Demonstration: Binary Decision Diagrams

Shannon expansion of a formula ϕ with respect to variable x is

$$(x \wedge \phi[\top/x]) \vee (\neg x \wedge \phi[\bot/x])$$

where $\phi[\psi/x]$ denotes substituting all occurrences of x in ϕ by ψ.

Binary Decision Diagrams (BDD) are a representation of propositional formulas with many useful properties, including constant-time test for logical equivalence.

We will illustrate properties of BDDs, including their construction by repeated applications of Shannon expansion.

1 Exercise

A literal is an atomic proposition x or a negated atomic proposition $\neg x$. A clause is a disjunction of literals $l_1 \vee \cdots \vee l_n$.

Following Boole's representation of propositional formulas as numeric expressions, sketch a general translation from clauses into linear integer inequalities. You can use inequalities of the following forms (the first two are special cases of the third.)

$$\begin{aligned} &x \geq c \\ &c \geq x \\ &c_1 \cdot x_1 + c_2 \cdot x_2 + \cdots + c_n \cdot x_n \geq c \end{aligned}$$

Here c and $c_i, i \geq 1$ are integer constants and x and $x_i, i \geq 1$ are integer variables.

Hint: Limit all your integer variables to have value 0 or 1.

2 Exercise

By using the previous representation, do the following.

1. Verify that the integer representation of the satisfiable set of clauses

   $$\begin{aligned} &\neg a \vee b \vee c \\ &\neg b \vee \neg c \\ &a \end{aligned}$$

   is a solvable set of inequalities.

2. The following set is unsatisfiable. Construct the corresponding set of integer inequalities, and show that it has no solutions.

   $$\begin{aligned} &a \vee b \\ &a \vee \neg b \\ &\neg a \vee b \\ &\neg a \vee \neg b \end{aligned}$$

   Does it have solutions if the variables are real-valued? Explain.

3 Exercise

Consider a general modeling language that allows formalizing actions' effects as simple imperative programs, similar to conventional programming languages, like the program notation in Definition 4.15 in *"Propositional Logic and Its Applications in Artificial Intelligence"* available from the course's supplementary materials page.

1. Consider a goal $x = 10$ and an action with effect $x := 2 \times x + 5$. Use the weakest precondition predicate (Definition 4.16) to compute the condition that must hold before taking the action so that the goal will be achieved. (You can ignore the possible precondition of the action.) Simplify the condition to as simple form as possible.
2. Similarly, consider goal $x = 10$ and effect IF $y = 1$ THEN x := $x + 5$ ELSE x := $5x$.

4 Exercise

Let $X = \{C, D, E, F\}$. What sets do the following formulas correspond to?

1. $(C \rightarrow D) \wedge (D \rightarrow E) \wedge (E \rightarrow F)$
2. $E \wedge (C \vee D)$
3. $(C \leftrightarrow D) \wedge (D \leftrightarrow E) \wedge (E \leftrightarrow F)$
4. $(C \leftrightarrow D) \wedge (D \leftrightarrow E)$

5 Exercise

Let $X = \{C, D, E, F\}$. Give formulas that represent the following sets. Give as small (simple, intuitive) a formula as possible.

1. $\{0000, 0001\}$
2. $\{0011, 1100\}$
3. $\{0000, 0010, 0110, 1110\}$