Advanced Microeconomics 1: Problem set 4 Solutions

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Problem 1

Problem 1

Prove that if a risk averse decision maker rejects a fixed favorable bet at all levels of wealth, then the Bernoulli utility of the decision maker is bounded from above.

- At any wealth w, the agent rejects a bet that wins g w.p. p and loses l w.p. $1-p$:

$$u(w) \geq pu(w+g)+(1-p)u(w-l) \iff \frac{1-p}{p}\Big(u(w)-u(w-l)\Big) \geq \Big(u(w+g)-u(w)\Big).$$

- By the previous inequality and concavity,

$$u'(w+g) \leq \frac{u(w+g)-u(w)}{g} \leq \frac{1-p}{p}\frac{l}{g}\frac{u(w)-u(w-l)}{l} \leq \frac{1-p}{p}\frac{l}{g}u'(w-l).$$

- Since the bet is also rejected at wealth $w+g+l$ etc., we get similarly

$$u'(w+2g+l) \leq \frac{1-p}{p}\frac{l}{g}u'(w+g) \leq \left(\frac{1-p}{p}\frac{l}{g}\right)^2 u'(w-l), \text{ and generally}$$

$$u'(w+n(g+l)-l) \leq \left(\frac{1-p}{p}\frac{l}{g}\right)^n u'(w-l) \text{ for any } n \in \mathbb{N}. \qquad (1)$$

Problem 1

- By concavity, for any $n \in \mathbb{N}$,

$$\begin{aligned} u(w+n(g+l)-l) &\leq u(w-l)+\sum_{k=0}^{n-1}(g+l)u'(w+k(g+l)-l) \\ &\leq u(w-l)+\sum_{k=0}^{n-1}(g+l)\left(\frac{1-p}{p}\frac{l}{g}\right)^k u'(w-l) \end{aligned}$$

  where the second inequality uses inequality (1).
- Since the bet is favorable, $\frac{1-p}{p}\frac{l}{g} < 1$, so the geometric series on the RHS converges and we find that u is bounded above by

$$u(w-l)+\frac{(g+l)u'(w-l)}{1-\frac{1-p}{p}\frac{l}{g}}.$$

Problem 2

Problem 2

Consider the savings and consumption model analyzed in lectures. There are two periods, $t = 0, 1$. The decision maker has a strictly concave separable Bernoulli utility function

$$u(c_0, c_1) = u_0(c_0) + \delta u_1(c_1),$$

where c_t denotes consumption in period t. Assume that the consumer receives a certain income w_0 in period 0 and a random income $\tilde{w}_1$ in period 1. The only means for transferring wealth between periods for the consumer is by either borrowing or lending at a risk free rate r.

Problem 2

Problem 2

(a) Set up the consumer's intertemporal budget constraint and characterize the solution to the savings problem through first order conditions (are these also sufficient conditions?).

- Denote savings by $s := w_0 - c_0$. The budget constraint is then $c_1 \leq \tilde{w}_1 + s(1+r)$. If u_0, u_1 are strictly increasing, the constraint binds, and the problem can be written as

$$\max_s u_0(w_0 - s) + \delta \mathbb{E}[u_1(\tilde{w}_1 + s(1+r))]$$

- It might be also reasonable to restrict $s \in [-\underline{\tilde{w}}_1/(1+r), w_0]$ where $\underline{\tilde{w}}_1$ is a lower bound of period-1 income (if such exists), so that the consumer cannot borrow money she is not able to repay or lend money she does not have.
- Verify that strict concavity of the objective function in s follows from strict concavity of u_0 and u_1. Then, assuming interior solution, the following FOC is sufficient for optimality :

$$u_0'(w_0 - s) = \delta \mathbb{E}[u_1'(\tilde{w}_1 + s(1+r))](1+r)$$

Problem 2

Problem 2

(b) Consider the changes in optimal savings resulting from changes in interest rate r. Can you find an income and a substitution effect in your expression for $\frac{ds}{dr}$?

Apply the implicit function theorem to the FOC to get:

$$\frac{ds}{dr} = \frac{\delta\mathbb{E}[u_1''(\tilde{w}_1 + s(1+r))]s(1+r) + \delta\mathbb{E}[u_1'(\tilde{w}_1 + s(1+r))]}{-u_0''(w_0 - s) - \delta\mathbb{E}[u_1''(\tilde{w}_1 + s(1+r))](1+r)^2}$$

This is positive if the numerator is positive.

- Term $\delta\mathbb{E}[u_1''(\tilde{w}_1 + s(1+r))]s(1+r)$ in the numerator can be positive or negative depending on whether the consumer is a lender, $s < 0$, or borrower, $s > 0$. For a lender (borrower), the term is negative (positive): increase in r increases (decreases) her consumption in period 1 for given s. This is an income effect.
- Term $\delta\mathbb{E}[u_1'(\tilde{w}_1 + s(1+r))]$ in the numerator is always positive and relates to the substitution effect: an increase in r decreases the relative price of period-2 consumption, incentivizing higher savings.

Problem 2

Problem 2

(c) Show that when the Arrow-Pratt coefficient of relative risk aversion is less than unity, savings increase in interest rate.

- The Arrow-Pratt coefficient of relative risk aversion being less than unity means

$$-\frac{(\tilde{w}_1 + s(1+r))u_1''(\tilde{w}_1 + s(1+r))}{u'(\tilde{w}_1 + s(1+r))} < 1 \iff u_1''(\tilde{w}_1 + s(1+r)) > -\frac{u'(\tilde{w}_1 + s(1+r))}{\tilde{w}_1 + s(1+r)}$$

- Then, using the formula for ds/dr in (b):

$$\frac{ds}{dr} > \frac{\delta \mathbb{E}[u_1'(\tilde{w}_1 + s(1+r))(1 - \frac{s(1+r)}{\tilde{w}_1 + s(1+r)})]}{-u_0''(w_0 - s) - \delta \mathbb{E}[u_1''(\tilde{w}_1 + s(1+r))](1+r)^2}$$

where the denominator is still positive and the numerator is positive when $\tilde{w}_1 > -s(1+r)$, which must be true when the consumer is not allowed to borrow money she is not able to repay.

Problem 3

Problem 3

Consider the model of the previous exercise. Assume that $u_i(c_i) = \alpha + \beta c_i + (\gamma - c_i)^2$.

(a) What is the range for possible consumptions where utility is increasing in consumption?

$$u_i'(c_i) > 0 \iff c_i > \gamma - \beta/2$$

(b) Assume that all the possible realizations from $\tilde{w}_1$ lie in the range found in part a. Does the demand for savings depend on the riskiness of the distribution of $\tilde{w}_1$?

[Edited.] Now u_0 and u_1 are convex and therefore $u_0(w_0 - s) + \delta\mathbb{E}[u_1(\tilde{w}_1 + s(1+r))]$ is strictly *convex* in savings s. The optimal s is an extreme point of the feasible set, i.e. the consumer chooses the least or greatest s she is allowed: $s = -\underline{w}_1/(1+r)$ or $s = w_0$. Riskiness of $\tilde{w}_1$ can affect the optimal s by changing the feasible set of s and the expected utilities at extreme points.

Problem 4

Problem 4

Consider the following model of criminal behavior due to Becker. An individual's income is w, and her monetary benefit from crime is b, where b is a random variable coming from a continuously differentiable strictly positive density function (on the entire real line) $g(b)$. If the individual commits a crime, then she will be caught with probability π and in this case she must pay a fine F.

(a) Show first that there is a unique cutoff level b^* such that the individual commits the crime if and only if $b > b^*$.

The expected utility from committing the crime is $(1-\pi)u(w+b)+\pi u(w+b-F)$. At cutoff level $b = b^*$, $(1-\pi)u(w+b)+\pi u(w+b-F)-u(w)=0$. There is a unique solution to the equation as the LHS continuous and strictly increasing in b, positive at $b > F$ and negative at $b < 0$.

Problem 4

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(b) Show next that b^* is increasing in π and F.

By the implicit function theorem,

$$\frac{db^*}{d\pi} = \frac{u(w+b^*) - u(w+b^*-F)}{(1-\pi)u'(w+b^*) + \pi u'(w+b^*-F)},$$

$$\frac{db^*}{dF} = \frac{\pi u'(w+b^*-F)}{(1-\pi)u'(w+b^*) + \pi u'(w+b^*-F)}$$

which are both positive since u_i is strictly increasing.

Problem 4

Problem 4

(c) Suppose next that if caught in a crime, the individual must go to jail for fraction f of her total labor time. Then we have $F = fw$. Show that if the coefficient of relative risk aversion is less than 1, then b^* is increasing in w.

Now cutoff b^* satisfies

$$(1-\pi)u(w+b^*) + \pi u((1-f)w+b^*) = u(w)$$

By the implicit function theorem,

$$\frac{db^*}{dw} = \frac{u'(w) - (1-\pi)u'(w+b^*) - \pi(1-f)u'((1-f)w+b^*)}{(1-\pi)u'(w+b^*) + \pi u'((1-f)w+b^*)} \tag{2}$$

where the denominator is always positive.

Problem 4

By concavity,

$$(1-\pi)\,u'(w) > (1-\pi)\,u'(w+b^*)\,. \tag{3}$$

On the other hand, $r_R < 1$ implies that

$$\frac{d}{da}au'(ax) = u'(ax) + au''(ax)\,x = u'(ax)\left[1 - \left(-\frac{axu''(ax)}{u'(ax)}\right)\right] > 0.$$

Thus we have

$$\pi u'(w) > \pi(1-f)\,u'((1-f)\,w)\,. \tag{4}$$

Summing up equations (3) and (4) gives

$$u'(w) > (1-\pi)\,u'(w+b^*) + \pi(1-f)\,u'((1-f)\,w)\,.$$

so the numerator in (2) is positive, therefore $db^*/dw > 0$.

Problem 5

Problem 5

Consider an economy where all agents face an independent risk to lose 100 with probability p. N agents decide to create a mutual agreement where the aggregate loss in the pool is equally split among its members.

(a) Describe the change in the lotteries facing individuals in the pool when N is changed from 2 to 3.

With $N = 2$, an individual engaging in the agreement faces the following lottery:

- W. p. p^2, the risk is realized for both agents $\rightarrow$ the individual loses $200/2$.
- W. p. $2p(1-p)$, the risk is realized for one agent $\rightarrow$ the individual loses $100/2$.
- W. p. $(1-p)^2$, the risk is realized for no agent $\rightarrow$ the individual loses $0/2$.

The expected utility at wealth w is $p^2u(w-100) + 2p(1-p)u(w-50) + (1-p)^2u(w)$.

Problem 5

With $N = 3$, an individual engaging in the agreement faces the following lottery:

- W. p. p^3, the risk is realized for 3 agents $\rightarrow$ the individual loses $300/3$.
- W. p. $3p^2(1-p)$, the risk is realized for two agents $\rightarrow$ the individual loses $200/3$.
- W. p. $3p(1-p)^2$, the risk is realized for one agent $\rightarrow$ the individual loses $100/3$.
- W. p. $(1-p)^3$, the risk is realized for no agent $\rightarrow$ the individual loses $0/3$.

The expected utility at wealth w is
$p^3 u(w-100) + 3p^2(1-p)u(w-200/3) + 3p(1-p)^2 u(w-100/3) + (1-p)^3 u(w)$.

Problem 5

Problem 5

(b) Show that the risk with $N = 3$ is smaller in the sense of second order stochastic dominance that the risk with $N = 2$.

With $N = 2$, cdf F_2 of an individual final outcome x is given by

$$F_2(x) = \begin{cases} 0 \text{ if } x < w - 100 \\ p^2 \text{ if } x \in [w-100, w-50) \\ p^2 + 2p(1-p) \text{ if } x \in [w-50, w) \\ 1 \text{ if } x \geq w \end{cases}$$

$$\int_{w-100}^{x} F_2(s)ds = \begin{cases} p^2\big(x - (w-100)\big) \text{ if } x \in [w-100, w-50) \\ 50p^2 + \big(p^2 + 2p(1-p)\big)\big(x - (w-50)\big) \text{ if } x \in [w-50, w) \\ 100p^2 + 50 \cdot 2p(1-p) = 100p \text{ if } x = w \end{cases}$$

Problem 5

With $N = 3$, cdf F_3 of an individual final outcome x is given by

$$F_3(x) = \begin{cases} 0 \text{ if } x < w - 100 \\ p^3 \text{ if } x \in [w - 100, w - 200/3) \\ p^3 + 3p^2(1-p) \text{ if } x \in [w - 200/3, w - 100/3) \\ 1 - (1-p)^3 \text{ if } x \in [w - 100/3, w) \\ 1 \text{ if } x \geq w \end{cases}$$

$$\int_{w-100}^{x} F_3(s)ds = \begin{cases} p^3\big(x - (w - 100)\big) \text{ if } x \in [w - 100, w - 200/3) \\ p^3(100/3) + \big(p^3 + 3p^2(1-p)\big)(x - (w - 200/3)) \text{ if } x \in [w - 200/3, w - 100/3) \\ p^3\frac{200}{3} + 3p^2(1-p)\frac{100}{3} + (1 - (1-p)^3)(x - (w - \frac{100}{3})) \text{ if } x \in [w - 100/3, w) \\ 100p^3 + \frac{200}{3}3p^2(1-p) + \frac{100}{3}3p(1-p)^2 = 100p \text{ if } x = w \end{cases}$$

We can verify that $\int_{w-100}^{x} F_3(s)ds \leq \int_{w-100}^{x} F_2(s)ds$ for all $x \in [w - 100, w]$ and $\int_{w-100}^{w} F_3(s)ds = \int_{w-100}^{w} F_2(s)ds$, so the risk with $N = 3$ is smaller in the sense of SOSD.

Bonus Problem 1

Problem 1

(a) In order to aid the poor, the Government introduces a scheme whereby the first 1kg of butter a family buys is subsidized and the remaining amounts are taxed. Consider a family which consumes butter and is made neither better off nor worse off as a result of this scheme. Is it correct to state that the total amount of tax this family pays cannot exceed the subsidy it receives? Explain your answer.

Yes, it is correct (given that the family has strictly monotonic rational preferences). Suppose that the tax exceeds the subsidy. Then, the cost of the new consumption choice must have been lower before the introduction of the government scheme than after. So the family must have been strictly better off before the scheme.

Bonus Problem 1

Problem 1

(b) A consumer buys one unit of a good when its price is €2 and two units when its price is €1. Is it correct to state that he would rather pay €2.80 for two units of the good than go without it altogether? Explain your answer.

No, because of income effects. For example, suppose the budget is 3 and there is a second good that the consumer needs one unit (price 1) to survive[1]. The observed consumption patterns are then perfectly possible but the consumer would clearly rather go without the first good than spend €2.80 on 2 units of it.

[1] So that any bundle with one unit of the second good is preferred to any bundle with a smaller amount of the second good.

Bonus Problem 1

Problem 1

(c) You can only adjust your consumption of x_2 in the long run, but x_1 is flexible in the short run. Is it true that if x_1 is normal, then the demand for x_1 is more elastic in the long run than in the short run? Explain your answer.

Yes. Given that x_1 is normal, its demand decreases in its own price. In the long run, there is a positive adjustment in x_2 if the goods are substitutes and a negative adjustment if the goods are complements. In either case, the adjustment in x_2 further decreases the demand for x_1.

Bonus Problem 2

Problem 2

A consumer in a three-commodity environment (x, y, z) behaves as follows. *i)* When prices are $p_x = 1, p_y = 1$ and $p_z = 1$ the consumer buys $x = 1, y = 2$ and $z = 3$;
ii) When prices are $p_x = 4, p_y = 6$ and $p_z = 4$ the consumer buys $x = 3, y = 2$ and $z = 1$.
Does the consumer maximize a strictly quasi-concave utility function?

No. Maximizing a strictly quasiconcave utility function implies that there is a unique optimal choice. However, both bundles $(1, 2, 3)$ and $(3, 2, 1)$ are budget feasible in both budget situations (i) and (ii), so the consumer must be indifferent between the two bundles to choose $(1, 2, 3)$ in situation (i) and $(3, 2, 1)$ in situation (ii).

Bonus Problem 3

Problem 3

Suppose that the expenditure function of a consumer is of Gorman polar form:

$$e\left(p,u\right)=a\left(p\right)+ub\left(\boldsymbol{p}\right).$$

Derive the demands for each good and calculate also the income shares that each good receives. Can you find an economic interpretation for your results.

- By duality, we have with Gorman polar form $v(\boldsymbol{p},w)=\frac{w-a(\boldsymbol{p})}{b(\boldsymbol{p})}$.
- By Shephard's lemma,

$$h_i(\boldsymbol{p},\bar{u})=\frac{\partial e\left(\boldsymbol{p},u\right)}{\partial p_i}=\frac{\partial a(\boldsymbol{p})}{\partial p_i}+\bar{u}\frac{\partial b(\boldsymbol{p})}{\partial p_i}.$$

- Then

$$x_i(\boldsymbol{p},w)=h_i(\boldsymbol{p},v(\boldsymbol{p},w))=\frac{\partial a(\boldsymbol{p})}{\partial p_i}+\frac{w-a(\boldsymbol{p})}{b(\boldsymbol{p})}\frac{\partial b(\boldsymbol{p})}{\partial p_i}.$$

Bonus Problem 3

- Demand is linear in wealth, so if consumers' expenditure functions take the same Gorman polar form, the distribution of wealth does not matter for aggregate demand.
- The income share received by good x_i is

$$\frac{p_i x_i}{w} = \frac{p_i}{w}\left(\frac{\partial a(\boldsymbol{p})}{\partial p_i} + \frac{w - a(\boldsymbol{p})}{b(\boldsymbol{p})}\frac{\partial b(\boldsymbol{p})}{\partial p_i}\right)$$

which is constant in w only in a special case (homothetic preferences).

Bonus Problem 4

Problem 4

Consider the aggregate demand in a model where individual consumers behave in a random manner (and thus do not satisfy any of the rationality criteria that we had for individual choice). To be more specific, assume that a consumer with wealth w facing prices $\boldsymbol{p}$ picks a consumption vector at random from the budget set $B(\boldsymbol{p}, w) = \{x : p \cdot x = w\}$ according to the uniform distribution. Suppose furthermore that there are a continuum of such consumers (and assume that you can apply the law of large numbers for this setting, i.e. the distribution of realized choices in the population coincides with the distribution of a single consumer's choice).

(a) Denote the individual (random) demand by $x^i(p, w)$. Compute the average demand

$$\overline{\boldsymbol{x}}(\boldsymbol{p}, w) = \int \boldsymbol{x}^i(\boldsymbol{p}, w)\, di.$$

Average expenditure shares of goods must be equal, so with L goods, for each good $k \in \{1, \ldots, L\}$, $\bar{x}_k(\boldsymbol{p}, w) = \frac{w}{Lp_k}$.

Bonus Problem 4

Problem 4

(b) Does this average demand satisfy weak axiom of revealed preference?

Yes. Suppose that $\boldsymbol{p}' \cdot \overline{\boldsymbol{x}}(\boldsymbol{p}, w) \leq w'$, that is,

$$\frac{w}{L} \sum_{k=1}^{L} \frac{p_k'}{p_k} \leq w' \tag{5}$$

$$\Longrightarrow \; w \leq L \frac{w'}{\sum_{k=1}^{L}(p_k'/p_k)} \leq \frac{w' \sum_{k=1}^{L}(p_k/p_k')}{L} = \boldsymbol{p} \cdot \overline{\boldsymbol{x}}(\boldsymbol{p}', w') \tag{6}$$

where the last inequality is the "AM-HM inequality" and holds as a strict inequality except if for some $a \in \mathbb{R}_+$ and all $k \in \{1, \ldots, L\}$, $p_k'/p_k = a$.

- IF for some $a \in \mathbb{R}_+$ and all $k \in \{1, \ldots, L\}$, $p_k'/p_k = a$, then we have both $\boldsymbol{p}' \cdot \overline{\boldsymbol{x}}(\boldsymbol{p}, w) \leq w'$ and $\boldsymbol{p} \cdot \overline{\boldsymbol{x}}(\boldsymbol{p}', w') \leq w$ if and only if also $w' = aw$. In this case, $\boldsymbol{x}(\boldsymbol{p}, w) = \boldsymbol{x}(\boldsymbol{p}', w')$.

So we proved that if $\boldsymbol{p}' \cdot \overline{\boldsymbol{x}}(\boldsymbol{p}, w) \leq w'$ and $\boldsymbol{p} \cdot \overline{\boldsymbol{x}}(\boldsymbol{p}', w') \leq w$ then $\overline{\boldsymbol{x}}(\boldsymbol{p}, w) = \overline{\boldsymbol{x}}(\boldsymbol{p}', w')$, i.e., $\overline{\boldsymbol{x}}$ satisfies WARP.

Bonus Problem 4

Problem 4

(c) Can you find a utility function such that $\overline{\boldsymbol{x}}(\boldsymbol{p}, w)$ is the Walrasian demand function for that utility function?

Since expenditure shares of goods are equal, $1/L$, this average demand would result from maximization of specific Cobb-Douglas utility $u(\boldsymbol{x}) = x_1 x_2 \cdots x_L$ (see the Lecture Notes on Cobb-Douglas utility).

In fact, this would also suffice to prove part (b).