1. Consider the two fuzzy sets

$$\begin{aligned} A =& \{(x_1, 1), (x_2, 0.5), (x_3, 0.3), (x_4, 0.2)\} \\ B =& \{(x_1, 0.5), (x_2, 0.7), (x_3, 0.2), (x_4, 0.4)\} \end{aligned}$$

Find, $\overline{A}$, $A \cup B$, $A \cap B$, $\overline{A \cup B}$ and $\overline{A} \cap \overline{B}$.

2. Consider the fuzzy sets A ('close to $a \in \mathbb{R}$') and B ('very close to $b \in \mathbb{R}$') defined on $\mathbb{R}$ by the following membership functions

$$A(x) = \begin{cases} 1 - \frac{|a-x|}{\alpha}, & if|a - x| \leq \alpha \\ 0, & otherwise \end{cases} \qquad B(x) = \begin{cases} 1 - \frac{|b-x|}{\beta}, & if|b - x| \leq \beta \\ 0, & otherwise \end{cases}$$

Draw $A \cup B$ and $A \cap B$ for $a = 0$ and $\alpha = 2$, and $b = 2$ and $\beta = 1$. What is the $x \in \mathbb{R}$ which is the most 'close to a' *and* 'very close to b' at the same time?

3. Compute the (scalar) cardinality of the fuzzy sets defined as

$$\begin{aligned} A =& \{(v, 0.4), (w, 0.2), (x, 0.5), (y, 0.4), (z, 1)\} \\ \mu_B(x) =& \frac{x}{x+1} \text{ for } x \in \{0, 1, \dots, 10\} \\ \mu_C(x) =& 1 - \frac{x}{10} \text{ for } x \in \{0, 1, \dots, 10\} \end{aligned}$$

Can you think of the concept of fuzzy cardinality? What would it be for A? (It is defined in the literature!)

4. Find at least a function to measure the fuzziness of fuzzy sets and measure the fuzziness of A in the previous exercise. Can you extend the concept of that function to fuzzy sets defined on an infinite domain, for example $U = [a, b] \subset \mathbb{R}$? Consider $U = [0, 4]$ and calculate the fuzziness of the following two fuzzy sets:

$$A(x) = \begin{cases} 1 - \frac{|2-x|}{2}, & if|2 - x| \leq 2 \\ 0, & otherwise \end{cases} \qquad A(x) = \begin{cases} 1 - \frac{|2-x|}{1}, & if|2 - x| \leq 1 \\ 0, & otherwise \end{cases}$$

5. Let $f(x) = 2x$ and A be a fuzzy number in the previous exercise, use the *extension principle* to find the membership function of $B = f(A)$. And what if $f(x) = x^2$. This is an exercise on the so-called extension principle.

6. Consider the fuzzy set A and its level sets $A_\alpha = \{x \in U | \mu_A(x) \geq \alpha\}$. Prove that

$$A = \bigcup_{\alpha \in [0,1]} \alpha \cdot A_\alpha$$

This is the *decomposition theorem*.