CS-E4800 AI Exercises 9: Decision-Making Multi-Agent Settings March 14, 2017

1 Exercise

Find an example where 2-Candidate Runoff with plurality rule and Single Transferable Vote with plurality rule lead to different results.

Hint: Construct rankings so that A and B go to the second round in the 2-candidate runoff but C beats them under STV after lower-ranked candidates have been eliminated.

Solution

Let the preferences be as follows.

2×	2×	1×	1×	1×
A	*B*	*C*	*E*	*F*
C	*C*	*D*	*C*	*C*
D	*E*	*E*	*F*	*D*
E	*F*	*F*	*D*	*E*
F	*D*	*A*	*A*	*B*
B	*A*	*B*	*B*	*A*

A and B both have most first votes, and will go to the second round (A winning).

With STV, D will be first eliminated, then E and F, then B and A, and, finally, C will be the winner.

2 Exercise

Does voting with the Borda count have the following properties? If so, give a proof, or otherwise give a counter-example.

1. The winner is the top choice for at least one voter.
2. Assume candidate X wins. If some other candidate had not been running, and the individual orderings of the candidates had otherwise remained the same, then X would still necessarily have won.

Solution

1. *Does* **not** *hold.*

   Consider two candidates A and B, and two voters, with the following preference ordering.

points	*voter 1*	*voter 2*
3	*A*	*B*
2	*C*	*C*
1	*D*	*D*
0	*B*	*A*

   (Hence A and B are both very popular with some of the voters, but also very unpopular with others, while C is consistently appreciated by all voters, but not the top candidate for anybody.)

   This means the following total points for the 4 candidates.

candidate	*points*
A	*3*
B	*3*
C	*4*
D	*2*

   Hence C is the winner, but not anybody's top candidate.

2. *Does* **not** *hold.*

   Consider a vote with three candidates A, B and C. There are only two types of voters (respectively 3 and 2 of each type), which respectively have preference ordering B, A, C and A, C, B, leading to the following points according to the Borda rule.

	3×	2×
2 points	*B*	*A*
1 point	*A*	*C*
0 points	*C*	*B*

   The total scores for the three candidates are therefore the following.

candidate	*points*
A	*7 points*
B	*6 points*
C	*2 points*

Which means that A wins.

Let the voters and preference orderings be the same, but assume candidate C is not running. This affects the points, because the top candidate is now getting relatively more than the other candidates.

	$3\times$	$2\times$
1 point	*B*	*A*
0 points	*A*	*B*

Relatively speaking, the first one now gets more points than the second one, leading to a advantage to B over A. This results in the following total scores for the two candidates, meaning that B wins.

candidate	*points*
B	*3 points*
A	*2 points*

3 Exercise

Consider the Vickrey-Clarke-Groves mechanism with the Clarke pivot rule, as described in the lecture.

1. Consider a very small town that wants to decide between
   (a) building a new road,
   (b) building a bridge over a river that passes through the town.

   The three inhabitants of the town value these two projects differently, based on their need for the road and the bridge. They express their preferenes by declaring the following valuations for the two projects.

person	Bridge	Road
A	10	-5
B	3	3
C	4	6

   The collective value of the bridge – based on the public declarations – is 10+3+4=17, and the collective value of the road is -5+3+6=4. So it is the bridge that will be built.
   Calculate the payments and total utilities for all agents.
2. Show that in VCG, all the payments will be zero or negative.
3. Show that in VCG, if all declared valuations are non-negative, an agent's value+payment (for the chosen outcome) is always non-negative.
4. Explain what can happen, if an agent declares a value of some alternative un-truthfully, A) too high, B) too low.

Solution

1. *Without A, Bridge would have a lower utility 7=3+4 than the Road's 9=3+6, and the difference between these two is -2=7-9. Hence agent A has to pay 2.*
   Ignoring either B or C alone would not have changed the result, so neither has to pay (essentially, because Bridge was chosen because of A).
   So with the Bridge, the payments and utilities are as follows.

agent	*payment*	*utility*
A	-2	*8*
B	*0*	*3*
C	*0*	*4*

2. *The chosen outcome is*

$$x^{opt} = \arg\max_{x \in X} \sum_{i=1}^{n} v_i(x).$$

Agent i is paid (must pay if < 0)

$$\sum_{j \neq i} v_j(x^{opt}) - \max_{x \in X} \sum_{j \neq i} v_j(x)$$

Assume

$$x^{opt} = \arg\max_{j \neq i} \sum_{i=1}^{n} v_i(x).$$

Hence the payment is trivially zero. Otherwise, it must be that

$$x^{opt} \neq \arg\max_{j \neq i} \sum_{i=1}^{n} v_i(x).$$

Hence

$$\sum_{j \neq i} v_j(x^{opt}) < \max_{x \in X} \sum_{j \neq i} v_j(x)$$

Therefore

$$\sum_{j \neq i} v_j(x^{opt}) - \max_{x \in X} \sum_{j \neq i} v_j(x) \leq 0$$

3. *We sketch the solution.*
   Consider a given agent a. Let x^{opt} be the chosen outcome and x^{2nd} the possibly-different outcome that would have been reached without a.
   The payment (the difference in the values of x^{opt} and x^{2nd} for the other agents)

$$p = \sum_{j \neq i} v_j(x^{opt}) - \max_{x \in X} \sum_{j \neq i} v_j(x)$$

   satisfies

$$p \geq v_a(x^{2nd}) - v_a(x^{opt}).$$

   Since both $v_a(x^{2nd})$ and $v_a(x^{opt})$ are positive, $v_a(x^{opt}) + p$ must be positive as well.
4. *We give a sketch of the proof.*
   Declaring too high: If alternative A was chosen with truthful declaration and B with a too-high value declaration, the cost to the society may be the higher than the difference in values to the agent, leading to a high payment, which exceeds the difference in the values, and therefore a smaller utility.
   Declaring too low: An alternative may be chosen that has a lower utility to the agent than the alternative that would have been chosen with a truthful declaration.