L3: Stochastic processes
Definitions and Necessary Assumptions

MEC-E1030 Random Loads and Processes
Literature

Field specific, e.g. marine:
- Waves
- Wind
- Ice
- Earthquakes

Theoretical
- Dynamics
- Probability
- Tools
The Big Picture – Linear World

Mission Definition (e.g. route, cargo, schedule)

Characterization of environment (e.g. weather, location, duration)

Response Amplitude Operators (e.g. experiments, simulations, FEA, analytical)

Responses (e.g. motions, stresses, shear forces, moments)

Criteria (e.g. short term for ultimate, long-term for fatigue)

Design (e.g. material selection, risk-analysis, optimization, production)
The Big Picture – Linear World

Characterization of environment (e.g. weather, location, duration, probability)

Weather at given area

Wave spectrum and components at time

Fourier transformation

Probability of sea state

Table 5—Observed Percentage Frequency of Occurrence of Wave Heights and Periods (Hogben and Lumb data)

<table>
<thead>
<tr>
<th>significant wave height m</th>
<th>2.5</th>
<th>6.5</th>
<th>8.5</th>
<th>10.5</th>
<th>12.5</th>
<th>14.5</th>
<th>16.5</th>
<th>18.5</th>
<th>20.5</th>
<th>Over 21</th>
<th>Total</th>
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<tr>
<td>Wave Period T, sec</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>0-1</td>
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<td>8.8559</td>
<td>0.3301</td>
<td>0.1217</td>
<td>0.0438</td>
<td>0.0249</td>
<td>0.0172</td>
<td>0.0723</td>
<td>0.3584</td>
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<td>0.5807</td>
<td>0.1883</td>
<td>0.0671</td>
<td>0.0254</td>
<td>0.0203</td>
<td>0.0763</td>
<td>36.2941</td>
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<td>2-3</td>
<td>1.5944</td>
<td>7.8962</td>
<td>8.0854</td>
<td>3.7270</td>
<td>1.1790</td>
<td>0.3713</td>
<td>0.1092</td>
<td>0.0321</td>
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<td>0.0027</td>
<td>0.0038</td>
<td>0.4555</td>
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<td>0.0550</td>
<td>0.0303</td>
<td>0.0173</td>
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<td>10-11</td>
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<td>0.0001</td>
<td>0.0019</td>
<td>0.0035</td>
<td>0.0002</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
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<tr>
<td>Totals</td>
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<td>39.3043</td>
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<td>0.2080</td>
<td>0.1306</td>
<td>0.4691</td>
<td>100.000</td>
</tr>
</tbody>
</table>
The Big Picture – Linear World

Characterization of environment (e.g. weather, location, duration, probability)

Probability of sea state, $p_1(x)$

Probability of exceeding certain wave height, $p_2(x)$, from spectra by using spectral moments, $m_k$

Total conditional probability, $p=p_1(x) \cdot p_2(x)$

Link to probability distribution type, e.g. Gaussian
The Big Picture – Linear World

Response Amplitude Operators (e.g. experiments, simulations, FEA, analytical)

Type of response e.g. bending moment, $M(x,t)$

Time history, $M(x=L/2, t)$

Fourier transformation
- Load
- Response
- RAO computed
<table>
<thead>
<tr>
<th>Week</th>
<th>Topic 1</th>
<th>Topic 2</th>
<th>Contribution to big picture</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Definitions and mechanics</td>
<td>Definitions and probability</td>
<td>To understand the basic concepts need in treatment of random loads and processes. General description of random vibration. General description of random variables and their properties and continuous and discrete probability distributions. Define the random process of your engineering project.</td>
</tr>
<tr>
<td>2</td>
<td>Stochastic process and simplifications done</td>
<td>Representation of random time signal in frequency domain</td>
<td>To understand under which circumstances we can present random signal using variance spectrum and probability distributions, i.e. Gaussian signal in time domain. Define the conditions for the engineering project when random treatment can be carried out with the tools provided.</td>
</tr>
<tr>
<td>3</td>
<td>Environmental loads and their formation</td>
<td>Representation of environment as random process</td>
<td>To identify the kind of environmental loads relevant to your engineering project. The select the proper spectra design. To discuss under what circumstances the approach is feasible.</td>
</tr>
<tr>
<td>4</td>
<td>Response spectrum</td>
<td>Response statistics for fatigue and extreme loading</td>
<td>To determine the response spectrum for selected application case with approximated response amplitude operator. Fourier-transformation of for random vibrations. Single degree of freedom system in frequency domain. Determine from these the design loads for the engineering project. Random vibrations of beams and strings. Peak and extreme value statistics.</td>
</tr>
<tr>
<td>5</td>
<td>Review of the course contents</td>
<td>Questions and answers before exam</td>
<td>To make sure that the different elements learned in order to make a solid basis for the direct assessment of design loads.</td>
</tr>
<tr>
<td>6</td>
<td>Exam reviewing the learned contents</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Exercise**

**Grade 1:** Based on the 1st week report, describe the random variable formation from physical viewpoint (i.e. what causes the random nature).

**Grade 3:** Describe the ensemble you could have for your application case. Under what circumstances you can define the random load and response for your application case using stochastic methods? Is the response of your system narrow or broad-banded – how can you make it narrow-banded?

**Grade 5:** Describe the process to measure the random excitations and responses. What are the sensors needed? Create and test a matlab script that calculates the time average and standard deviation for random signal. What effects do you see in the script that affect the results (learning diary)?

**Grade 6:** Find article related to your engineering problem that has discussed the random events from physical viewpoint. Reflect it in learning diary.

**Submission**
- Report the work done. Show clearly the roles of the group members in the report (leader, algorithms, data, reporting). Submit to jani.romanoff@aalto.fi
- The grading is based 50% on technical contents, 20% on using technical aids, 15% on reporting and 15% on reflection previous studies (individual 0.5-1 page learning diary).
Contents

• Motivation
• Theory
  • Statistics
    - Stochastic Processes
    - Statistical Properties
    - Covariance & Correlation Factor
    - Stationary Processes
    - Ergodic Processes
  • Process in Frequency domain
  • Autocovariance
  • Variance Spectrum
  • Variance Spectrum directly by FFT
  • Narrow vs. broadband
• Summary
Motivation

- Environmental loads are not fully known at every time step and are called stochastic loads.
- It is necessary to use statistical and probabilistic methods to describe these loads.
- Change of any variable $x(t)$ in time interval $T$ can be presented in statistical terms:
  - Stress, velocity, temperature...
  - Measured, simulated...
- It is impossible to predict the value outside time interval $T$. 
Motivation

• Variance spectrum can be used to describe properties of natural Stochastic Process in frequency domain (multiple sinusoidal components at the same time)
  • e.g. wave spectrum (variance spectrum) can be used to assess response due to random sea
  • Important to understand Autocovariance function

\[
\begin{align*}
\text{Variance} & \quad \sigma^2_X = E[(X - m_X)(X - m_X)] \\
\text{Covariance} & \quad c_{XY} = E[(X - m_X)(Y - m_Y)] \\
\text{Autocovariance} & \quad C_X(\tau) = E[(X(\tau) - m_X)(X(t - \tau) - m_X)]
\end{align*}
\]

\[
S^+_X(\omega) = \begin{cases} 
2S_X(\omega), & \omega > 0 \\
0, & \omega < 0
\end{cases}
\]
Stochastic Process

• Seemingly identical conditions (road, wave, wind) exhibit different behaviour from one recorded experiment to the next: $x_1(t)$, $x_2(t)$, $x_3(t)$…
• The ensemble of all these signals (i.e. entire set) makes the random process $x(t)$
• Time history of random variable can be presented as statistical distributions
Stochastic Process

- The special situation where the conditions remain steady, we call the process stationary and ergodic.
- *Stationary* means that probability distribution of \( x_1(t), x_2(t), x_3(t) \ldots \) at any time instant \( t_1 \) is independent of the choice of \( t_1 \).
- *Ergodic* means that probability distribution of \( x(t) \) is equal to all of the members that contribute to the process.
- Thus, under these special conditions we can determine the \( x(t) \) from single member probability distribution.
Stochastic Process

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• **Stationary** means that probability distribution of $x_1(t)$, $x_2(t)$, $x_3(t)$… at any time instant $t_1$ is independent of the choice of $t_1$
• **Ergodic** means that probability distribution of $x(t)$ is equal to all of the members that contribute to the process
• Thus, under these special conditions we can determine the $x(t)$ from single member probability distribution
Ensemble and Time Averages, $k^{th}$ Moments

- For probability estimates we need to assess the statistical properties (averages, moments)
- We can do this in two ways:
  - The ensemble average is obtained by carrying out the experiment/measurement $N$ time at time instant $t$ and taking the average from that, i.e.
    \[
    \langle x(t) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i(t) \quad \langle x^k(t) \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i^k(t)
    \]
  - The time average is determined by taking single signal and averaging that, i.e.
    \[
    \bar{x}(t) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} x_i(t)dt \quad \bar{x^k}(t) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} x_i^k(t)dt
    \]
  - For stationary ergodic process these produce the same result:
    \[
    \langle x^k(t) \rangle = \bar{x^k}(t)
    \]
Ensemble and Time Averages, $k^{th}$ Moments

- The probability $\Pr(X)=P(X)$ is determined by:
  1. Considering the number of realizations when $x(t)$ is smaller than the prescribed value $X$,
  2. Dividing this by number of realizations $N$
  3. Letting $N$ increase to large number
  4. Carrying out this for any other values of $X$ to form the entire distribution

- When the process is stationary, this forms the probability distribution of the random process
Definition: Ergodic processes

Ergodic process is subclass of stationary process

It is defined as such process:
Ensemble mean equals time average of single realization

Ergodic process → Data from single measurement is sufficient

Ensemble mean = mean of all realizations at $t$
Cumulative and probability distribution functions

Cumulative Density Function

\[ F_{X(t)}(x) = \lim_{T \to \infty} \frac{T[x(t) \leq x]}{T} \]

Probability Distribution Function

\[ f_{X(t)}(x) = \frac{dF_{X(t)}(x)}{dx} \]
Statistical properties of random variable

Mean value (expected value) $m_X = E[X]$ 

$$m_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx \quad m_X = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} x_j = \sum_{k=1}^{n} x^{(k)} p_k$$

Variance $\sigma^2_X$

$$\sigma^2_X = Var(X) = E[(X - m_X)^2] = \int_{-\infty}^{\infty} (x - m_X)^2 f_X(x) \, dx$$

Standard deviation $\sigma_X$

$$\sigma_X = \sqrt{Var[X]}$$
Correlations in Time

- The correlation between two random variables \( x(t) \) and \( y(t) \) can be expressed by covariance \( \sigma_{xy}^2 = \langle x(t)y(t) \rangle \)
- When the two variables are completely independent covariance is 0
- The same concept can be used in time by looking at the covariance between different times (cross-correlation)
  \[ R_{xy}(\tau) = \langle x(t)y(t+\tau) \rangle \]
- For stationary process this should vary only due to variation of \( \tau \) as the statistical properties should not change
- We can do the same for the signal itself (auto-correlation)
  \[ R_{xx}(\tau) = \langle x(t)x(t+\tau) \rangle \]
- For large time separation the \( R_{xx} \) must be zero, thus the shape of the curve tells about the random signal
  - The lower the value \( t=0 \), the more sudden the signal is
  - The shape varies depending if scaling has been carried out

For multivariable stochastic process, connection between statistically independent random variables \( X_1, X_2 \)

Joint CDF
\[
F_{X_1,X_2}(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{X_1,X_2}(s_1, s_2) ds_1 ds_2
\]

Joint PDF
\[
f_{X_1,X_2}(x_1, x_2) = \frac{\partial^2 F_{X_1,X_2}(x_1, x_2)}{\partial x_1 \partial x_2}
\]

Figure 1.2. Example of a joint Gaussian PDF of two random variables \( x \) and \( y \).
Correlations in Time

- The correlation between two random variables \( x(t) \) and \( y(t) \) can be expressed by covariance \( \sigma_{xy}^2 = \langle x(t)y(t) \rangle \).
- When the two variables are completely independent, covariance is 0.
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- We can do the same for the signal itself (autocorrelation):
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  - The lower the value \( t=0 \), the more sudden the signal is.
  - The shape varies depending if scaling has been carried out.

\[
R_{xx}(\tau) = \frac{\langle x(t)x(t+\tau) \rangle}{\sigma_{xx}^2}
\]

For stationary process Autocovariance function \( C(\tau) \) and autocorrelation \( R_{xx}(\tau) \):
- Covariance between values of the same variable at two moments of time: \( t & t + \tau \)
  \[
  C_x(\tau) = E[(X(t) - m_X)(X(t+\tau) - m_X)]
  \]
  \[
  C_x(\tau) = C_x(-\tau)
  \]
  \[
  R_x(\tau) = C_x(\tau) + m_X^2
  \]
Stationary processes

- Mean value & Standard deviation are not functions of time → Process is stationary
- Stationary process is idealization of real process, and thus always approximation
- Slowly changing processes, such as sea waves can be considered piecewise stationary
L4: Stochastic processes
Time Signal in Frequency Domain

MEC-E1030 Random Loads and Processes
Contents and Aim

- The aim of the lecture is to understand how spectral methods can be used to assess stochastic loads and response on structures and apply this knowledge to derivation to short term design loads
  - Lecture uses marine structures as example
  - The ideas can be used for other stochastic loads as well (wind, etc)

Session 1:
- Motivation and different aspects of designing structures for lifetime service
  - Extreme loads for ultimate strength
  - Lifetime loads for fatigue strength
- Representing irregular sea with spectrum
  - Fourier series
  - Calculation of the spectral parameters and properties

Session 2:
- Concepts of probability theory applied to time history
- Link between spectrum and probability
- Probability of certain load level exceedance

Literature
Motivation
Design of Structures

• As engineers our duty is to:
  – Guarantee safety of structures/products we design
  – Maximize the performance, e.g. wave energy absorption
  – ...

• The process involves:
  – Determination of loads (e.g. environmental such as wind, mechanical, thermal…)
  – Defining response for given loads (e.g. stress, strain, displacement…)
  – Checking if the strength is adequate (maximum stress, buckling etc)

• The lecture focuses on loads and assumptions made in the analysis:
  – Deterministic: load history is fully known (not valid in nature)
  – Stochastic: we can only evaluate statistics related to the loads and responses, i.e. probability of exceedance of certain load level
Stochastic Process

Let's consider quantity $x(t)$ randomly varying – variable $t$ is independent (time)

$x(t)$ is considered random process
- Force on car’s suspension system
- Stress in hull girder of a ship
- Pressure around aeroplane wing
- ...

The problem is that although we can measure the signal, we cannot define the certain value outside ($t > T_1$) the measured range $t = 0 \ldots T_1$

What is needed is statistical treatment of the signal(s), e.g. consideration of several measurements under identical conditions, i.e. $x_1(t), x_2(t), x_3(t), x_4(t), \ldots, x_N(t)$.

The ensemble of all these random signals is said to be random process $x(t)$

Stationary means that probability distribution of all these measurements is independent of the time instance $t_1$ when the probability is measured

Ergodic means that probability distribution $x(t)$ is equal to distributions of all the member functions that contribute to the process
Stochastic Process

- Let’s consider quantity \( x(t) \) randomly varying – variable \( t \) is independent (time)
- \( x(t) \) is considered random process
  - Force on car’s suspension system
  - Stress in hull girder of a ship
  - Pressure around aeroplane wing
  - ...
- The problem is that although we can measure the signal, we cannot define the certain value outside \( (t>T_1) \) the measured range \( t=0…T_1 \)
- What is needed is statistical treatment of the signal(s), e.g. consideration of several measurements under identical conditions, i.e. \( x_1(t), x_2(t), x_3(t), x_4(t),..., x_N(t) \),
- The ensemble of all these random signals is said to be random process \( x(t) \)
- Stationary means that probability distribution of all these measurements is independent of the time instance \( t_1 \) when the probability is measured
- Ergodic means that probability distribution \( x(t) \) is equal to distributions of all the member functions that contribute to the process
Environmental Loads

- Load vs. time data measured for a structure detail during operation is the “exact” form of a load spectrum – however it is also deterministic
- Quality of the spectrum highly depends on the measurement system, e.g. on the frequency of the data measurement
- Measurement frequency must be high enough to catch short load peaks
- It is necessary to identify different loading modes, e.g. ground, flight, landing in aircraft and to treat these separately
- Loads can be due to
  - Wind and wave
  - Current
  - Ice
  - Earthquakes
  - …
Process in frequency domain

• Realizations needs to be decomposed into Fourier series in order to process it in frequency domain.
• Periodic Process can be approximated by Fourier series
• Fourier series decomposition can be presented in frequency domain
• This is only possible if time history is periodic OR it dies out in time.
Energy in Wave Component

- Contents of waves
  - Waves can be considered to be composed as sum of large (infinite) number of regular sinusoidal waves
  - We can represent each component by its’ frequency ($\omega$) rather than time ($t$)
  - For this we need Fourier transformation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) e^{i\omega t} d\omega$$

$$A(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

- The result is wave spectrum that describes the energy contents of given wave conditions
- Analogously (different process)
  - Wind spectrum
  - Ice load spectrum
  - ...

Fig. 8 Typical variance spectrum of waves, showing approximation by a finite sum of components
Calculation of Random Response due to Stochastic Loads with Spectral Method

- Many of the loads are caused by random processes
  - Load spectrum contains the information of energy available to produce a response
  - Response spectrum contains "what happens" due to this energy
- Therefore, following components of applied mathematics are needed (complicated)
  - Statistics and probability concepts
  - Spectrum techniques & and stochastic processes
- The process of application theory of applied mathematics is easy
  1. Describe the waves as sum of sinusoidal functions for single sea state
  2. Derive the wave spectrum (function of angular velocity of the wave) $S_{xx}(\omega)$
  3. Obtain the response amplitude operator RAO, $H(\omega)$, "how is energy consumed"
     - For global hull girder bending moment, shear force, torsion moment
     - Local pressures
     - Stresses
     - Motions
     - Etc
  4. Obtain the response $S_{yy}(\omega)$

\[ S_{yy}(\omega) = |H_{yy}(\omega)|^2 S_{xx}(\omega) \]
The Basic Idea

• Consider everything in frequency domain instead of time domain – Fourier transformation
  – Obtain responses for all waves with one analysis Response=Wake*RAO (=H(\omega)^2)
  – Theory is heavy, but
  – The application is extremely simple

• What is needed is
  – Wave spectra
  – Response Amplitude Operator – RAO
    • Moment
    • Shear force
    • Motions
    • etc

• After this statistical considerations are carried out to find out
  – Short term response
  – Long term response
Parameters/Characteristics of

The $k$\textsuperscript{th} moment (analogous to mechanics)

$$m_k = \int_0^\infty \omega^k S(\omega) d\omega$$

The average angular velocity

$$\bar{\omega}_1 = \frac{m_1}{m_0} = \frac{\int_0^\infty \omega S(\omega) d\omega}{\int_0^\infty S(\omega) d\omega}$$

The $k$\textsuperscript{th} moment with respect to average angular velocity

$$\mu_k = \int_0^\infty (\omega - \bar{\omega})^k S(\omega) d\omega$$

The bandwidth parameter
(0 for narrow band and 1 for broad band)

$$\varepsilon = \sqrt{1 - \frac{m_2^2}{m_0 m_4}}$$

$0 \leq \varepsilon \leq 1$
Estimate for Extreme Value of Gaussian Process

• Let $x(t)$ be stationary Gaussian process with zero mean and spectral area $m_0$ while the objective is to define the most probable extreme value at certain time

$$\bar{z} = \sqrt{\ln n} \sqrt{2m_0} \quad \bar{N}_{0+} = \frac{1}{2\pi} \sqrt{\frac{m_2}{m_0}}$$

• The expected value for zero-crossings for certain time is calculated by

$$n = \frac{T}{2\pi} (60)^2 \sqrt{\frac{m_2}{m_0}}$$

• and the extreme value

$$\bar{z} = \sqrt{2 \ln \left( \frac{T}{2\pi} (60)^2 \sqrt{\frac{m_2}{m_0}} \right)} \sqrt{m_0}$$

This works for one sea state for example!
Variance Spectrum Directly from data

Normally only one realization is available
Measured data
Assumption of ergodicy is required

Matlab, excel etc. provide tools to calculate variance spectrum by using FFT (Fast Fourier Transform)
Matlab example - FFT

fft

Fast Fourier transform

Syntax

Y = fft(X)
Y = fft(X,n)
Y = fft(X,n,dim)

Description

Y = fft(X) computes the discrete Fourier transform (DFT) of X using a fast Fourier transform (FFT) algorithm.

- If X is a vector, then fft(X) returns the Fourier transform of the vector.
- If X is a matrix, then fft(X) treats the columns of X as vectors and returns the Fourier transform of each column.
- If X is a multidimensional array, then fft(X) treats the values along the first array dimension whose size does not equal 1 as vectors and returns the Fourier transform of each vector.

Y = fft(X,n) returns the n-point DFT. If no value is specified, Y is the same size as X.

- If X is a vector and the length of X is less than n, then X is padded with trailing zeros to length n.
- If X is a vector and the length of X is greater than n, then X is truncated to length n.
- If X is a matrix, then each column is treated as in the vector case.
- If X is a multidimensional array, then the first array dimension whose size does not equal 1 is treated as in the vector case.

Y = fft(X,n,dim) returns the Fourier transform along the dimension dim. For example, if X is a matrix, then fft(X,n,2) returns the n-point Fourier transform of each row.
Matlab example – FFT

\begin{verbatim}
Fs = 1000;     % Sampling frequency
T = 1/Fs;     % Sampling period
L = 1000;     % Length of signal
t = (0:L-1)*T; % Time vector

Form a signal containing a 50 Hz sinusoid of amplitude 0.7 and a 120 Hz sinusoid of amplitude 1.
\[ S = 0.7\sin(2\pi\cdot50\cdot t) + \sin(2\pi\cdot120\cdot t); \]

Corrupt the signal with zero-mean white noise with a variance of 4.
\[ X = S + 2\cdot\text{randn}(\text{size}(t)); \]

Plot the noisy signal in the time domain. It is difficult to identify the frequency components by looking at the signal \( X(t) \).
\begin{verbatim}
plot(1000*t(1:50),X(1:50))
title('Signal Corrupted with Zero-Mean Random Noise')
xlabel('t (milliseconds)')
ylabel('X(t)')
\end{verbatim}
\end{verbatim}
Matlab example - FFT

Compute the Fourier transform of the signal.

```matlab
Y = fft(X);
```

Compute the two-sided spectrum P2. Then compute the single-sided spectrum P1 based on P2 and the even-valued signal length L.

```matlab
P2 = abs(Y/L);
P1 = P2(1:L/2+1);
P1(2:end-1) = 2*P1(2:end-1);
```

Define the frequency domain f and plot the single-sided amplitude spectrum P1. The amplitudes are not exactly at 0.7 and 1, as expected, because of the added noise. On average, longer signals produce better frequency approximations.

```matlab
f = Fs*(0:(L/2))/L;
plot(f,P1);
title('Single-Sided Amplitude Spectrum of X(t)')
xlabel('f (Hz)')
ylabel('|P1(f)|')
```

![Single-Sided Amplitude Spectrum of X(t)](image)
Matlab example - FFT

Now, take the Fourier transform of the original, uncorrupted signal and retrieve the exact amplitudes, 0.7 and 1.0.

```matlab
Y = fft(S);
P2 = abs(Y/L);
P1 = P2(1:L/2+1);
P1(2:end-1) = 2*P1(2:end-1);

plot(f,P1)
title('Single-Sided Amplitude Spectrum of S(t)')
xlabel('f (Hz)')
ylabel('|P1(f)|')
```

![Single-Sided Amplitude Spectrum of S(t)](image-url)
Matlab example - FFT

Now, take the Fourier transform of the original, uncorrupted signal and retrieve the exact amplitudes, 0.7 and 1.0.

```matlab
Y = fft(S);
P2 = abs(Y/L);
P1 = P2(1:L/2+1);
P1(2:end-1) = 2*P1(2:end-1);
plot(f,P1)
title('Single-Sided Amplitude Spectrum of S(t)')
xlabel('f (Hz)')
ylabel('|P1(f)|')
```

If we have narrow banded process, we have zero mean!

We can extend this to more components, add phase angle and do the inverse to get time signal from spectrum

\[ \varphi(t) = \sum_{i=1}^{N} A_i \cos(\omega_i t + \epsilon_i) \]

Fig. 1 Wave pattern combining four regular waves

Fig. 2 Spectrum of four waves
Probability of Certain Surface Elevation

• If we take a set of surface elevation measurements with the same statistical properties, i.e. ensemble, the wave elevation at time $t_1$ can be considered as random variable
  - It is different for different measurements
  - The value is both positive and negative with respect to mean (typically set to zero)

• If we take $X(t_1)$ is random variable of surface elevation, the estimate of the cumulative the probability of occurrence is

$$F_{X(t_1)}(x) = \frac{N[X(t_1) \leq x]}{N}$$

where $N$ is the number of measurements considered

• If we have “really” a random variable in question, we can assume that $N$ is infinite giving (N should be reasonable large 😊)

$$F_{X(t_1)}(x) = \lim_{N \to \infty} \frac{N[X(t_1) \leq x]}{N}$$
**Probability of Certain Surface Elevation**

- Using same logic we can derive the cumulative density function for any arbitrary time point, i.e.

\[
F_{X(t)}(x) = \frac{N[X(t) \leq x]}{N} \quad F_{X(t)}(x) = \lim_{N \to \infty} \frac{N[X(t) \leq x]}{N}
\]

- Then we can consider surface elevation as random function. Often term **“random process”** is used instead.

- Typically we consider the entire time history and several of them. All these make the sample space.

- For natural phenomena the “measurements” have been ongoing for billions of years and will go on so also in the future - we have only recorded few of those.

- Thus we can assume infinite time series.
Probability of Certain Surface Elevation

- It is impossible to repeat exactly what happens in nature
- In practise we can assume that the time history that is available is (part of) ergodic process
  - Statistical information is contained in a single representative measurement
  - Strong assumption that allows relatively short measurements, e.g. 0.5-3 hours
  - Practically the statistical treatment becomes feasible
- Then we can write
  \[ F_{X(t)}(x) = \lim_{T \to \infty} \frac{T[X(t) \leq x]}{T} \]
  \[ F_{X(t)}(x) = \lim_{N \to \infty} \frac{N[X(t) \leq x]}{N} \]
- That is perform the determination of cumulative probability distribution over time
Mathematics on obtained probability

• The probability density function is obtained by differentiation (F needs to be differentiable)

\[ f_X(x) = \frac{dF_X(x)}{dx}, \quad F_X(x) = \int_{-\infty}^{x} f_X(s) \, ds \]

• The mean value (expected value) is

\[ m_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) \, dx \quad m_X = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^{N} x_j \]

• which for discrete set is

\[ m_X = \sum_{k=1}^{n} x^{(k)} p_k = \frac{1}{N} \sum_{j=1}^{N} x_j = \sum_{k=1}^{n} x^{(k)} \frac{p_k}{\sum_{k=N_k \cdot N}^{N_k}} = \sum_{k=1}^{n} x^{(k)} \frac{N_k}{N} \]
Gaussian Random Process

- The wave system can be described as sum of very large number of random, independent, contributions having sinusoidal form – Fourier summation
  \[ h(x, y, t) = \sum_{i=1}^{\infty} h_i(x, y, t), k \to \infty \]

- The average of each of these component \( h_i \) are zero over infinitely long time, thus their sum is also zero

- However, the variance, i.e. elevation from \( h^2 \) is not and this measures physically the severity of the sea

- Random process is Gaussian if \( x(t) \) at any time instance \( t \) is normally distributed

- Physical phenomena such as wave elevation at sea are normally distributed according to Central Limit Theorem of statistical mathematics
  - If we know the variance and average of each wave component \( h_i \), we can calculate those also for the entire wave system
  - These statistical properties are used in connection with probability if the probability distribution is known

- The wave measurements have proven this
  - Normal distribution
  - Average is zero
Gaussian Random Process

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  - These statistical properties are used in connection with probability if the probability distribution is known
- The wave measurements have proven this
  - Normal distribution
  - Average is zero

Wave period 10s
Total time 3*60*60s=10800s
=>Cycles 1080
Continuous Probability Distributions and Their Relations on Surface Waves

\[ f(x) = \frac{1}{\Gamma(m)} \lambda^m x^{m-1} e^{-\lambda x} \]

**Long Term Predictions**

**Wave Height** \( H, H_{1/3}, \ldots \)

**Wave Elevation** \( x(t), x(\omega) \)

**Figure 6.8** Linear and asymptotic relationship between some continuous-type random variables.
Autocovariance

Property of stationary process (not single realization)

Autocovariance function $C(\tau)$:

Covariance between values of the same variable at two moments of time: $t \& t + \tau$

\[
C_X(\tau) = E[(X(t) - m_X)(X(t+\tau) - m_X)]
\]

$C_X(\tau) = C_X(-\tau)$

$C_X(0) = \sigma_x^2$
Variance Spectrum

Variance spectrum $S_X(\omega)$ shows division of process’ total variance in different frequencies.

Alternative names: Energy Spectrum, Power Spectral Density...

\[ C_X(0) = \int_{-\infty}^{\infty} S_X(\omega) d\omega = \sigma^2_x \]

\[ S^+_X(\omega) = \begin{cases} 2S_X(\omega), & \omega > 0 \\ 0, & \omega < 0 \end{cases} \]
It can be shown that symmetry \( S_X(\omega) = S_X(-\omega) \) holds. If \( S_X(\omega) \) is a "reasonably nice" function, autocovariance function \( C_X(\tau) \) can be approximated as a sum.

**Wiener-Kinchinine:**

\[
S_X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_X(\tau) e^{-i\omega \tau} d\tau
\]

\[
C_X(\tau) = \int_{-\infty}^{\infty} S_X(\omega) e^{i\omega \tau} d\omega
\]

**Symmetry of \( S_X(\omega) \):**

\[
S_X(\omega) = \frac{1}{\pi} \int_{0}^{\infty} C_X(\tau) \cos(\omega \tau) d\tau
\]

\[
C_X(\tau) \approx \sum_{j=1}^{N} 2S_X(\omega_j) \Delta \omega \cos(\omega_j \tau_j)
\]
Variance Spectrum

Negative frequencies have no practical meaning

\[
S_X^+(\omega) = \begin{cases} 
2S_X(\omega), & \omega > 0 \\
0, & \omega < 0
\end{cases}
\]
Parameters/Characteristics of Response Spectrum

The $k^{th}$ moment (analogous to mechanics)

\[ m_k = \int_0^\infty \omega^k S(\omega) d\omega \]

The average angular velocity

\[ \bar{\omega}_1 = \frac{m_1}{m_0} = \frac{\int_0^\infty \omega S(\omega) d\omega}{\int_0^\infty S(\omega) d\omega} \]

The $k^{th}$ moment with respect to average angular velocity

\[ \mu_k = \int_0^\infty (\omega - \bar{\omega})^k S(\omega) d\omega \]

The bandwidth parameter (0 for narrow band and 1 for broad band)

\[ \epsilon = \sqrt{1 - \frac{m_2}{m_0 m_4}} \]

\[ 0 \leq \epsilon \leq 1 \]
Units of Variance Spectra

Care need to be taken when changing units of variance spectrum

\[ \int_{0}^{\pi} S_X^+ (\omega) \, d\omega = \sigma_X^2 = \int_{0.5}^{2.0} G_X^+ (f) \, df \]

\[ 2\pi S_X^+ (\omega) = G_X^+ (f) \]
Narrow and broad banded process

Narrow band

Broadband