Advanced Microeconomics 1: Problem set 3 Solutions

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Problem 1

Problem 1

Find the expenditure function in the following cases.

(a) A consumer with preferences represented by

$$u(x_1, x_2) = 2x_1 + 3x_2.$$

- Perfect substitutes preferences. The MRS is constant, so the indifference curves are linear. Corner solutions are generic: only good 1 is consumed when $MRS_{1,2} > p_1/p_2$ (and to reach utility $\bar{u}$, she'll choose $x_1 = \bar{u}/2$), and only good 2 when $MRS_{1,2} < p_1/p_2$ (and to reach utility $\bar{u}$, she'll choose $x_2 = \bar{u}/3$).
- The expenditure function is

$$e(\boldsymbol{p}, \bar{u}) = \begin{cases} \frac{\bar{u}p_1}{2} & \text{if } p_1/p_2 < 2/3 \\ \frac{\bar{u}p_2}{3} & \text{if } p_1/p_2 \geq 2/3 \end{cases}$$

Problem 1

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Find the expenditure function in the following cases.

(b) A consumer with preferences represented by

$$u(x_1, x_2) = \min\{2x_1, 3x_2\}.$$

- Perfect complements preferences. The cheapest way to reach utility $\bar{u}$ is to consume so that $\bar{u} = 2x_1 = 3x_2$.
- The expenditure function is

$$e(\mathbf{p}, \bar{u}) = \bar{u}\left(\frac{p_1}{2} + \frac{p_2}{3}\right)$$

Problem 1

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(c) A consumer with preferences represented by

$$u(x_1, x_2) = \min\{2x_1 + 3x_2, 3x_1 + 2x_2\}.$$

- When $2x_1 + 3x_2 \leq 3x_1 + 2x_2$, i.e. $x_2 \leq x_1$, the utility is given by $2x_1 + 3x_2$, and in this case only good 1 is consumed if $p_1/p_2 < 2/3$ (cf. part a).
- On the other hand, when $x_2 \geq x_1$, the utility is given by $3x_1 + 2x_2$, and in this case only good 2 is consumed if $p_1/p_2 > 3/2$.
- When $2/3 \leq p_1/p_2 \leq 3/2$, optimal to consume reach $\bar{u}$ by consuming so that $\bar{u} = 2x_1 + 3x_2 = 3x_1 + 2x_2$ (cf. part b).
- The expenditure function is

$$e(\boldsymbol{p}, \bar{u}) = \begin{cases} p_1\bar{u}/2 \text{ if } p_1/p_2 < 2/3 \\ (\bar{u}/5)(p_1 + p_2) \text{ if } 2/3 \leq p_1/p_2 \leq 3/2 \\ p_2\bar{u}/2 \text{ if } p_1/p_2 > 3/2 \end{cases}$$

Problem 2

Problem 2

(a) Suppose that a consumer splits her income w between two goods x and y. Assume that she has twice differentiable strictly concave utility function $u(x, y)$. The government can finance government expenditures $g > 0$ by choosing either a proportional tax t_w on income or by taxing consumption of good x by rate t_x. The government budget constraint for the two cases reads: $t_w w = g$ and $t_x x\left(p_x, p_y, t_x\right) = g$. Show that the consumer prefers an income tax in this case.

Take choice (x^*, y^*) under a consumption tax. Given the government's budget constraint, $x^* p_x + y^* p_y = w - g$. Then (x^*, y^*) is also feasible under a lump-sum income tax g and no consumption tax, so the consumer must be weakly better off under the income tax. (If solutions are interior, the consumer must be strictly better off under the lump-sum income tax because of substitution.)

Problem 2

Problem 2

(b) Suppose now that there is no exogenous income in the model and good y is now interpreted as leisure. Assume that the consumer has an initial endowment y^e of leisure that she may sell to buy the other good. Hence the consumer's budget constraint is now:

$$p_x x + p_y y = p_y y^e.$$

Compare now the effect of taxes on x and y as in the previous part. Can you relate the comparison to the price elasticities of demand?

- Suppose that consumption of x is taxed at rate t_x and leisure y is taxed at rate t_y. (The analysis of a tax on work $y^e - y$ would be technically similar.)
- The consumer's utility maximization problem is then

$$\max_{x,y} u(x, y)$$
$$\text{s.t. } (p_x + t_x)x + (p_y + t_y)y = p_y y^e$$

Problem 2

- Solve the consumer's most preferred tax system that raises government revenue g:

$$\max_{t_x, t_y} v(p_x + t_x, p_y + t_y, p_y y^e)$$

$$\text{s.t.} \;\; t_x x(p_x + t_x, p_y + t_y, p_y y^e) + t_y y(p_x + t_x, p_y + t_y, p_y y^e) = g$$

  where $v(\cdot)$ is the indirect utility and $x(\cdot)$ and $y(\cdot)$ are the consumption choices when the consumer faces prices $p_x + t_x$ and $p_y + t_y$ and has endowment $p_y y^e$.

- Assuming an interior solution, the first-order conditions w.r.t. t_x and t_y are

$$-\frac{dv}{dt_x} = \mu\left(x + t_x \frac{dx}{dt_x} + t_y \frac{dy}{dt_x}\right) = 0, \qquad -\frac{dv}{dt_y} = \mu\left(y + t_x \frac{dx}{dt_y} + t_y \frac{dy}{dt_y}\right) = 0, \quad (1)$$

  where μ is the Lagrange multiplier of the government's budget constraint.

Problem 2

- By the envelope theorem, $dv/dt_x = -\lambda x$ and $dv/dt_x = -\lambda y$ where λ is the multiplier in the consumer's UMP. Plug into (1) and combine the equations in (1) to get

$$\frac{t_x}{x}\frac{dx}{dt_x} + \frac{t_y}{x}\frac{dy}{dt_x} = \frac{t_x}{y}\frac{dx}{dt_y} + \frac{t_y}{y}\frac{dy}{dt_y}$$

where we have the elasticity of x w.r.t. tax t_x on the LHS, and the elasticity of y w.r.t. tax t_y on the RHS. The condition hints that the consumer prefers to have a higher tax on the good whose consumption is not so elastic.

Problem 3

Problem 3

Show that for normal goods, the Hicksian demand for a good as a function of its own price (i.e. with all other prices and target utility fixed) is steeper than the Walrasian demand.

- The Slutsky equation gives us

$$\frac{\partial h_i(\boldsymbol{p}, \bar{u})}{\partial p_i} = \frac{\partial x_i(\boldsymbol{p}, w)}{\partial p_i} + \frac{\partial x_i(\boldsymbol{p}, w)}{\partial w} x_i. \tag{2}$$

- Both sides of (2) are non-positive since the Slutsky matrix is the Hessian of the expenditure function and therefore negative semi-definite, so the diagonal elements $\frac{\partial h_i(\boldsymbol{p},\bar{u})}{\partial p_i}$ are non-positive. Furthermore, for normal goods, $\frac{\partial x_i(\boldsymbol{p},w)}{\partial w} \geq 0$ and therefore

$$0 \geq \frac{\partial h_i(\boldsymbol{p}, \bar{u})}{\partial p_i} \geq \frac{\partial x_i(\boldsymbol{p}, w)}{\partial p_i}$$

Problem 4

Problem 4

Preferences are said to be *additively separable* if they can be represented by a utility function of the form: $u(\boldsymbol{x}) = \sum_{i=1}^{L} u_i(x_i)$. Suppose that $u_i(x_i)$ is strictly concave and twice differentiable and that the optimal consumption is interior (so that the demands are differentiable in prices).

(a) Show that all goods are normal.

- Clearly there must exist good k^* s.t. $\frac{\partial x_{k^*}}{\partial w} \geq 0$.
- For every good i, we must have the following satisfied

$$\frac{u_i'(x_i)}{u_{k^*}'(x_{k^*})} = \frac{p_i}{p_{k^*}}. \tag{3}$$

Increase in w increases x_{k^*} and therefore by concavity of u_{k^*} decreases the denominator on the LHS of (3). So increase in w must also decrease the numerator for the condition to continue to hold, implying $\frac{\partial x_i}{\partial w} \geq 0$ by the concavity of u_i.

Problem 4

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(b) Show also that for all i, j, k:

$$\frac{\partial x_i(\mathbf{p}, w)/\partial p_k}{\partial x_j(\mathbf{p}, w)/\partial p_k} = \frac{\partial x_i(\mathbf{p}, w)/\partial w}{\partial x_j(\mathbf{p}, w)/\partial w}.$$

- Given differentiability and interior solution, choice $x_i(\mathbf{p}, w)$ satisfies for all i

$$u_i'(x_i) - \lambda p_i = 0$$

- Totally differentiate w.r.t. p_k and w to get

$$u_i'(x_i)\frac{dx_i}{dp_k} = \frac{d\lambda}{dp_k}p_i, \qquad u_i'(x_i)\frac{dx_i}{dw} = \frac{d\lambda}{dw}p_i$$

Combine these with the same conditions for x_j to get the result.

Problem 5

Problem 5

(a) A monopolist choosing the profit maximizing price is facing a linear demand function $q = d(p) = a - p$, where $q = d(p)$ is the maximal quantity that can be sold at output price p. Her fixed cost is given by f and the constant marginal cost is $c > 0$. Solve the problem and find the value function.

The monopolist's problem is

$$\max_{q \geq 0} \mathbb{1}_{q>0}(q(a-q-c)-f)$$

Take the FOCs to find that whenever it is optimal to produce a positive quantity, the optimal quantity is $q = (a-c)/2$. The corresponding price is $p = (a+c)/2$ and profit is $((a-c)/2)^2 - f$. So the value function is

$$\pi(a,c,f) = \begin{cases} \left(\frac{a-c}{2}\right)^2 - f & \text{if } ((a-c)/2)^2 - f \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 5

Problem 5

(b) A profit maximizing monopolist facing a downward sloping demand $q = a - p$, and marginal cost $c(\beta)$, where β is the level of investment in cost reduction, and the cost of investment is $\gamma\beta^2$. What would you assume on the shape of $c(\beta)$? Write the first-order condition for the problem and compute the derivative of the value function to the problem (with respect to the parameter).

- The monopolist's problem is now $\max_{q,\beta\geq 0} q(a - q - c(\beta)) - \gamma\beta^2$
- It's natural to assume $c(\beta)$ to be decreasing and convex in β (although this doesn't guarantee necessity/sufficiency of FOCs).
- The FOCs w.r.t. q, β are

$$a - 2q - c(\beta) + \lambda_q = 0, \qquad -qc'(\beta) - 2\gamma\beta + \lambda_\beta = 0$$

  where λ_q, λ_β are non-negative Lagrange multipliers of the constraints $q, \beta \geq 0$.
- By the envelope theorem, for value function $\pi(a, \gamma)$: $\partial\pi/\partial a = q$, $\partial\pi/\partial\gamma = -\beta^2$.

Problem 6

Problem 6

A real valued function $f : \mathbb{R}^L_+ \to$ is called superadditive if for all z^1, z^2,

$$f\left(z^1 + z^2\right) \geq f\left(z^1\right) + f\left(z^2\right).$$

(a) Show that every cost function is superadditive in input prices.

- Let $\boldsymbol{z}^*(\boldsymbol{w}, q)$ be some solution to $\min_{\boldsymbol{z}:f(\boldsymbol{z})\geq q} \boldsymbol{w} \cdot \boldsymbol{z}$.
- By optimality of $\boldsymbol{z}^*(\boldsymbol{w}^1, q)$ and $\boldsymbol{z}^*(\boldsymbol{w}^2, q)$, we have

$$\boldsymbol{w}^1 \cdot \boldsymbol{z}^*(\boldsymbol{w}^1, q) \leq \boldsymbol{w}^1 \cdot \boldsymbol{z}^*(\boldsymbol{w}^1 + \boldsymbol{w}^2, q) \text{ and } \boldsymbol{w}^2 \cdot \boldsymbol{z}^*(\boldsymbol{w}^2, q) \leq \boldsymbol{w}^2 \cdot \boldsymbol{z}^*(\boldsymbol{w}^1 + \boldsymbol{w}^2, q)$$

$$\implies \boldsymbol{w}^1 \cdot \boldsymbol{z}^*(\boldsymbol{w}^1, q) + \boldsymbol{w}^2 \cdot \boldsymbol{z}^*(\boldsymbol{w}^2, q) \leq (\boldsymbol{w}^1 + \boldsymbol{w}^2) \cdot \boldsymbol{z}^*(\boldsymbol{w}^1 + \boldsymbol{w}^2, q)$$

$$\iff c(\boldsymbol{w}^1, q) + c(\boldsymbol{w}^2, q) \leq c(\boldsymbol{w}^1 + \boldsymbol{w}^2, q)$$

Problem 6

Problem 6

(b) Using this fact, show that the cost function is nondecreasing in input prices.

Take input price vectors $\boldsymbol{w}^1$, $\boldsymbol{w}^2$ such that $\boldsymbol{w}^1 \geq \boldsymbol{w}^2$. We show that $c(\boldsymbol{w}^1, q) \geq c(\boldsymbol{w}^2, q)$:

$$c(\boldsymbol{w}^1, q) = c(\boldsymbol{w}^1 + \boldsymbol{w}^2 - \boldsymbol{w}^2, q) \geq c(\boldsymbol{w}^1 - \boldsymbol{w}^2, q) + c(\boldsymbol{w}^2, q) \geq c(\boldsymbol{w}^2, q)$$

where the inequality follows from superadditivity.

Problem 7

Problem 7

An expected utility maximizing decision maker has a Bernoulli utility function for final wealth x given by $u(x) = -\frac{1}{x}$. Suppose her initial wealth is w and she is offered a gamble winning g with probability p and losing l with probability $(1 - p)$.

(a) What is her final wealth and expected utility if she accepts the gamble?

- If she wins, her final wealth is $w + g$. If she loses, her final wealth is $w - l$.
- Therefore, her expected wealth is $p(w + g) + (1 - p)(w - l) = w + pg - (1 - p)l$.
- Her expected utility is

$$pu(w + g) + (1 - p)u(w - l) = -p\frac{1}{w + g} - (1 - p)\frac{1}{w - l}$$

Problem 7

Problem 7

(b) What is her certainty equivalent to accepting the gamble?

The certainty equivalent c is such that

$$u(c) = pu(w+g) + (1-p)u(w-l) \iff \frac{1}{c} = p\frac{1}{w+g} + (1-p)\frac{1}{w-l}$$

The certainty equivalent is $c = \frac{(w+g)(w-l)}{w+(1-p)g-pl}$.

Problem 7

Problem 7

(c) Compute the certainty equivalent to another gamble that wins $g + \frac{\Delta}{p}$ with probability p and loses $l + \frac{\Delta}{1-p}$ with probability $1 - p$ with $\Delta > 0$. Compare to the previous part.

The new certainty equivalent c' satisfies

$$u(c') = pu(w+g+\frac{\Delta}{p})+(1-p)u(w-l-\frac{\Delta}{1-p}) \iff \frac{1}{c'} = p\frac{1}{w+g+\Delta/p}+(1-p)\frac{1}{w-l-\Delta}$$

We can solve that the new certainty equivalent is $c' = \frac{(w+g+\Delta/p)(w-l-\Delta/(1-p))}{w+(1-p)(g+\Delta/p)-p(l+\Delta/(1-p))}$. Note that

$$\frac{1}{c} - \frac{1}{c'} = \frac{\Delta}{(w+g)(w+g+\Delta/p)} - \frac{\Delta}{(w-l)(w-l-\Delta/(1-p))}$$

which is negative, that is, the CE of part (c), c', is smaller than the CE of part (b), c.

Problem 8

Problem 8 (Bonus)

A rational preference relation $\succeq$ satisfies betweenness if for all $p, q \in \mathcal{L}$ and all $\alpha \in (0, 1)$, we have

$$p \succ q \;\Rightarrow\; p \succ \alpha p + (1 - \alpha)\, q \succ q.$$

Show that for continuous rational preference relations betweenness implies the following condition: For all $p, q \in \mathcal{L}$ and all $\alpha \in (0, 1)$, we have

$$p \sim q \;\Rightarrow\; p \sim \alpha p + (1 - \alpha)\, q \sim q.$$

In other words, betweenness implies linearity of indifference curves in the Machina triangle.

Problem 8

Consider a continuous, rational preference relation that satisfies betweenness.

- Take any $p, q \in \mathcal{L}$ s.t. $p \sim q$, and take any $\tilde{x}$ s.t. $\tilde{x} \succ p$.
  - If such p, q don't exist, the proof is complete; if such $\tilde{x}$ doesn't exist, take $\tilde{y}$ s.t. $p \succ \tilde{y}$ and work analogously; and if such $\tilde{y}$ doesn't exist either, the proof is complete.
- Define sequence $x_n = (1/n)\tilde{x} + (1 - 1/n)q$.
- By betweenness and rationality, $\tilde{x} \succ x_n \succ p$ and $x_n \succ \alpha p + (1-\alpha)x_n \succ p$ for any $n \geq 2$ and $\alpha \in (0, 1)$. Then by continuity of preferences,

$$\lim_{n\to\infty} x_n = q \succeq \alpha p + (1-\alpha) \lim_{n\to\infty} x_n = \alpha p + (1-\alpha)q \succeq p$$

so by rationality, $p \sim \alpha p + (1-\alpha)q \sim q$.