Model Solutions to Exercises of Week 10

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Exercise 10.4

Consider a 10-node graph $G = (V, E)$ illustrated in Figure 1. The edges in E are coloured blue and the missing edges are in red. It is a completely symmetric graph, where each node node is connected in a cycle, and is also connected to the neighbours of its neighbours. It is immediately clear, that the blue edges do not form a 4-clique. It remains to show that there is no independent set of size four in G.

Consider an arbitray node v. The neighbourhood $N(v)$ of v is coloured blue in Figure 1. Now it is easy to see by going through all the possibilities that we cannot choose a set of three independent nodes in $V \setminus (\{v\} \cup N(v))$.

Exercise 10.5

We will use asymmetric Ramsey numbers to prove the claim. These are much more general than what is necessary in the exercise. Let $R(s, t) = R_2(s, t; 2)$ denote the smallest number N such that if we colour the edges of a complete graph with N nodes either blue or red, then there is either a blue clique with s nodes or a red clique with t nodes. These are the generalisation of the Ramsey numbers we have seen on the lectures: $R_2(n; 2) = R(n, n)$.

In the context of the asymmetric Ramsey numbers, we need to prove that

$$R(4, 4) \leq 100.$$

We begin with a few simple lemmas.

Lemma 1. *For every $s \geq 2$, it holds that $R(s, 2) = s$.*

Proof. Consider K_n with $n < s$. Labelling all edges blue, we have no blue s-clique and no red edge. If $n = s$, we must have either a red edge or a s-clique. □

We know from Exercise 10.3 that $R(3, 3) = 6$.

The proof itself will come from the following inductive argument.

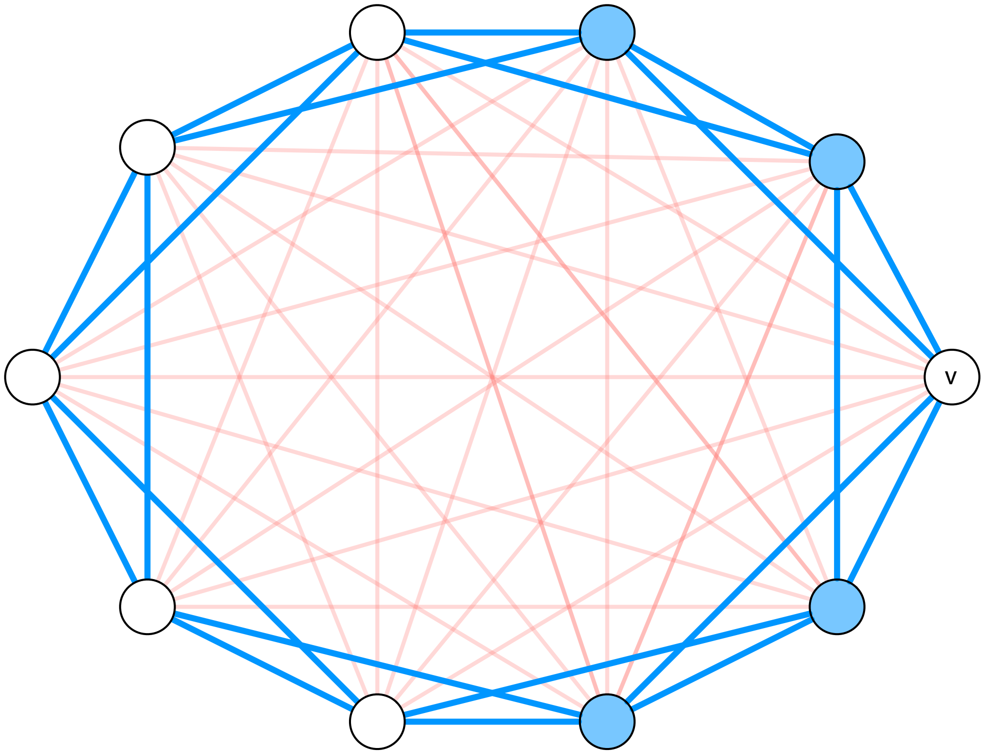

Figure 1: Illustration for Exercise 10.4. A graph $G = (V, E)$, where $|V| = \{1, 2, \ldots, 10\}$, the blue edges induce a subgraph with no 4-cliques and no independents sets of size 4.

Lemma 2. $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

Proof. Let $N = R(s-1,t) + R(s,t-1)$. We consider the complete graph on N nodes. We pick a special node w. Consider an arbitrary labelling of the edges. We split the remaining nodes into two sets $B$and R: $v \in B$ if the edge $\{w, v\}$ is labelled blue and $v \in R$ otherwise.

Now consider the sets B and R. Since $|B| + |R| + 1 = N$, we must have that either $|B| \geq R(s-1,t)$ or $|R| \geq R(s,t-1)$. In the former case, we either have a K_t inside the subgraph induced by B, or there is a K_{s-1} connected to the node w with edges labelled blue. The latter case is symmetric. □

We still need the values of $R(4,3)$ and $R(3,4)$. But by the previous Lemma we have that

$$R(4,3) = R(3,4) \leq R(4,2) + R(3,3) = 10.$$

Now we are ready to state the result.

Theorem 3. $R(4,4) \leq R(4,3) + R(3,4) = 20$.