Equations for fluid flow

Navier-Stokes equation and continuity equation for an incompressible fluid

\[ \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + 2\nabla \cdot (\eta \nabla_s \mathbf{v}) + \mathbf{f} \]

\[ \nabla \cdot \mathbf{v} = 0 \]

where \( \rho \) is density, \( \mathbf{v} \) speed, \( p \) pressure, \( \eta \) dynamic viscosity, \( \nabla_s \mathbf{v} \) symmetric part of dyad \( \nabla \mathbf{v} \) and \( \mathbf{f} \) volume force acting on the fluid.

Solving this equation is problematic because of the non-linear inertia term which generates turbulence.
An equation can be derived for the thickness of the boundary layer as a function of the length $L$ of the plane

$$
\delta = \sqrt{\frac{32\eta L}{\rho v_\infty}} \quad \Rightarrow \quad \frac{\delta}{L} = \sqrt{\frac{32\eta}{\rho v_\infty L}} = \sqrt{32} \frac{1}{\sqrt{Re}}
$$

where $Re = \frac{\rho v_\infty L}{\eta}$ is Reynolds number.

$\rho$ = density, $\eta$ = dynamic viscosity
The flow in the boundary layer defines the viscous force acting on the plane.

\[ \tau = \eta \frac{dv}{dy} \bigg|_{y=0} \quad F = \int_{A} \tau dA = \int_{A} \eta \frac{dv}{dy} dA \]

The friction losses are often expressed using a friction coefficient \( c_f \)

\[ \tau = \frac{1}{2} c_f \rho v_{\infty}^2 \]
Turbulence appears in a plane flow approximately at a Reynolds number

\[ \text{Re} \approx 3 \cdot 10^5 \]

Friction coefficient \( c_f \) for a turbulent flow is typically obtained experimentally and expressed as function of Reynolds number.

When the speed increases, the flow becomes turbulent. However, there is always a thin laminar layer on the plane surface.
Example: Friction loss in air gap

Friction losses on a cylinder surface

\[ P_{fr} = \frac{1}{2} C_f \pi \rho \omega^3 R_r^4 l \]

\[ C_f = \begin{cases} 
10.0 Re^{-1} \left( \frac{\delta}{R_r} \right)^{0.3}, & Re < 64 \\
2.00 Re^{-0.6} \left( \frac{\delta}{R_r} \right)^{0.3}, & 64 < Re < 500 \\
1.03 Re^{-0.5} \left( \frac{\delta}{R_r} \right)^{0.3}, & 500 < Re < 10000 \\
0.065 Re^{-0.2} \left( \frac{\delta}{R_r} \right)^{0.3}, & Re > 10000 
\end{cases} \]

\[ Re = \frac{\rho R_r \omega_m \delta}{\eta} \]

Circumferential flow pattern in air-gap

Reynolds number

$$Re = \frac{\rho v \delta}{\eta} = \frac{\rho R_r \omega_m \delta}{\eta}$$

$\rho = \text{density}, \ \delta = \text{radial air-gap length}, \ R_r = \text{rotor radius}, \ \eta = \text{dynamic viscosity}$

$Re < 2000 \Rightarrow \text{laminar flow}; \ \ Re > 5000 \Rightarrow \text{turbulent flow}$
Flow distribution in $r,z$-plane

Couette flow with Taylor vortices:

Taylor number

$$Ta = Re^2 \frac{\delta}{R_r} = \frac{\rho_0^2 \omega_m^2 \delta^3 R_r}{\eta^2}$$

$\rho$ = density, $\delta$ = radial air-gap length, $R_r$ = rotor radius, $\eta$ = dynamic viscosity

Taylor vortices appear in air-gap flow, if $Ta > 1700$
Characters for air-gap flow

\[ Re = \frac{\rho R_r \omega_m \delta}{\eta} \]

- \( Re < 2000 \) => laminar flow;
- \( Re > 5000 \) => turbulent flow

\[ \rho = \text{density}, \quad R_r = \text{rotor radius}, \quad \omega_m = \text{angular speed}, \quad \delta = \text{air gap}, \quad \eta = \text{dynamic viscosity} \]

Machine A: Turbo-generator 270 MVA, 3000 1/min
Machine B: Hydro-generator 21 MVA, 125 1/min
Machine C: Cage induction motor 4.75 MW, 1500 1/min
Machine D: Cage induction motor 37 kW, 1500 1/min

<table>
<thead>
<tr>
<th></th>
<th>Machine A</th>
<th>Machine B</th>
<th>Machine C</th>
<th>Machine D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re</td>
<td>660219</td>
<td>21678</td>
<td>13838</td>
<td>702</td>
</tr>
<tr>
<td>Ta</td>
<td>5.7E+10</td>
<td>1.6E+06</td>
<td>2.5E+06</td>
<td>3941</td>
</tr>
</tbody>
</table>
The thermal and flow boundary layers are of the same order of magnitude. An equation between the two boundary layers

$$\delta_T = \delta Pr^{\frac{1}{3}}; \quad Pr = \frac{c\eta}{\lambda}$$

where $Pr$ is Prandtl number for the flowing fluid, $c$ is specific heat. For gases, Prandtl number is close to one.
In general, the temperature distribution cannot be solved analytically. The heat-transfer through the thermal boundary layer is expressed using a heat-transfer coefficient $\alpha$

$$q = \alpha (T_w - T_\infty)$$

The heat-transfer coefficient is often expressed using a dimensionless Nusselt number

$$Nu = \frac{\alpha L}{\lambda}$$
Dimensionless characteristic numbers

Reynolds number \( \text{Re} = \frac{\rho v_\infty x}{\eta} \)

Prandtl number \( \text{Pr} = \frac{c\eta}{\lambda} \)

Nusselt number \( \text{Nu} = \alpha \frac{x}{\lambda} \)

where \( x \) is a characteristic length related to the flow. In the plane flow discussed previously, the value of \( x \) is set to the length \( L \), in a tube flow equal to the diameter \( D \). In the air-gap flow of an electrical machine, \( x \) is the radial air-gap length.
Properties of cooling gases

The Nusselt number for turbulent plane flow is expressed in Wagner (1994)

\[ \text{Nu} = 0.037 \text{Re}^{0.8} \text{Pr}^{0.48}, \quad \text{when} \ 5 \cdot 10^5 < \text{Re} < 10^7 \]

The relative heat-transfer coefficients for air, helium and hydrogen at a constant speed are

<table>
<thead>
<tr>
<th></th>
<th>1 bar, 50 °C</th>
<th>Air</th>
<th>Helium</th>
<th>Hydrogen</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density [kg/m³]</td>
<td>1.078</td>
<td>0.149</td>
<td>0.075</td>
<td></td>
</tr>
<tr>
<td>Thermal conductivity [W/mK]</td>
<td>0.0282</td>
<td>0.160</td>
<td>0.191</td>
<td></td>
</tr>
<tr>
<td>Viscosity [μNs/m²]</td>
<td>19.3</td>
<td>21</td>
<td>9.4</td>
<td></td>
</tr>
<tr>
<td>Relative heat-transfer coefficient</td>
<td>1.0</td>
<td>1.1</td>
<td>1.4</td>
<td></td>
</tr>
</tbody>
</table>

Hydrogen has the lowest density. It has less friction and cooling losses, which often makes the main difference.
Heat-transfer in the air-gap flow

- Heat-transfer coefficient for the air-gap surfaces

From one surface to the air flow in the middle of air gap

\[ \alpha_c = \frac{\lambda}{\delta} Nu ; \quad \begin{cases} 
Nu = 2, & Ta_m < 1700 \\
Nu = 0.128 \cdot Ta_m^{0.367}, & 1700 \leq Ta_m \leq 10^4 \\
Nu = 0.409 \cdot Ta_m^{0.241}, & 10^4 < Ta_m < 10^7 
\end{cases} \]

\( Nu = \) Nusselt number
\( Ta_m = \) Modified Taylor number \( (Ta_m \approx Ta \) for air-gap flow \)

Modelling the air gap in a thermal FE code

If the code allows setting a heat transfer coefficient inside the solution domain, the equations of the previous slide can be directly applied. If not, the conductivity of the air should be modified. The power flow to the middle of the air gap is

\[ P = qA = \alpha_c \left( T - T_a \right) A = \frac{\lambda}{\delta} Nu \left( T - T_a \right) A \]

\[ = \frac{\lambda}{2} \frac{1}{Nu} A \left( T - T_a \right) = \frac{\lambda_{eff} A}{l} \left( T - T_a \right) \]

From this we get an effective conductivity for the air gap

\[ \lambda_{eff} = \frac{1}{2} Nu \lambda \]
Contact between the frame and core in FEA

<table>
<thead>
<tr>
<th>Contact</th>
<th>$\alpha$ [W/m²K]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminium frame – Stator core</td>
<td>350 - 550</td>
</tr>
<tr>
<td>Cast iron frame – Stator core</td>
<td>650 - 870</td>
</tr>
<tr>
<td>Rotor bar – Rotor core</td>
<td>430 - 2600</td>
</tr>
</tbody>
</table>

Modelling the tiny crack with elements would mean tens of thousands of extra unknowns. One possibility is to increase the size of the crack and change the conductivity.
Contact between the frame and core in FEA II

From the equations for modelling convection in the air gap

\[ P = \alpha_c (T - T_a) A = \frac{\lambda_{\text{eff}} A}{l} (T - T_a) = \frac{1}{R_{\text{ag}}} (T - T_a) \]

\[ \Rightarrow R_{\text{ag}} = \frac{1}{\alpha_c A} \]

If the thickness of the enlarged crack is \( d \), its effective conductivity is

\[ R = \frac{d}{\lambda_{\text{eff}} A} = \frac{1}{\alpha_c A} + \frac{d}{\lambda_{\text{Fe}} A} \]

\[ \Rightarrow \frac{1}{\lambda_{\text{eff}}} = \frac{1}{\alpha_c d} + \frac{1}{\lambda_{\text{Fe}}} \]
A significant part of the losses of an electrical machine is extracted from the end-winding space.

For thermal modelling, the heat-transfer coefficients are needed on the surfaces of the end windings, stator and rotor cores and support structures.

The speed distribution at those surfaces should be known.
Literature


Problem 1  Temperature distribution in a solid rotor

Calculate the steady-state temperature distribution in a homogeneous solid rotor (radius $R$, length $l$). The thermal conductivity $\lambda$ and loss density $p_h$ are constants, and the possible temperature difference in the axial direction ($z$-axis) is neglected. The heat-transfer coefficient on the rotor surface is $\alpha$, and the temperature in the middle of the air gap is $T_{ag}$. The gradient and Laplace operators expressed in polar coordinates ($r, \varphi$) are

\[
\nabla T = \frac{\partial T}{\partial r} e_r + \frac{1}{r} \frac{\partial T}{\partial \varphi} e_\varphi
\]

\[
\nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \varphi^2}
\]
Problem 1  Temperature distribution in a solid rotor

\[ \lambda \nabla^2 T = -p \]

\[ \nabla T = \frac{\partial T}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial T}{\partial \varphi} \mathbf{e}_\varphi \]

\[ \nabla^2 T = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \partial^2 \varphi^2} \frac{\partial^2 T}{\partial r^2} \]

Symmetry: \[ \frac{\partial T}{\partial \varphi} = 0 \]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = -\frac{p}{\lambda} \quad \Rightarrow \quad \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = -\left( \frac{p}{\lambda} \right) r \quad \Rightarrow \quad r \frac{\partial T}{\partial r} = -\frac{p}{2\lambda} r^2 + C
\]

\[ \Rightarrow \frac{\partial T}{\partial r} = -\frac{p}{2\lambda} r + \frac{C}{r} \quad \Rightarrow \quad T = -\frac{p}{4\lambda} r^2 + C \ln(r) + D \]

**Boundary conditions:** The temperature in the centre of the rotor is finite and all the loss power is transferred from the rotor through its outer surface.
Problem 1    Temperature distribution in a solid rotor II

Finite temperature in the centre: \( C = 0 \)

Total power: \( P_{\text{tot}} = p\pi R^2 l = qA = q2\pi Rl \)
\[ \Rightarrow q = \frac{1}{2} pR \]

On the other hand \( q = \alpha [T(R) - T_{ag}] \)

Temperature on the outer surface:
\[ T(R) = T_{ag} + \frac{pR}{2\alpha} = -\frac{p}{4\lambda} R^2 + D \]
\[ \Rightarrow D = T_{ag} + \frac{pR}{2\alpha} + \frac{p}{4\lambda} R^2 \]

\[ T(r) = \frac{p}{4\lambda} (R^2 - r^2) + \frac{pR}{2\alpha} + T_{ag} \]
Total loss in solid rotor:

\[ P_{rt} = 2.7 \text{ kW} \]

Heat transfer coeff. on rotor surface:

\[ \alpha = 350 \text{ W/Km}^2 \]

Temperature in air-gap:

\[ T_{amp} = 60 \text{ °C} \]
Problem 2  Heat transfer from a high-speed rotor

Calculate the heat-transfer coefficient on the air-gap surfaces of a high-speed induction motor and the friction loss in the air gap. The diameter of the rotor is 76 mm, the rotation speed 100 000 rpm and the radial and axial air-gap lengths 3 mm and 150 mm, respectively.
Problem 2     Heat transfer from a high-speed rotor

Equations for heat-transfer coefficient

\[ \alpha_c = \frac{\lambda}{\delta} N u \]

\[
\begin{cases}
N u = 2, & T a_m < 1700 \\
N u = 0.128 \cdot T a_m^{0.367}, & 1700 \leq T a_m \leq 10^4 \\
N u = 0.409 \cdot T a_m^{0.241}, & 10^4 < T a_m < 10^7
\end{cases}
\]

\[
T a = R e^2 \frac{\delta}{R_r} = \frac{\rho^2 \omega_m^2 \delta^3 R_r}{\eta^2} \quad R e = \frac{\rho v \delta}{\eta} = \frac{\rho R_r \omega_m \delta}{\eta}
\]
Problem 2  Heat transfer from a high-speed rotor

Substituting the values

\[ Re = \frac{\rho v \delta}{\eta} = 80\,000 \]

\[ Ta = Re^2 \frac{\delta}{R_r} = 512 \cdot 10^6 \]

\[ Nu = 0.409 \cdot Ta_m^{0.241}, \quad 10^4 < Ta_m < 10^7 \]

\[ Nu = 51.3 \]

\[ \alpha_c = \frac{\lambda}{\delta} Nu = 440 \text{ W/ K m}^2 \]
Problem 2  Friction loss in the air gap

Equations for friction loss

\[ P_{fr} = \frac{1}{2} C_f \pi \rho \omega^3 R_r^4 l \]

\[
C_f = \begin{cases} 
10.0 \text{Re}^{-1} \left( \frac{\delta}{R_r} \right)^{0.3}, & \text{Re} < 64 \\
2.00 \text{Re}^{-0.6} \left( \frac{\delta}{R_r} \right)^{0.3}, & 64 < \text{Re} < 500 \\
1.03 \text{Re}^{-0.5} \left( \frac{\delta}{R_r} \right)^{0.3}, & 500 < \text{Re} < 10 000 \\
0.065 \text{Re}^{-0.2} \left( \frac{\delta}{R_r} \right)^{0.3}, & \text{Re} > 10 000 
\end{cases}
\]
Problem 2  Friction loss in the air gap

Substituting the values

\[ C_f = 0.065Re^{-0.2} \left( \frac{\delta}{R_r} \right)^{0.3} = 0.00317, \quad Re > 10000 \]

\[ P_{fr} = \frac{1}{2} C_f \pi \rho \omega^3 R_r^4 l = 2.15 \text{ kW} \]
Problem 3  Temperature of a cooling rib

Estimate the temperature distribution in the thin cooling rib shown below. The ambient temperature is $T_0$, the temperature of the main body $T_1$, and the rib is assumed to be so long that the temperature at the end of the rib is equal to the ambient temperature. The thermal conductivity of the rib material is $\lambda$ and the heat-transfer coefficient on the surface of the rib is $\alpha$. 
Problem 3  Temperature of a cooling rib

Power balance in a slice of rib

Input power from the left

\[ P(x) = qA = -\lambda A \frac{dT}{dx} \]

Output power to the right

\[ P(x+\Delta x) = P(x) + \frac{dP}{dx} \Delta x = -\lambda A \frac{dT}{dx} - \lambda A \frac{d^2T}{dx^2} \Delta x \]

The difference \( P(x) - P(x+\Delta x) \) is transferred to ambient air by convection

\[ P(x) - P(x+\Delta x) = \lambda A \frac{d^2T}{dx^2} \Delta x = \alpha (T - T_0) C \Delta x \]

where \( C \) is the circumference (length) of the rib.
Problem 3  Temperature of a cooling rib

Differential equation for solving the temperature

\[
\frac{d^2 T}{dx^2} - \frac{\alpha C}{\lambda A} (T - T_0) = 0
\]

Assuming a constant ambient temperature and changing the temperature variable to the temperature rise \( \Theta = T - T_0 \)

\[
\frac{d^2 \Theta}{dx^2} - m^2 \Theta = 0 \quad \text{where} \quad m^2 = \frac{\alpha C}{\lambda A}
\]

Solution

\[
\Theta = C_1 e^{-mx} + C_2 e^{mx}
\]

where \( C_1 \) and \( C_2 \) are integration constants.
Problem 3   Temperature of a cooling rib

Boundary conditions for defining $C_1$ and $C_2$

\[
\begin{align*}
T(x = 0) &= T_1 \\
T(x \to \infty) &= T_0 \\
\Theta(x = 0) &= T_1 - T_0 \\
\Theta(x \to \infty) &= 0
\end{align*}
\]

\[
\Theta(x = 0) = C_1e^{-m0} + C_2e^{m0} = C_1 + C_2 = T_1 - T_0
\]

The second condition requires that $C_2$ is equal to zero. Otherwise, the temperature at the end of the rib would approach infinity.

\[
\begin{align*}
C_1 &= T_1 - T_0 \\
C_2 &= 0
\end{align*}
\]

\[
\Theta(x) = (T_1 - T_0)e^{-mx}
\]

\[
T(x) = (T_1 - T_0)e^{-mx} + T_0
\]