AALTO UNIVERSITY
Systems analysis laboratory

Mat-2.4136 Special Topics in Decision Making: Fuzzy Sets
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Exercise 4: Possibility theory

1. Prove the following two implications:

$$\mathrm{Pos}(A) < 1 \Rightarrow \mathrm{Nec}(A) = 0 \tag{1}$$
$$\mathrm{Nec}(A) > 0 \Rightarrow \mathrm{Pos}(A) = 1 \tag{2}$$

2. Suppose that you have a piece of DNA of a killer and a set of four suspects $X = \{a, b, c, d\}$ and you know that the killer is one of them. The degrees of compatibility of the DNAs of the four suspects with the one of the real killer are the following (already expressed as degrees of possibility with the highest equal to 1):

$$\mathrm{Pos}(\{a\}) = 0.8, \quad \mathrm{Pos}(\{b\}) = 0.5 \quad \mathrm{Pos}(\{c\}) = 1 \quad \mathrm{Pos}(\{d\}) = 0.2$$

Reconstruct the possibility distribution Pos : $\mathcal{P}(X) \to [0, 1]$, where $\mathcal{P}(X)$ is the power set of X, and answer the following questions.

   (a) What is $\mathrm{Nec}(\{a, b, c\})$?

   (b) What does it say about the probability that the killer is one among a, b, and c?

   (c) What is the probability interval that the killer is c or d?

3. Every possibility measure Pos on a finite $\mathcal{P}(X)$ is uniquely determined by a function

$$r : X \to [0, 1] \text{ such that } \exists x \in X \text{ with } r(x) = 1$$

by means of the formula $\mathrm{Pos}(A) = \max_{x \in A} r(x)\ \forall A \in \mathcal{P}(X)$.

4. Evidence theory is based on the *basic belief assignment* function which assigns degrees of evidence to the elements of $\mathcal{P}(X)$ and not only on X. Suppose that you have a old electronic device with three critical and degrading components $\{a, b, c\}$. The device does not work, so we know that some component must be broken.

   (a) Considering that the device is very old, this seems to favor that *all* three components are broken;

   (b) However, an expert confirms that, at least few years ago, component b was working, which seems to favor that it would still be working now and so the problem should be in a or c;

   (c) Finally a superficial inspection of the components reveals that a looks better than b, which, in turn looks better than c, so following this indication is seems that there is more evidence that c is broken, than b and even more than a.

Consider the indications given in (a)–(c) and build a reasonable basic belief assignment (the precise values are not important, just follow the indications). Then, find the Belief and Plausibility functions defining lower and upper probabilities.

5. We saw that normalized monotone measures have their duals. In the case of probability we are lucky and $\Pr(A) = 1 - \Pr(\bar{A})$. Prove that in the more general case of Bel, Pl is its dual, i.e. $\mathrm{Bel}(A) = 1 - \mathrm{Pl}(\bar{A})$.

6. Prove that:
There is a probability measure (and Bel = Pl) $\Leftrightarrow$ there is a basic belief assignment on the singletons $\{x\} \subseteq X$ (i.e. $\sum_{x \in X} \phi(x) = 1$).

7. Prove that when the basic belief assignment is on a family of nested sets, i.e. $A_1 \subset A_2 \subset \cdots \subset A_n$, then Pl = Pos.