Ingredients for deriving $\widehat{\beta}_1$

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Starting point

The expression for $\widehat{\beta_1}$ we start from is

$$\widehat{\beta_1} = \frac{\frac{1}{n}\sum(X_i - \overline{X})(Y_i - \overline{Y})}{\frac{1}{n}\sum(X_i - \overline{X})^2} \tag{1}$$

NOTE: β_0 does not enter the equation for $\widehat{\beta_1}$.

Step #1

We then note that by definition (taking our linear specification as given):

$$Y_i = \beta_0 + \beta_1 X_i + u_i \tag{2}$$

Also it is true (taking averages of the above), that

$$\overline{Y} = \beta_0 + \beta_1 \overline{X} + \overline{u} \tag{3}$$

Taking the difference of these yields

$$Y_i - \overline{Y} = \beta_1 (X_i - \overline{X}) + (u_i - \overline{u}) \tag{4}$$

We then note that equation (4) can be substituted into the numerator of equation (1) in place of $Y_i - \overline{Y}$.

Step #2

After reorganizing the numerator of equation (1) we get

$$\frac{1}{n}\sum(X_i-\overline{X})(Y_i-\overline{Y})=\beta_1\frac{1}{n}\sum(X_i-\overline{X})^2 \tag{5}$$

$$+\frac{1}{n}\sum(X_i-\overline{X})u_i-\frac{1}{n}\sum(X_i-\overline{X})\overline{u}$$

Notice that

$$\frac{1}{n}\sum(X_i-\overline{X})=\frac{1}{n}\sum X_i-n\overline{X}=0$$

Therefore we can write

$$\frac{1}{n}\sum(X_i-\overline{X})(Y_i-\overline{Y})=\beta_1\frac{1}{n}\sum(X_i-\overline{X})^2 \tag{6}$$

$$+\frac{1}{n}\sum(X_i-\overline{X})u_i$$

Step #3

Plug the right hand side of equation (6) into equation (1) (i.e., replace the numerator in (1) with the RHS of (6)).
We then get

$$\widehat{\beta_1} = \beta_1 + \frac{\frac{1}{n}\sum(X_i - \overline{X})u_i}{\frac{1}{n}\sum(X_i - \overline{X})^2} \tag{7}$$

Step #4

What is the value of $\widehat{\beta}_1$ in expectation?

$$E[\widehat{\beta}_1] = \beta_1 + E[\frac{\frac{1}{n}\sum(X_i - \overline{X})u_i}{\frac{1}{n}\sum(X_i - \overline{X})^2}] = \beta_1 \tag{8}$$

This is true as the term

$$E[\frac{\frac{1}{n}\sum(X_i - \overline{X})u_i}{\frac{1}{n}\sum(X_i - \overline{X})^2}]$$

is zero **when the OLS assumptions hold**. The reason for this is that then $E[u_i|X] = 0$.

- Because $E[\widehat{\beta}_1] = \beta_1$, $\widehat{\beta}_1$ is unbiased.