

MS-E2112 Multivariate Statistical Analysis (5cr)

Lecture 8: Canonical Correlation Analysis

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Canonical Correlation Analysis

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Canonical correlation analysis involves partition of variables into two vectors x and y . The aim is to find linear combinations $\alpha^T x$ and $\beta^T y$ that have the largest possible correlation.

Canonical Correlation Analysis

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Let x be a p -variate random vector and let y be a q -variate random vector. The object in canonical correlation analysis is to find linear combinations

$$u_k = \alpha_k^T x$$

and

$$v_k = \beta_k^T y$$

that maximizes the correlation $|\text{corr}(u_k, v_k)|$ between u_k and v_k subject to

$$\text{var}(u_k) = \text{var}(v_k) = 1,$$

and

$$\text{corr}(u_k, u_t) = 0, \text{corr}(v_k, v_t) = 0, t < k.$$

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Quick reminder:

$$\text{corr}(w_1, w_2) = \frac{E[(w_1 - \mu_{w_1})(w_2 - \mu_{w_2})]}{\sigma_{w_1} \sigma_{w_2}}.$$

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The vectors α_k and β_k are called the k th canonical vectors and

$$\rho_k = |\text{corr}(u_k, v_k)|$$

are called canonical correlations.

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Whereas **principal component analysis** considers interrelationships **within a set of variables**, **canonical correlation analysis** considers relationships **between two groups of variables**.

Canonical Correlation Analysis, Examples

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- Exercise — health.
- Open book exams — closed book exams.
- Job satisfaction — performance.

Canonical Correlation Analysis, Regression Analysis

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Canonical correlation analysis can be seen as an extension of multivariate regression analysis. However, note that in canonical correlation analysis there is **no assumption of causal asymmetry** - x and y are treated symmetrically!

Canonical Correlation Analysis, Solution

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Let $z = (x^T, y^T)^T$, and let

$$\text{cov}(z) = \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

Define

$$M_1 = \Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21},$$

and

$$M_2 = \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}.$$

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Now, the canonical vectors α_k are the eigenvectors of M_1 (α_k corresponds to the k th largest eigenvalue), the canonical vectors β_k are the eigenvectors of M_2 , and ρ_k^2 are the eigenvalues of the matrix M_1 (and of M_2 as well). The proof of this solution can be found from pages 283-284 of [1].

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Note that the eigenvectors α_k and β_k do not have length= 1!

Requirements

$$\text{var}(u_k) = \text{var}(\alpha_k^T x) = 1$$

and

$$\text{var}(v_k) = \text{var}(\beta_k^T y) = 1$$

define the lengths of the eigenvectors.

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If the covariance matrices Σ_{11} and Σ_{22} are not full rank, similar results may be obtained using generalized inverses. One may also consider dimension reduction as a first step.

Canonical Correlation Analysis, Sample Version

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Sample estimates $\hat{\alpha}_k$, $\hat{\beta}_k$ and $\hat{\rho}_k$ of α_k , β_k and ρ_k , respectively, are obtained by using sample covariance matrices calculated from the samples x_1, x_2, \dots, x_n , y_1, y_2, \dots, y_n and z_1, z_2, \dots, z_n .

Canonical Correlation Analysis, Standardization

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As in PCA, also in canonical correlation analysis, the data is sometimes standardized first.

Testing Independence

Testing Independence

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Assume that $z = (x^T, y^T)^T \sim N_{p+q}(\mu, \Sigma)$. Consider testing

H_0 : x and y are independent,

against

H_1 : x and y are not independent.

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Let $m = \min\{p, q\}$, and let

$$T = -\left(n - \frac{1}{2}(p + q + 3)\right) \ln\left(\prod_{k=1}^m (1 - \hat{\rho}_k^2)\right).$$

Now, under H_0 (and under the assumption of multivariate normality), the test statistic T is asymptotically distributed as $\chi^2(pq)$.

Testing Partial Independence

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Assume that $z = (x^T, y^T)^T \sim N_{p+q}(\mu, \Sigma)$. Consider testing

H_0 : Only s of the canonical correlation coefficients are nonzero,

against

H_1 : The number of nonzero canonical correlation coefficients is larger than s .

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Testing Partial Independence

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Let $m = \min\{p, q\}$, and let

$$T_s = -\left(n - \frac{1}{2}(p + q + 3)\right) \ln\left(\prod_{k=s+1}^m (1 - \hat{\rho}_k^2)\right).$$

Now, under H_0 (and under the assumption of multivariate normality), the test statistic T is asymptotically distributed as $\chi^2((p - s)(q - s))$.

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Let X and Y denote the $n \times p$ and $n \times q$ data matrices for n individuals, and let $\hat{\alpha}_k$ and $\hat{\beta}_k$ denote the k th (sample) canonical vectors. Then the $n \times 1$ vectors

$$\eta_k = X\hat{\alpha}_k$$

and

$$\phi_k = Y\hat{\beta}_k$$

denote the scores of the n individuals on the k th canonical correlation variables.

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If the x and y variables are interpreted as the "predictor" and "predicted" variables, respectively, then the η_k score vector can be used to predict the ϕ_k score vector by using least square regression:

$$(\tilde{\phi}_k)_i = \hat{\rho}_k((\eta_k)_i - \hat{\alpha}_k^T \bar{x}) + \hat{\beta}_k^T \bar{y}.$$

The canonical correlation $\hat{\rho}_k$ estimates the proportion of the variance of ϕ_k that is explained by the regression on x .

Example

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Example: closed book exams — open book exams.

Example

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Marks in open-book (O) and closed-book (C) exams:

i	Mechanics (C)	Vectors (C)	Algebra (O)	Analysis (O)	Statistics (O)
1	77	82	67	67	81
2	63	78	80	70	81
3	75	73	71	66	81
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
100	46	52	53	41	40

Source: K. V. Mardia, J. T. Tent, J. M. Bibby, Multivariate analysis, Academic Press, London, 2003 (reprint of 1979).

Example

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Means:

Variable	Mean
x_1	38.9545
x_2	50.5909
y_1	50.6023
y_2	46.6818
y_3	42.3068

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Covariance matrix

$$\Sigma =$$

	Σ_{11}		Σ_{12}		
	302.3	125.8	100.4	105.1	116.1
		170.9	84.2	93.6	97.9
			111.6	110.8	120.5
				217.9	153.8
					294.4
		Σ_{21}		Σ_{22}	

Example

Calculate the eigenvectors

$$M_1 = \Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \Rightarrow \hat{\alpha}_k$$

and

$$M_2 = \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12} \Rightarrow \hat{\beta}_k.$$

Here

$$\hat{\alpha}_1 = \begin{bmatrix} 0.0260 \\ 0.0518 \end{bmatrix}$$

and

$$\hat{\beta}_1 = \begin{bmatrix} 0.0824 \\ 0.0081 \\ 0.0035 \end{bmatrix}.$$

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$$u_1 = 0.0260x_1 + 0.0518x_2$$

and

$$v_1 = 0.0824y_1 + 0.0081y_2 + 0.0035y_3.$$

The highest correlation occurs between an average of x_1 and x_2 weighted on x_2 and an average of y_1 , y_2 and y_3 , heavily weighted on y_1

The canonical correlations

$$\rho_1 = 0.6630$$

and

$$\rho_2 = 0.0412.$$

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Predicting

$$\left(\tilde{\phi}_k\right)_i = \hat{\rho}_k \left((\eta_k)_i - \hat{\alpha}_k^T \bar{x} \right) + \hat{\beta}_k^T \bar{y}.$$

Here

$$\begin{aligned} \left(\tilde{\phi}_1\right)_i &= 0.6630 \left((\eta_1)_i - (0.0260 * 38.9545 + 0.0518 * 50.5909) \right) \\ &\quad + (0.0824 \cdot 50.6023 + 0.0081 \cdot 46.6818 + 0.0035 \cdot 42.3068) \\ &\approx 0.6630(\eta_1)_i + 2.2905 \\ &\approx 0.6630(0.0260(x_1)_i + 0.0518(x_2)_i) + 2.2905 \\ &\approx 0.0172(x_1)_i + 0.0343(x_2)_i + 2.209. \end{aligned}$$

Note that this almost predicts y_1 .

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- The procedure maximizes the correlation between the linear combination of variables — it can be more than difficult to interpret the results.
- Correlation does not automatically imply causality.
- Normality assumption is required in testing.

Next Week

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Next week we will talk about discriminant analysis and classification.

References

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References

 K. V. Mardia, J. T. Kent, J. M. Bibby, *Multivariate Analysis*, Academic Press, London, 2003 (reprint of 1979).

References II

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