Appendix C

Matrix Factorizations

1. $A = LU = \begin{pmatrix} \text{lower triangular } L \\ \text{1s on the diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{pivots on the diagonal} \end{pmatrix}$

Requirements: No row exchanges as Gaussian elimination reduces A to U.

2. $A = LDU = \begin{pmatrix} \text{lower triangular } L \\ \text{1s on the diagonal} \end{pmatrix} \begin{pmatrix} \text{pivot matrix} \\ D \text{ is diagonal} \end{pmatrix} \begin{pmatrix} \text{upper triangular } U \\ \text{1s on the diagonal} \end{pmatrix}$

Requirements: No row exchanges. The pivots in D are divided out to leave 1s in U. If A is symmetric, then U is L^{T} and $A = LDL^{\mathrm{T}}$.

3. $PA = LU$ (permutation matrix P to avoid zeros in the pivot positions).

Requirements: A is invertible. Then P, L, U are invertible. P does the row exchanges in advance. Alternative: $A = L_1 P_1 U_1$.

4. $EA = R$ ($m \times m$ invertible E) (any A) $= \mathrm{rref}(A)$.

Requirements: None! *The reduced row echelon form* R has r pivot rows and pivot columns. The only nonzero in a pivot column is the unit pivot. The Last $m - r$ rows of E are a basis for the left nullspace of A. and the first r columns of E^{-1} are a basis for the column space of A.

5. $A = CC^{\mathrm{T}} = \big(\text{lower triangular matrix } C\big) \big(\text{transpose is upper triangular}\big)$

Requirements: A is symmetric and positive definite (all n pivots in D are positive). This *Cholesky factorization* has $C = L\sqrt{D}$.

6. $A = QR = \big(\text{orthonormal columns in } Q\big) \big(\text{upper triangular } R\big)$

Requirements: A has independent columns. Those are *orthogonalized* in Q by the Gram-Schmidt process. If A is square, then $Q^{-1} = Q^{\mathrm{T}}$.

7. $A = S\Lambda S^{-1} = \big(\text{eigenvectors in } S\big) \big(\text{eigenvalues in } \Lambda\big) \big(\text{left eigenvectors in } S^{-1}\big)$.

Requirements: A must have n linearly independent eigenvectors.

8. $A = Q\Lambda Q^{\mathrm{T}} = \big(\text{orthogonal matrix } Q\big) \big(\text{real eigenvalue matrix } \Lambda\big) \big(Q^{\mathrm{T}} \text{ is } Q^{-1}\big)$.

Requirements: A is *symmetric*. This is the Spectral Theorem.

9. $A = MJM^{-1} = \Big(\text{generalized eigenvectors in } M\Big)\Big(\text{Jordan blocks in } J\Big)\Big(M^{-1}\Big).$

Requirements: A is any square matrix. *Jordan form* J has a block for each independent eigenvector of A. Each block has one eigenvalue.

10. $A = U\Sigma V^{\mathrm{T}} = \begin{pmatrix}\text{orthogonal}\\ U \text{ is } m\times m\end{pmatrix}\begin{pmatrix} m\times n \text{ matrix } \Sigma \\ \sigma_1,\ldots,\sigma_r \text{ on diagonal}\end{pmatrix}\begin{pmatrix}\text{orthogonal}\\ V \text{ is } n\times n\end{pmatrix}.$

Requirements: None. This *singular value decomposition* (SVD) has the eigenvectors of AA^{T} in U and of $A^{\mathrm{T}}A$ in V; $\sigma_i = \sqrt{\lambda_i(A^{\mathrm{T}}A)} = \sqrt{\lambda_i(AA^{\mathrm{T}})}$.

11. $A^+ = V\Sigma^+U^{\mathrm{T}} = \begin{pmatrix}\text{orthogonal}\\ n\times n\end{pmatrix}\begin{pmatrix}\text{diagonal } n\times m\\ 1/\sigma_1,\ldots,1/\sigma_r\end{pmatrix}\begin{pmatrix}\text{orthogonal}\\ m\times m\end{pmatrix}.$

Requirements: None. The *pseudoinverse* has $A^+A =$ projection onto row space of A and $AA^+ =$ projection onto column space. The shortest least-squares solution to $Ax = b$ is $\widehat{x} = A^+b$. This solves $A^{\mathrm{T}}A\widehat{x} = A^{\mathrm{T}}b$.

12. $A = QH = \Big(\text{orthogonal matrix } Q\Big)\Big(\text{symmetric positive definite matrix } H\Big).$

Requirements: A is invertible. This *polar decomposition* has $H^2 = A^{\mathrm{T}}A$. The factor H is semidefinite if A is singular. The reverse polar decomposition $A = KQ$ has $K^2 = AA^{\mathrm{T}}$. Both have $Q = UV^{\mathrm{T}}$ from the SVD.

13. $A = U\Lambda U^{-1} = \Big(\text{unitary } U\Big)\Big(\text{eigenvalue matrix } \Lambda\Big)\Big(U^{-1} = U^H = \overline{U}^{\mathrm{T}}\Big).$

Requirements: A is *normal*: $A^{\mathrm{H}}A = AA^{\mathrm{H}}$. Its orthonormal (and possibly complex) eigenvectors are the columns of U. Complex λ's unless $A = A^{\mathrm{H}}$.

14. $A = UTU^{-1} = \Big(\text{unitary } U\Big)\Big(\text{triangular } T \text{ with } \lambda\text{'s on diagonal}\Big)\Big(U^{-1} = U^{\mathrm{H}}\Big).$

Requirements: *Schur triangularization* of any square A. There is a matrix U with orthonormal columns that makes $U^{-1}AU$ triangular.

15. $F_n = \begin{bmatrix} I & D \\ I & -D\end{bmatrix}\begin{bmatrix} F_{n/2} & \\ & F_{n/2}\end{bmatrix}\begin{bmatrix}\text{even-odd}\\ \text{permutation}\end{bmatrix} =$ one step of the **FFT**.

Requirements: $F_n =$ Fourier matrix with entries w^{jk} where $w^n = 1$, $w = e^{2\pi i/n}$. Then $F_n\overline{F}_n = nI$. D has $1, w, w^2, \ldots$ on its diagonal. For $n = 2^\ell$ the *Fast Fourier Transform* has $\frac{1}{2}n\ell$ multiplications from ℓ stages of D's.