 4 per unit of output.
- After the labor price change, production cost 12 exceeds output price 10 with technology (i). So before capital can be adjusted, optimal to produce nothing.
- In the long run, capital can be adjusted to produce with technology (ii) at cost 9.
- This goes against Le Chatelier's principle, as the short-run adjustment of the variable factor is more extreme than the long-run reaction.

Problem 9

Problem 9 (Bonus)

(b) Let's try to find conditions where Le Chatelier's principle holds. Let $f(k,l)$ be a competitive firm's production function and consider parametric changes in the wage rate (i.e. we are fixing p, r throughout). Write the short-run and long-run problems as follows:

$$l(k;w) = \operatorname*{argmax}_{l \geq 0} pf(k,l) - wl - rk; \qquad k(w) = \operatorname*{argmax}_{k \geq 0} f(k, l(k;w)) - wl(k;w) - rk.$$

Assume that the maximizers are unique for both problems and show that if the cross partial derivative satisfies $f_{kl}(k,l) \geq 0$, then for $w' \geq w$, we have:

$$l(k(w'), w') \leq l(k(w), w') \leq l(k(w), w).$$

- The profit is supermodular in (k,l), since $\frac{\partial^2}{\partial k \partial l}(pf(k,l) - wl - rk) = pf_{kl}(k,l) \geq 0$.
- The profit has increasing differences in $((k,l), -w)$ since $\frac{\partial^2}{\partial k \partial(-w)}(pf(k,l) - wl - rk) = 0 \geq 0$ and $\frac{\partial^2}{\partial l \partial(-w)}(pf(k,l) - wl - rk) = 1 \geq 0$.

$\rightarrow$ By Topkis's thm, $l(k;w)$ is non-decreasing in k, and $l(k;w)$ and $k(w)$ are non-increasing in w. So $w' \geq w \implies k(w) \geq k(w')$ and $l(k(w'), w') \leq l(k(w), w') \leq l(k(w), w)$.