4. Photometric Concepts and Magnitudes

Most astronomical observations utilize electromagnetic radiation in one way or another. We can obtain information on the physical nature of a radiation source by studying the energy distribution of its radiation. We shall now introduce some basic concepts that characterize electromagnetic radiation.

4.1 Intensity, Flux Density and Luminosity

Let us assume we have some radiation passing through a surface element $dA$ (Fig. 4.1). Some of the radiation will leave $dA$ within a solid angle $d\omega$; the angle between $d\omega$ and the normal to the surface is denoted by $\theta$. The amount of energy with frequency in the range $[\nu, \nu + d\nu]$ entering this solid angle in time $dt$ is

$$dE_\nu = I_\nu \cos \theta \, dA \, d\nu \, d\omega \, dt.$$  \hfill (4.1)

Here, the coefficient $I_\nu$ is the specific intensity of the radiation at the frequency $\nu$ in the direction of the solid angle $d\omega$. Its dimension is $W \, m^{-2} \, Hz^{-1} \, sterad^{-1}$.

![Fig. 4.1. The intensity $I_\nu$ of radiation is related to the energy passing through a surface element $dA$ into a solid angle $d\omega$, in a direction $\theta$.](image)

The projection of the surface element $dA$ as seen from the direction $\theta$ is $dA_\parallel = dA \cos \theta$, which explains the factor $\cos \theta$. If the intensity does not depend on direction, the energy $dE_\nu$ is directly proportional to the surface element perpendicular to the direction of the radiation.

The intensity including all possible frequencies is called the total intensity $I$, and is obtained by integrating $I_\nu$ over all frequencies:

$$I = \int_0^\infty I_\nu \, d\nu.$$  

More important quantities from the observational point of view are the energy flux ($L_\nu, L$) or, briefly, the flux and the flux density ($F_\nu, F$). The flux density gives the power of radiation per unit area; hence its dimension is $W \, m^{-2} \, Hz^{-1}$ or $W \, m^{-2}$, depending on whether we are talking about the flux density at a certain frequency or about the total flux density.

Observed flux densities are usually rather small, and $W \, m^{-2}$ would be an inconveniently large unit. Therefore, especially in radio astronomy, flux densities are often expressed in Janskys; one Jansky (Jy) equals $10^{-26} \, W \, m^{-2} \, Hz^{-1}$.

When we are observing a radiation source, we in fact measure the energy collected by the detector during some period of time, which equals the flux density integrated over the radiation-collecting area of the instrument and the time interval.

The flux density $F_\nu$ at a frequency $\nu$ can be expressed in terms of the intensity as

$$F_\nu = \frac{1}{dA \, d\nu \, dt} \int_S dE_\nu = \int_S I_\nu \cos \theta \, d\omega,$$ \hfill (4.2)

where the integration is extended over all possible directions. Analogously, the total flux density is

$$F = \int_S I \cos \theta \, d\omega.$$
For example, if the radiation is *isotropic*, i.e. if $I$ is independent of the direction, we get

$$F = \int_S I \cos \theta \, d\omega = I \int_S \cos \theta \, d\omega . \quad (4.3)$$

The solid angle element $d\omega$ is equal to a surface element on a unit sphere. In spherical coordinates it is (Fig. 4.2; also c.f. Appendix A.5):

$$d\omega = \sin \theta \, d\theta \, d\phi .$$

Substitution into (4.3) gives

$$F = I \int_0^\pi \int_0^{2\pi} \cos \theta \sin \theta \, d\theta \, d\phi = 0 ,$$

so there is no net flux of radiation. This means that there are equal amounts of radiation entering and leaving the surface. If we want to know the amount of radiation passing through the surface, we can find, for example, the radiation leaving the surface. For isotropic radiation this is

$$F_l = I \int_0^{\pi/2} \int_0^{2\pi} \cos \theta \sin \theta \, d\theta \, d\phi = \pi I . \quad (4.4)$$

In the astronomical literature, terms such as intensity and brightness are used rather vaguely. Flux density is hardly ever called flux density but intensity or (with luck) flux. Therefore the reader should always carefully check the meaning of these terms.

*Flux* means the power going through some surface, expressed in watts. The flux emitted by a star into a solid angle $\omega$ is $L = \omega r^2 F$, where $F$ is the flux density observed at a distance $r$. *Total flux* is the flux passing through a closed surface encompassing the source. Astronomers usually call the total flux of a star the *luminosity* $L$. We can also talk about the luminosity $L_v$ at a frequency $v$ 

$$[L_v] = W \text{ Hz}^{-1} .$$

If the source (like a typical star) radiates isotropically, its radiation at a distance $r$ is distributed evenly on a spherical surface whose area is $4\pi r^2$ (Fig. 4.3). If the flux density of the radiation passing through this surface is $F$, the total flux is

$$L = 4\pi r^2 F . \quad (4.5)$$
If we are outside the source, where radiation is not created or destroyed, the luminosity does not depend on distance. The flux density, on the other hand, falls off proportional to $1/r^2$.

For extended objects (as opposed to objects such as stars visible only as points) we can define the *surface brightness* as the flux density per unit solid angle (Fig. 4.4). Now the observer is at the apex of the solid angle. The surface brightness is independent of distance, which can be understood in the following way. The flux density arriving from an area $A$ is inversely proportional to the distance squared. But also the solid angle subtended by the area $A$ is proportional to $1/r^2$ ($\omega = A/r^2$). Thus the surface brightness $B = F/\omega$ remains constant.

The energy density $u$ of radiation is the amount of energy per unit volume ($\text{J m}^{-3}$):

$$u = \frac{1}{c} \int S \, I \, d\omega.$$  \hspace{1cm} (4.6)

This can be seen as follows. Suppose we have radiation with intensity $I$ arriving from a solid angle $d\omega$ perpendicular to the surface $dA$ (Fig. 4.5). In the time $dt$, the radiation travels a distance $c \, dt$ and fills a volume $dV = c \, dt \, dA$. Thus the energy in the volume $dV$ is

$$dE = I \, dA \, d\omega \, dt = \frac{1}{c} \, I \, d\omega \, dV.$$

Hence the energy density $du$ of the radiation arriving from the solid angle $d\omega$ is

$$du = \frac{dE}{dV} = \frac{1}{c} \, I \, d\omega,$$

and the total energy density is obtained by integrating this over all directions. For isotropic radiation we get

$$u = \frac{4\pi}{c} I.$$ \hspace{1cm} (4.7)

### 4.2 Apparent Magnitudes

As early as the second century B.C., Hipparchos divided the visible stars into six classes according to their apparent brightness. The first class contained the brightest stars and the sixth the faintest ones still visible to the naked eye.

The response of the human eye to the brightness of light is not linear. If the flux densities of three stars are in the proportion $1:10:100$, the brightness difference of the
first and second star seems to be equal to the difference of the second and third star. Equal brightness ratios correspond to equal apparent brightness differences: the human perception of brightness is logarithmic.

The rather vague classification of Hipparcos was replaced in 1856 by Norman R. Pogson. The new, more accurate classification followed the old one as closely as possible, resulting in another of those illogical definitions typical of astronomy. Since a star of the first class is about one hundred times brighter than a star of the sixth class, Pogson defined the ratio of the brightnesses of classes \( n \) and \( n+1 \) as \( \sqrt[100]{100} = 2.512 \).

The brightness class or magnitude can be defined accurately in terms of the observed flux density \( F \) \((F \text{ in } W \text{ m}^{-2})\). We decide that the magnitude 0 corresponds to some preselected flux density \( F_0 \). All other magnitudes are then defined by the equation

\[ m = -2.5 \log \frac{F}{F_0} \]  

(4.8)

Note that the coefficient is exactly 2.5, not 2.512! Magnitudes are dimensionless quantities, but to remind us that a certain value is a magnitude, we can write it, for example, as 5 mag or 5

It is easy to see that (4.8) is equivalent to Pogson’s definition. If the magnitudes of two stars are \( m \) and \( m+1 \) and their flux densities \( F_m \) and \( F_{m+1} \), respectively, we have

\[ m - (m + 1) = -2.5 \log \frac{F_m}{F_0} + 2.5 \log \frac{F_{m+1}}{F_0} \]

whence

\[ \frac{F_m}{F_{m+1}} = \sqrt[100]{100} \]

In the same way we can show that the magnitudes \( m_1 \) and \( m_2 \) of two stars and the corresponding flux densities \( F_1 \) and \( F_2 \) are related by

\[ m_1 - m_2 = -2.5 \log \frac{F_1}{F_2} \]

(4.9)

Magnitudes extend both ways from the original six values. The magnitude of the brightest star, Sirius, is in fact negative – 1.5. The magnitude of the Sun is – 26.8 and that of a full moon – 12.5. The magnitude of the faintest objects observed depends on the size of the telescope, the sensitivity of the detector and the exposure time. The limit keeps being pushed towards fainter objects; currently the magnitudes of the faintest observed objects are over 30.

### 4.3 Magnitude Systems

The apparent magnitude \( m \), which we have just defined, depends on the instrument we use to measure it. The sensitivity of the detector is different at different wavelengths. Also, different instruments detect different wavelength ranges. Thus the flux measured by the instrument equals not the total flux, but only a fraction of it. Depending on the method of observation, we can define various magnitude systems. Different magnitudes have different zero points, i.e. they have different flux densities \( F_0 \) corresponding to the magnitude 0. The zero points are usually defined by a few selected standard stars.

In daylight the human eye is most sensitive to radiation with a wavelength of about 550 nm, the sensitivity decreasing towards red (longer wavelengths) and violet (shorter wavelengths). The magnitude corresponding to the sensitivity of the eye is called the visual magnitude \( m_v \).

Photographic plates are usually most sensitive at blue and violet wavelengths, but they are also able to register radiation not visible to the human eye. Thus the photographic magnitude \( m_\text{pg} \) usually differs from the visual magnitude. The sensitivity of the eye can be simulated by using a yellow filter and plates sensitised to yellow and green light. Magnitudes thus observed are called photovisual magnitudes \( m_\text{pv} \).

If, in ideal case, we were able to measure the radiation at all wavelengths, we would get the bolometric magnitude \( m_\text{bol} \). In practice this is very difficult, since part of the radiation is absorbed by the atmosphere; also, different wavelengths require different detectors. (In fact there is a gadget called the bolometer, which, however, is not a real bolometer but an infrared detector.) The bolometric magnitude can be derived from the visual magnitude if we know the bolometric correction BC:

\[ m_\text{bol} = m_v - BC \]  

(4.10)

By definition, the bolometric correction is zero for radiation of solar type stars (or, more precisely, stars of the spectral class F5). Although the visual and bolometric
magnitudes can be equal, the flux density corresponding to the bolometric magnitude must always be higher. The reason of this apparent contradiction is in the different values of $F_0$.

The more the radiation distribution differs from that of the Sun, the higher the bolometric correction is. The correction is positive for stars both cooler or hotter than the Sun. Sometimes the correction is defined as $m_{bol} = m_v + BC$ in which case $BC \leq 0$ always. The chance for errors is, however, very small, since we must have $m_{bol} \leq m_v$.

The most accurate magnitude measurements are made using photoelectric photometers. Usually filters are used to allow only a certain wavelength band to enter the detector. One of the multicolour magnitude systems used widely in photoelectric photometry is the UBV system developed in the early 1950’s by Harold L. Johnson and William W. Morgan. Magnitudes are measured through three filters, $U =$ ultraviolet, $B =$ blue and $V =$ visual. Figure 4.6 and Table 4.1 give the wavelength bands of these filters. The magnitudes observed through these filters are called $U$, $B$ and $V$ magnitudes, respectively.

The UBV system was later augmented by adding more bands. One commonly used system is the five colour UBVRI system, which includes $R =$ red and $I =$ infrared filters.

There are also other broad band systems, but they are not as well standardised as the UBV, which has been defined moderately well using a great number of standard stars all over the sky. The magnitude of an object is obtained by comparing it to the magnitudes of standard stars.

In Strömgren’s four-colour or uvby system, the bands passed by the filters are much narrower than in the UBV system. The uvby system is also well standardized, but it is not quite as common as the UBV. Other narrow band systems exist as well. By adding more filters, more information on the radiation distribution can be obtained.

In any multicolour system, we can define colour indices; a colour index is the difference of two magnitudes. By subtracting the $B$ magnitude from $U$ we get the colour index $U - B$, and so on. If the UBV system is used, it is common to give only the $V$ magnitude and the colour indices $U - B$ and $B - V$.

The constants $F_0$ in (4.8) for $U$, $B$ and $V$ magnitudes have been selected in such a way that the colour indices $B - V$ and $U - B$ are zero for stars of spectral type A0 (for spectral types, see Chap. 8). The surface temperature of such a star is about 10,000 K. For example, Vega ($\alpha$ Lyr, spectral class A0V) has $V = 0.03$, $B - V = U - B = 0.00$. The Sun has $V = -26.8$, $-26.8 = 0.62$ and $U - B = 0.10$.

Before the UBV system was developed, a colour index C.I., defined as

\[ \text{C.I.} = m_{pg} - m_v , \]

was used. Since $m_{pg}$ gives the magnitude in blue and $m_v$ in visual, this index is related to $B - V$. In fact,

\[ \text{C.I.} = (B - V) - 0.11 . \]
4.4 Absolute Magnitudes

Thus far we have discussed only apparent magnitudes. They do not tell us anything about the true brightness of stars, since the distances differ. A quantity measuring the intrinsic brightness of a star is the absolute magnitude. It is defined as the apparent magnitude at a distance of 10 parsecs from the star (Fig. 4.7).

We shall now derive an equation which relates the apparent magnitude $m$, the absolute magnitude $M$ and the distance $r$. Because the flux emanating from a star into a solid angle $\omega$ has, at a distance $r$, spread over an area $\omega r^2$, the flux density is inversely proportional to the distance squared. Therefore the ratio of the flux density at a distance $r$, $F(r)$, to the flux density at a distance of 10 parsecs, $F(10)$, is

$$\frac{F(r)}{F(10)} = \left(\frac{10 \text{ pc}}{r}\right)^2.$$  

Thus the difference of magnitudes at $r$ and 10 pc, or the distance modulus $m - M$, is

$$m - M = -2.5 \log \frac{F(r)}{F(10)} = -2.5 \log \left(\frac{10 \text{ pc}}{r}\right)^2.$$  

or

$$m - M = 5 \log \frac{r}{10 \text{ pc}}.$$  

(4.11)

For historical reasons, this equation is almost always written as

$$m - M = 5 \log r - 5,$$  

(4.12)

which is valid only if the distance is expressed in parsecs. (The logarithm of a dimensional quantity is, in fact, physically absurd.) Sometimes the distance is given in kiloparsecs or megaparsecs, which require different constant terms in (4.12). To avoid confusion, we highly recommend the form (4.11).

Absolute magnitudes are usually denoted by capital letters. Note, however, that the $U$, $B$ and $V$ magnitudes are apparent magnitudes. The corresponding absolute magnitudes are $M_U$, $M_B$ and $M_V$.

The absolute bolometric magnitude can be expressed in terms of the luminosity. Let the total flux density at a distance $r = 10$ pc be $F$ and let $F_\odot$ be the equivalent quantity for the Sun. Since the luminosity is $L = 4\pi r^2 F$, we get

$$M_{\text{bol}} - M_{\text{bol,}\odot} = -2.5 \log \frac{F}{F_\odot} = -2.5 \log \frac{L/4\pi r^2}{L_\odot/4\pi r^2},$$

or

$$M_{\text{bol}} - M_{\text{bol,}\odot} = -2.5 \log \frac{L}{L_\odot}.$$  

(4.13)

The absolute bolometric magnitude $M_{\text{bol}} = 0$ corresponds to a luminosity $L_0 = 3.0 \times 10^{38}$ W.

4.5 Extinction and Optical Thickness

Equation (4.11) shows how the apparent magnitude increases (and brightness decreases!) with increasing distance. If the space between the radiation source and the observer is not completely empty, but contains some interstellar medium, (4.11) no longer holds, because part of the radiation is absorbed by the medium (and usually re-emitted at a different wavelength, which may be outside the band defining the magnitude), or scattered away from the line of sight. All these radiation losses are called the extinction.

Now we want to find out how the extinction depends on the distance. Assume we have a star radiating a flux $L_0$ into a solid angle $\omega$ in some wavelength range. Since the medium absorbs and scatters radiation, the
flux $L$ will now decrease with increasing distance $r$ (Fig. 4.8). In a short distance interval $[r, r + dr]$, the extinction $dL$ is proportional to the flux $L$ and the distance travelled in the medium:

$$dL = -\alpha L \, dr.$$  \hspace{1cm} (4.14)

The factor $\alpha$ tells how effectively the medium can obscure radiation. It is called the opacity. From (4.14) we see that its dimension is $[\alpha] = \text{m}^{-1}$. The opacity is zero for a perfect vacuum and approaches infinity when the substance becomes really murky. We can now define a dimensionless quantity, the optical thickness $\tau$ by

$$d\tau = \alpha \, dr.$$  \hspace{1cm} (4.15)

Substituting this into (4.14) we get

$$dL = -L \, d\tau.$$  

Next we integrate this from the source (where $L = L_0$ and $r = 0$) to the observer:

$$\int_{L_0}^{L} \frac{dL}{L} = -\int_{0}^{r} d\tau ,$$

which gives

$$L = L_0 e^{-\tau}.$$ \hspace{1cm} (4.16)

Here, $\tau$ is the optical thickness of the material between the source and the observer and $L$, the observed flux. Now, the flux $L$ falls off exponentially with increasing optical thickness. Empty space is perfectly transparent, i.e. its opacity is $\alpha = 0$; thus the optical thickness does not increase in empty space, and the flux remains constant.

Let $F_0$ be the flux density on the surface of a star and $F(r)$, the flux density at a distance $r$. We can express the fluxes as

$$L = \omega r^2 F(r) , \quad L_0 = \omega R^2 F_0 ,$$

where $R$ is the radius of the star. Substitution into (4.16) gives

$$F(r) = F_0 \frac{R^2}{r^2} e^{-\tau} .$$

For the absolute magnitude we need the flux density at a distance of 10 parsecs, $F(10)$, which is still evaluated without extinction:

$$F(10) = F_0 \frac{R^2}{(10 \text{ pc})^2} .$$

The distance modulus $m - M$ is now

$$m - M = -2.5 \lg \frac{F(r)}{F(10)}$$

$$= 5 \lg \frac{r}{10 \text{ pc}} - 2.5 \lg e^{-\tau}$$

$$= 5 \lg \frac{r}{10 \text{ pc}} + (2.5 \lg e) \tau$$

or

$$m - M = 5 \lg \frac{r}{10 \text{ pc}} + A ,$$ \hspace{1cm} (4.17)

![Fig. 4.8. The interstellar medium absorbs and scatters radiation; this usually reduces the energy flux $L$ in the solid angle $\omega$ ($dL \leq 0$)](image-url)
where $A \geq 0$ is the extinction in magnitudes due to the entire medium between the star and the observer. If the opacity is constant along the line of sight, we have

$$\tau = \alpha \int_0^r dr = ar,$$

and (4.17) becomes

$$m - M = 5 \lg \frac{r}{10 \text{ pc}} + ar,$$  \hspace{1cm} (4.18)

where the constant $a = 2.5 \alpha \lg e$ gives the extinction in magnitudes per unit distance.

**Colour Excess.** Another effect caused by the interstellar medium is the *reddening of light*: blue light is scattered and absorbed more than red. Therefore the colour index $B - V$ increases. The visual magnitude of a star is, from (4.17),

$$V = M_V + 5 \lg \frac{r}{10 \text{ pc}} + A_V,$$  \hspace{1cm} (4.19)

where $M_V$ is the absolute visual magnitude and $A_V$ is the extinction in the V passband. Similarly, we get for the blue magnitudes

$$B = M_B + 5 \lg \frac{r}{10 \text{ pc}} + A_B.$$

The observed colour index is now

$$B - V = M_B - M_V + A_B - A_V,$$

or

$$B - V = (B - V)_0 + E_{B-V},$$  \hspace{1cm} (4.20)

where $(B - V)_0 = M_B - M_V$ is the *intrinsic colour* of the star and $E_{B-V} = (B - V) - (B - V)_0$ is the *colour excess*. Studies of the interstellar medium show that the ratio of the visual extinction $A_V$ to the colour excess $E_{B-V}$ is almost constant for all stars:

$$R = \frac{A_V}{E_{B-V}} \approx 3.0.$$

This makes it possible to find the visual extinction if the colour excess is known:

$$A_V \approx 3.0 E_{B-V}.$$  \hspace{1cm} (4.21)

When $A_V$ is obtained, the distance can be solved directly from (4.19), when $V$ and $M_V$ are known.

We shall study interstellar extinction in more detail in Sect. 15.1 (“Interstellar Dust”).

**Atmospheric Extinction.** As we mentioned in Sect. 3.1, the Earth’s atmosphere also causes extinction. The observed magnitude $m$ depends on the location of the observer and the zenith distance of the object, since these factors determine the distance the light has to travel in the atmosphere. To compare different observations, we must first *reduce* them, i.e. remove the atmospheric effects somehow. The magnitude $m_0$ thus obtained can then be compared with other observations.

If the zenith distance $z$ is not too large, we can approximate the atmosphere by a plane layer of constant thickness (Fig. 4.9). If the thickness of the atmosphere is used as a unit, the light must travel a distance

$$X = \frac{1}{\cos z} = \sec z$$  \hspace{1cm} (4.22)

in the atmosphere. The quantity $X$ is the *air mass*. According to (4.18), the magnitude increases linearly with the distance $X$:

$$m = m_0 + kX,$$  \hspace{1cm} (4.23)

where $k$ is the *extinction coefficient*.

The extinction coefficient can be determined by observing the same source several times during a night with as wide a zenith distance range as possible. The observed magnitudes are plotted in a diagram as a function of the air mass $X$. The points lie on a straight line the slope of which gives the extinction coefficient $k$. When

![Fig. 4.9. If the zenith distance of a star is z, the light of the star travels a distance H/cos z in the atmosphere; H is the height of the atmosphere](image-url)
this line is extrapolated to $X = 0$, we get the magnitude $m_0$, which is the apparent magnitude outside the atmosphere.

In practice, observations with zenith distances higher than $70^\circ$ (or altitudes less than $20^\circ$) are not used to determine $k$ and $m_0$, since at low altitudes the curvature of the atmosphere begins to complicate matters. The value of the extinction coefficient $k$ depends on the observation site and time and also on the wavelength, since extinction increases strongly towards short wavelengths.

### 4.6 Examples

**Example 4.1** Show that intensity is independent of distance.

Suppose we have some radiation leaving the surface element $dA$ in the direction $\theta$. The energy entering the solid angle $d\omega$ in time $dt$ is

$$dE = I \cos \theta \, dA \, d\omega \, dt,$$

where $I$ is the intensity. If we have another surface $dA'$ at a distance $r$ receiving this radiation from direction $\theta'$, we have

$$d\omega = dA' \cos \theta'/r^2.$$

The definition of the intensity gives

$$dE = I' \cos \theta' \, dA' \, d\omega' \, dt,$$

where $I'$ is the intensity at $dA'$ and

$$d\omega' = dA \cos \theta/r^2.$$

Substitution of $d\omega$ and $d\omega'$ into the expressions of $dE$ gives

$$I \cos \theta \, d\theta \, dA \frac{dA' \cos \theta'}{r^2} \, dr = I' \cos \theta' \, dA' \frac{dA \cos \theta}{r^2} \, dr \Rightarrow I' = I.$$

Thus the intensity remains constant in empty space.

**Example 4.2** *Surface Brightness of the Sun*

Assume that the Sun radiates isotropically. Let $R$ be the radius of the Sun, $F_\odot$ the flux density on the surface of the Sun and $F$ the flux density at a distance $r$. Since the luminosity is constant,

$$L = 4\pi R^2 F_\odot = 4\pi r^2 F,$$

the flux density equals

$$F = F_\odot \frac{R^2}{r^2}.$$

At a distance $r \gg R$, the Sun subtends a solid angle

$$\omega = \frac{A}{r^2} = \frac{\pi R^2}{r^2},$$

where $A = \pi R^2$ is the cross section of the Sun. The surface brightness $B$ is

$$B = \frac{F}{\omega} = \frac{F_\odot}{\pi}.$$

Applying (4.4) we get

$$B = I_\odot.$$

Thus the surface brightness is independent of distance and equals the intensity. We have found a simple
interpretation for the somewhat abstract concept of intensity.

The flux density of the Sun on the Earth, the solar constant, is \( S_\odot \approx 1370 \text{ W m}^{-2} \). The angular diameter of the Sun is \( \alpha = 32' \), whence
\[
\frac{R}{r} = \frac{\alpha}{2} = \frac{1}{2} \times \frac{32}{60} \times \frac{\pi}{180} = 0.00465 \text{ rad}.
\]
The solid angle subtended by the Sun is
\[
\omega = \pi \left( \frac{R}{r} \right)^2 = \pi \times 0.00465^2 = 6.81 \times 10^{-5} \text{ sterad}.
\]
and the surface brightness
\[
B = \frac{S_\odot}{\omega} = 2.01 \times 10^7 \text{ W m}^{-2} \text{ sterad}^{-1}.
\]

**Example 4.3  Magnitude of a Binary Star**

Since magnitudes are logarithmic quantities, they can be a little awkward for some purposes. For example, we cannot add magnitudes like flux densities. If the magnitudes of the components of a binary star are 1 and 2, the total magnitude is certainly not 3. To find the total magnitude, we must first solve the flux densities from
\[
1 = -2.5 \log \frac{F_1}{F_0}, \quad 2 = -2.5 \log \frac{F_2}{F_0},
\]
which give
\[
F_1 = F_0 \times 10^{-0.4}, \quad F_2 = F_0 \times 10^{-0.8}.
\]
Thus the total flux density is
\[
F = F_1 + F_2 = F_0(10^{-0.4} + 10^{-0.8})
\]
and the total magnitude,
\[
m = -2.5 \log \frac{F_0(10^{-0.4} + 10^{-0.8})}{F_0} = -2.5 \log 0.5566 = 0.64 .
\]

**Example 4.4** The distance of a star is \( r = 100 \text{ pc} \) and its apparent magnitude \( m = 6 \). What is its absolute magnitude?

Substitution into (4.11)
\[
m - M = 5 \log \frac{r}{10 \text{ pc}}
\]
gives
\[
M = 6 - 5 \log \frac{100}{10} = 1 .
\]

**Example 4.5** The absolute magnitude of a star is \( M = -2 \) and the apparent magnitude \( m = 8 \). What is the distance of the star?

We can solve the distance \( r \) from (4.11):
\[
r = 10 \text{ pc} \times 10^{(m - M)/5} = 10 \times 10^{10/5} \text{ pc} = 1000 \text{ pc} = 1 \text{ kpc} .
\]

**Example 4.6** Although the amount of interstellar extinction varies considerably from place to place, we can use an average value of 2 mag/kpc near the galactic plane. Find the distance of the star in Example 4.5, assuming such extinction.

Now the distance must be solved from (4.18):
\[
8 - (-2) = 5 \log \frac{r}{10} + 0.002 r ,
\]
where \( r \) is in parsecs. This equation cannot be solved analytically, but we can always use a numerical method. We try a simple iteration (Appendix A.7), rewriting the equation as
\[
r = 10 \times 10^{2 - 0.0004 r} .
\]
The value \( r = 1000 \text{ pc} \) found previously is a good initial guess:
\[
r_0 = 1000
\]
\[
r_1 = 10 \times 10^2 - 0.0004 \times 1000 = 398
\]
\[
r_2 = 693
\]
\[
\vdots
\]
\[
r_{12} = r_{13} = 584 .
\]
The distance is \( r \approx 580 \text{ pc} \), which is much less than our earlier value 1000 pc. This should be quite obvious,
since due to extinction, radiation is now reduced much faster than in empty space.

**Example 4.7** What is the optical thickness of a layer of fog, if the Sun seen through the fog seems as bright as a full moon in a cloudless sky?

The apparent magnitudes of the Sun and the Moon are \(-26.8\) and \(-12.5\), respectively. Thus the total extinction in the cloud must be \(A = 14.3\). Since

\[
A = (2.5 \log e) \tau ,
\]

we get

\[
\tau = A / (2.5 \log e) = 14.3 / 1.086 = 13.2 .
\]

The optical thickness of the fog is 13.2. In reality, a fraction of the light scatters several times, and a few of the multiply scattered photons leave the cloud along the line of sight, reducing the total extinction. Therefore the optical thickness must be slightly higher than our value.

**Example 4.8 Reduction of Observations**

The altitude and magnitude of a star were measured several times during a night. The results are given in the following table.

<table>
<thead>
<tr>
<th>Altitude</th>
<th>Zenith distance</th>
<th>Air mass</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>50°</td>
<td>40°</td>
<td>1.31</td>
<td>0.90</td>
</tr>
<tr>
<td>35°</td>
<td>55°</td>
<td>1.74</td>
<td>0.98</td>
</tr>
<tr>
<td>25°</td>
<td>65°</td>
<td>2.37</td>
<td>1.07</td>
</tr>
<tr>
<td>20°</td>
<td>70°</td>
<td>2.92</td>
<td>1.17</td>
</tr>
</tbody>
</table>

By plotting the observations as in the following figure, we can determine the extinction coefficient \(k\) and the magnitude \(m_0\) outside the atmosphere. This can be done graphically (as here) or using a least-squares fit.

Extrapolation to the air mass \(X = 0\) gives \(m_0 = 0.68\). The slope of the line gives \(k = 0.17\).

### 4.7 Exercises

**Exercise 4.1** The total magnitude of a triple star is 0.0. Two of its components have magnitudes 1.0 and 2.0. What is the magnitude of the third component?

**Exercise 4.2** The absolute magnitude of a star in the Andromeda galaxy (distance 690 kpc) is \(M = 5\). It explodes as a supernova, becoming one billion \((10^9)\) times brighter. What is its apparent magnitude?

**Exercise 4.3** Assume that all stars have the same absolute magnitude and stars are evenly distributed in space. Let \(N(m)\) be the number of stars brighter than \(m\) magnitudes. Find the ratio \(N(m + 1) / N(m)\).

**Exercise 4.4** The \(V\) magnitude of a star is 15.1, \(B - V = 1.6\), and absolute magnitude \(M_V = 1.3\). The extinction in the direction of the star in the visual band is \(a_V = 1\) mag kpc\(^{-1}\). What is the intrinsic colour of the star?

**Exercise 4.5** Stars are observed through a triple window. Each surface reflects away 15\% of the incident light.

a) What is the magnitude of Regulus \((M_V = 1.36)\) seen through the window?

b) What is the optical thickness of the window?
5. Radiation Mechanisms

In the previous chapters we have studied the physical properties and detection of electromagnetic radiation. Next we shall briefly discuss concepts related to emission and absorption of radiation. Since we can give here only a summary of some essential results without delving into quantum mechanical explanations, the reader interested in the details is advised to consult any good physics textbook.

5.1 Radiation of Atoms and Molecules

Electromagnetic radiation is emitted or absorbed when an atom or a molecule moves from one energy level to another. If the energy of the atom decreases by an amount $\Delta E$, the atom emits or radiates a quantum of electromagnetic radiation, called a photon, whose frequency $\nu$ is given by the equation

$$\Delta E = h\nu, \quad (5.1)$$

where $h$ is the Planck constant, $h = 6.6256 \times 10^{-34}$ J s. Similarly, if the atom receives or absorbs a photon of a frequency $\nu$, its energy increases by $\Delta E = h\nu$.

The classical model describes an atom as a nucleus surrounded by a swarm of electrons. The nucleus consists of $Z$ protons, each having a charge $+e$ and $N$ electrically neutral neutrons; $Z$ is the charge number of the atom and $A = Z + N$ is its mass number. A neutral atom has as many electrons (charge $-e$) as protons.

An energy level of an atom usually refers to an energy level of its electrons. The energy $E$ of an electron cannot take arbitrary values; only certain energies are allowed: the energy levels are quantized. An atom can emit or absorb radiation only at certain frequencies $\nu_{ij}$ corresponding to energy differences between some initial and final states $i$ and $f$: $|E_i - E_f| = h\nu_{ij}$. This gives rise to the line spectrum, specific for each element (Fig. 5.1). Hot gas under low pressure produces an emission spectrum consisting of such discrete lines. If the same gas is cooled down and observed against a source of white light (which has a continuous spectrum), the same lines are seen as dark absorption lines.

At low temperatures most atoms are in their lowest energy state, the ground state. Higher energy levels are excitation states; a transition from lower to higher state is called excitation. Usually the excited atom will return to the lower state very rapidly, radiating a photon (spontaneous emission); a typical lifetime of an excited state might be $10^{-8}$ seconds. The frequency of the emitted photon is given by (5.1). The atom may return to the lower state directly or through some intermediate states, emitting one photon in each transition.

Downward transitions can also be induced by radiation. Suppose our atom has swallowed a photon and become excited. Another photon, whose frequency $\nu$ corresponds to some possible downward transition from the excited state, can now irritate the atom, causing it to jump to a lower state, emitting a photon with the same frequency $\nu$. This is called induced or stimulated emission. Photons emitted spontaneously leave the atom randomly in all directions with random phases: the radiation is isotropic and incoherent. Induced radiation, on the other hand, is coherent; it propagates in the same direction as and in phase with the inducing radiation.

The zero level of the energy states is usually chosen so that a bound electron has negative energy and a free electron positive energy (cf. the energy integral of planetary orbits, Chap. 6). If an electron with energy $E < 0$ receives more energy than $|E|$, it will leave the atom, which becomes an ion. In astrophysics ionization is often called a bound-free transition (Fig. 5.2). Unlike in excitation all values of energy ($E > 0$) are now possible. The extraneous part of the absorbed energy goes to the kinetic energy of the liberated electron. The inverse process, in which an atom captures a free electron, is the recombination or free–bound transition.

When an electron scatters from a nucleus or an ion without being captured, the electromagnetic interaction can change the kinetic energy of the electron producing free–free radiation. In a very hot gas ($T > 10^6$ K) hydrogen is fully ionized, and the free–free radiation is the most important source of emission. It is then usually called thermal bremsstrahlung. The latter part of the name derives from the fact that decelerating electrons hitting the anode of an X-ray tube emit similar
Fig. 5.1a,b. Origin of line spectra. (a) Emission spectrum. Atoms of glowing gas returning from excited states to lower states emit photons with frequencies corresponding to the energy difference of the states. Each element emits its own characteristic wavelengths, which can be measured by spreading the light into a spectrum with a prism or diffraction grating. (b) Absorption spectrum. When white light containing all wavelengths travels through gas, the wavelengths characteristic of the gas are absorbed.

Fig. 5.2. Different kinds of transitions between energy levels. Absorption and emission occur between two bound states, whereas ionization and recombination occur between a bound and a free state. Interaction of an atom with an free electron can result in a free–free transition. Radiation. In an analogous way the absorption process can be called a bound–bound transition.

Electromagnetic radiation is transverse wave motion; the electric and magnetic fields oscillate perpendicular to each other and also perpendicular to the direction of propagation. The light of an ordinary incandescent lamp has a random distribution of electric fields vibrating in all directions. If the directions of electric fields in the plane perpendicular to the direction of propagation are not evenly distributed, the radiation is polarized (Fig. 5.3). The direction of polarization of linearly polarized light means the plane determined by the electric vector and the direction of the light ray. If the electric vector describes a circle, the radiation is circularly polarized. If the amplitude of the electric field varies at the same time, the polarization is elliptic.
Fig. 5.3. Polarization of light. The light of an incandescent bulb contains all possible directions of vibration and is therefore unpolarized. Some crystals, for example, pass electric fields oscillating only in certain directions, and the transmitted part of the light becomes linearly polarized. $E$ is the electric field and $B$ the magnetic field.

5.2 The Hydrogen Atom

The hydrogen atom is the simplest atom, consisting of a proton and an electron. According to the Bohr model the electron orbits the proton in a circular orbit. (In spite of the fact that this model has very little to do with reality, it can be successfully used to predict some properties of the hydrogen atom.) Bohr’s first postulate says that the angular momentum of the electron must be a multiple of $\hbar$:

$$mvr = n\hbar,$$  
(5.2)

where

$m = \text{mass of the electron},$

$v = \text{speed of the electron},$

$r = \text{radius of the orbit},$

$n = \text{the principal quantum number},$

$n = 1, 2, 3, \ldots ,$

$\hbar = \hbar/2\pi,$

$\hbar = \text{the Planck constant}.$

The quantum mechanical interpretation of Bohr’s first postulate is obvious: the electron is represented as a standing wave, and the “length of the orbit” must be a multiple of the de Broglie wavelength, $\lambda = \hbar/p = \hbar/m v$.

A charged particle in a circular orbit (and thus in accelerated motion) should emit electromagnetic radiation, losing energy, were it to obey the rules of classical electrodynamics. Therefore our electron should spiral down towards the nucleus. But obviously, Nature does not behave this way, and we have to accept Bohr’s second postulate, which says that an electron moving in an allowed orbit around a nucleus does not radiate. Radiation is emitted only when the electron jumps from a higher energy state to a lower one. The emitted quantum has an energy $h\nu$, equal to the energy difference of these states:

$$h\nu = E_{n_2} - E_{n_1}.$$  
(5.3)

We shall now try to find the energy of an electron in the state $E_n$. Coulomb’s law gives the force pulling the electron towards the proton:

$$F = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r_n^2},$$  
(5.4)

where

$\varepsilon_0 = \text{the vacuum permittivity}$

$= 8.85 \times 10^{-12} \text{ N}^{-1}\text{ m}^{-2}\text{ C}^2,$

$e = \text{the charge of the electron} = 1.6 \times 10^{-19} \text{ C},$

$r_n = \text{the distance between the electron and the proton}.$

The acceleration of a particle moving in a circular orbit of radius $r_n$ is

$$a = \frac{v_n^2}{r_n},$$

and applying Newton’s second law ($F = ma$), we get

$$\frac{m v_n^2}{r_n} = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r_n^2}.$$  
(5.5)
From (5.2) and (5.5) it follows that
\[
v_n = \frac{e^2}{4\pi\varepsilon_0 \hbar n}, \quad r_n = \frac{4\pi\varepsilon_0 \hbar}{m e^2 n^2}.
\]
The total energy of an electron in the orbit \(n\) is now
\[
E_n = T + V = \frac{1}{2} m v_n^2 - \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r_n} = -\frac{me^4}{32\pi^2\varepsilon_0^2 \hbar^2 n^2} \equiv -C \frac{1}{n^2},
\]
where \(C\) is a constant. For the ground state \((n = 1)\), we get from (5.6)
\[
E_1 = -2.18 \times 10^{-18} \text{ J} = -13.6 \text{ eV}.
\]

From (5.3) and (5.6) we get the energy of the quantum emitted in the transition \(E_{n_2} \rightarrow E_{n_1}\):
\[
h\nu = E_{n_2} - E_{n_1} = C \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right).
\]
In terms of the wavelength \(\lambda\) this can be expressed as
\[
\frac{1}{\lambda} = \frac{\nu}{c} = \frac{C}{hc} \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \equiv R \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right),
\]
where \(R\) is the Rydberg constant, \(R = 1.097 \times 10^7 \text{ m}^{-1}\).
Equation (5.8) was derived experimentally for \(n_1 = 2\) by Johann Jakob Balmer as early as 1885. That is why we call the set of lines produced by transitions \(E_n \rightarrow E_2\) the Balmer series. These lines are in the visible part of the spectrum. For historical reasons the Balmer lines are often denoted by symbols \(H_\alpha, H_\beta, H_\gamma\) etc. If the electron returns to its ground state \((E_n \rightarrow E_1)\), we get the Lyman

**Fig. 5.4.** Transitions of a hydrogen atom. The lower picture shows a part of the spectrum of the star HD193182. On both sides of the stellar spectrum we see an emission spectrum of iron. The wavelengths of the reference lines are known, and they can be used to calibrate the wavelengths of the observed stellar spectrum. The hydrogen Balmer lines are seen as dark absorption lines converging towards the Balmer ionization limit (also called the Balmer discontinuity) at \(\lambda = 364.7\, \text{nm}\) to the left. The numbers \((15, \ldots, 40)\) refer to the quantum number \(n\) of the higher energy level. (Photo by Mt. Wilson Observatory)
series, which is in the ultraviolet. The other series with specific names are the Paschen series \((n_1 = 3)\), Bracket series \((n_1 = 4)\) and Pfund series \((n_1 = 5)\) (see Fig. 5.4).

5.3 Line Profiles

The previous discussion suggests that spectral lines would be infinitely narrow and sharp. In reality, however, they are somewhat broadened. We will now consider briefly the factors affecting the shape of a spectral line, called a line profile. An exact treatment would take us too deep into quantum mechanics, so we cannot go into the details here.

According to quantum mechanics everything cannot be measured accurately at the same time. For example, even in principle, there is no way to determine the \(x\) coordinate and the momentum \(p_x\) in the direction of the \(x\) axis with arbitrary precision simultaneously. These quantities have small uncertainties \(\Delta x\) and \(\Delta p_x\), such that

\[
\Delta x \Delta p_x \approx \hbar .
\]

A similar relation holds for other directions, too. Time and energy are also connected by an uncertainty relation,

\[
\Delta E \Delta t \approx \hbar .
\]

The natural width of spectral lines is a consequence of this Heisenberg uncertainty principle.

If the average lifetime of an excitation state is \(T\), the energy corresponding to the transition can only be determined with an accuracy of \(\Delta E = \hbar / T = \hbar / (2\pi T)\). From (5.1) it follows that \(\Delta v = \Delta E / \hbar\). In fact, the uncertainty of the energy depends on the lifetimes of both the initial and final states. The natural width of a line is defined as

\[
\gamma = \frac{\Delta E_i + \Delta E_f}{\hbar} = \frac{1}{T_1} + \frac{1}{T_2}.
\] (5.9)

It can be shown that the corresponding line profile is

\[
I_v = \frac{\gamma}{2\pi} \frac{I_0}{(v - v_0)^2 + \gamma^2 / 4},
\] (5.10)

where \(v_0\) is the frequency at the centre of the line and \(I_0\) the total intensity of the line. At the centre of the line the intensity per frequency unit is

\[
I_{v_0} = \frac{2}{\pi \gamma} I_0 ,
\]

and at the frequency \(v = v_0 + \gamma / 2\),

\[
I_{v_0 + \gamma / 2} = \frac{1}{\pi \gamma} I_0 = \frac{1}{2} I_0 .
\]

Thus the width \(\gamma\) is the width of the line profile at a depth where the intensity is half of the maximum. This is called the full width at half maximum (FWHM).

**Doppler Broadening.** Atoms of a gas are moving the faster the higher the temperature of the gas. Thus spectral lines arising from individual atoms are shifted by the Doppler effect. The observed line consists of a collection of lines with different Doppler shifts, and the shape of the line depends on the number of atoms with different velocities.

Each Doppler shifted line has its characteristic natural width. The resulting line profile is obtained by giving each Doppler shifted line a weight proportional to the number of atoms given by the velocity distribution and integrating over all velocities. This gives rise to the Voigt profile (Fig. 5.5), which already describes most spectral lines quite well. The shapes of different profiles don’t

![Fig. 5.5](image-url) Each spectral line has its characteristic natural width (solid line). Motions of particles broaden the line further due to the Doppler effect, resulting in the Voigt profile (dashed line). Both profiles have the same area.
Fig. 5.6. The full width at half maximum (FWHM) of a spectral line is the width at the depth where the intensity is half of the maximum. The equivalent width $W$ is defined so that the line and the shaded rectangle have the same area in the picture. The two measures are not generally the same, although they usually are close to each other. See very different; the most obvious consequence of the broadening is that the maximum depth decreases.

One way to describe the width of a line is to give its full width at half maximum (Fig. 5.6). Due to Doppler broadening this is usually greater than the natural width. The equivalent width $W$ is defined so that the line and the shaded rectangle have the same area in the picture. The two measures are not generally the same, although they usually are close to each other.

### 5.4 Quantum Numbers, Selection Rules, Population Numbers

**Quantum Numbers.** The Bohr model needs only one quantum number, $n$, to describe all the energy levels of the electron. This can explain only the coarse features of an atom with a single electron.

Quantum mechanics describes the electron as a three dimensional wave, which only gives the probability of finding the electron in a certain place. Quantum mechanics has accurately predicted all the energy levels of hydrogen atoms. The energy levels of heavier atoms and molecules can also be computed; however, such calculations are very complicated. Also the existence of quantum numbers can be understood from the quantum mechanical point of view.

The quantum mechanical description involves four quantum numbers, one of which is our $n$, the principal quantum number. The principal quantum number describes the quantized energy levels of the electron. The classical interpretation of discrete energy levels allows only certain orbits given by (5.6). The orbital angular momentum of the electron is also quantized. This is described by the angular momentum quantum number $l$. The angular momentum corresponding to a quantum number $l$ is

$$L = \sqrt{l(l+1)}\hbar.$$ 

The classical analogy would be to allow some elliptic orbits. The quantum number $l$ can take only the values $l = 0, 1, \ldots, n-1$.

For historical reasons, these are often denoted by the letters $s$, $p$, $d$, $f$, $g$, $h$, $i$, $j$.

Although $l$ determines the magnitude of the angular momentum, it does not give its direction. In a magnetic field this direction is important, since the orbiting electron also generates a tiny magnetic field. In any experiment, only one component of the angular momentum can be measured at a time. In a given direction $z$ (e.g. in the direction of the applied magnetic field), the projection of the angular momentum can have only the values

$$L_z = m_l \hbar,$$

where $m_l$ is the magnetic quantum number

$$m_l = 0, \pm 1, \pm 2, \ldots, \pm l.$$ 

The magnetic quantum number is responsible for the splitting of spectral lines in strong magnetic fields, known as the Zeeman effect. For example, if $l = 1$, $m_l$ can have $2l + 1 = 3$ different values. Thus, the line arising from the transition $l = 1 \rightarrow l = 0$ will split into three components in a magnetic field (Fig. 5.7).

The fourth quantum number is the spin describing the intrinsic angular momentum of the electron. The spin of the electron is

$$S = \sqrt{s(s+1)}\hbar,$$
Fig. 5.7. The Zeeman effect. In strong magnetic fields each energy level of a hydrogen atom splits into \((2l + 1)\) separate levels, which correspond to different values of the magnetic quantum number \(m_l = l, l-1, \ldots, -l\). The energy differences of the successive levels have the same constant value \(\Delta E\). For example the \(p\) state \((l = 1)\) splits into three and the \(d\) state \((l = 2)\) into five sublevels. The selection rules require that in electric dipole transitions \(\Delta m_l\) equals 0 or \(\pm 1\), and only nine different transitions between \(p\) and \(d\) states are possible. Moreover, the transitions with the same \(\Delta m_l\) have the same energy difference. Thus the spectrum has only three separate lines

where the spin quantum number is \(s = \frac{1}{2}\). In a given direction \(z\), the spin is

\[ S_z = m_s \hbar, \]

where \(m_s\) can have one of the two values:

\[ m_s = \pm \frac{1}{2}. \]

All particles have a spin quantum number. Particles with an integral spin are called \(bosons\) (photon, mesons); particles with a half-integral spin are \(fermions\) (proton, neutron, electron, neutrino etc.).

Classically, spin can be interpreted as the rotation of a particle; this analogy, however, should not be taken too literally.

The total angular momentum \(J\) of an electron is the sum of its orbital and spin angular momentum:

\[ J = L + S. \]

Depending on the mutual orientation of the vectors \(L\) and \(S\) the quantum number \(j\) of total angular momentum can have one of two possible values,

\[ j = l \pm \frac{1}{2}, \]

(except if \(l = 0\), when \(j = \frac{1}{2}\)). The \(z\) component of the total angular momentum can have the values

\[ m_j = 0, \pm 1, \pm 2, \ldots \pm j. \]

Spin also gives rise to the fine structure of spectral lines. Lines appear as close pairs or doublets.

**Selection Rules.** The state of an electron cannot change arbitrarily; transitions are restricted by selection rules, which follow from certain conservation laws. The selection rules express how the quantum numbers must change in a transition. Most probable are the \(electric dipole transitions\), which make the atom behave like an oscillating dipole. The conservation laws require that in a transition we have

\[ \Delta l = \pm 1, \]
\[ \Delta m_l = 0, \pm 1, \]
\[ \Delta m_j = 0, \pm 1. \]

In terms of the total angular momentum the selection rules are

\[ \Delta l = \pm 1, \]
\[ \Delta j = 0, \pm 1, \]
\[ \Delta m_j = 0, \pm 1. \]

The probabilities of all other transitions are much smaller, and they are called \(forbidden transitions\); examples are magnetic dipole transitions and all quadrupole and higher multipole transitions.

Spectral lines originating in forbidden transitions are called \(forbidden lines\). The probability of such a transition is so low that under normal circumstances, the transition cannot take place before collisions force the electron to change state. Forbidden lines are possible only if the gas is extremely rarified (like in auroras and planetary nebulae).
The spins of an electron and nucleus of a hydrogen atom can be either parallel or antiparallel (Fig. 5.8). The energy of the former state is $0.0000059 \text{ eV}$ higher. But the selection rules make an electric dipole transition between these states impossible. The transition, which is a magnetic dipole transition, has a very low probability, $A = 2.8 \times 10^{-15} \text{ s}^{-1}$. This means that the average lifetime of the higher state is $T = 1/A = 11 \times 10^6$ years. Usually collisions change the state of the electron well before this period of time has elapsed. But in interstellar space the density of hydrogen is so low and the total amount of hydrogen so great that a considerable number of these transitions can take place.

The wavelength of the radiation emitted by this transition is 21 cm, which is in the radio band of the spectrum. Extinction at radio wavelengths is very small, and we can observe more distant objects than by using optical wavelengths. The 21 cm radiation has been of crucial importance for surveys of interstellar hydrogen.

Population Numbers. The population number $n_i$ of an energy state $i$ means the number of atoms in that state per unit volume. In thermal equilibrium, the population numbers obey the Boltzmann distribution:

$$\frac{n_i}{n_0} = \frac{g_i}{g_0} e^{-\Delta E/(kT)} , \quad (5.11)$$

where $T$ is the temperature, $k$ is the Boltzmann constant, $\Delta E = E_i - E_0 = h\nu$ is the energy difference between the excited and ground state, and $g_i$ is the statistical weight of the level $i$ (it is the number of different states with the same energy $E_i$). The subscript 0 always refers to the ground state. Often the population numbers differ from the values given by (5.11), but still we can define an excitation temperature $T_{\text{exc}}$ in such a way that (5.11) gives correct population numbers, when $T$ is replaced by $T_{\text{exc}}$. The excitation temperature may be different for different energy levels.

5.5 Molecular Spectra

The energy levels of an atom are determined by its electrons. In the case of a molecule, there are many more possibilities: atoms can vibrate around their equilibria and the molecule can rotate around some axis. Both vibrational and rotational states are quantized. Transitions between successive vibrational states typically involve photons in the infrared band, while transitions between rotational states involve photons in the microwave band. These combined with transitions of electrons produce a band spectrum, characteristic for molecules (Fig. 5.9). The spectrum has several narrow bands composed of a great number of lines.

5.6 Continuous Spectra

We have already mentioned some processes that produce continuous spectra. Continuous emission spectra can originate in recombinations and free–free transitions. In recombination, an atom captures a free packed that they overlap and at this resolution, the spectrum looks continuous. (R.W.B. Pearse, A.G. Gaydon: The Identification of Molecular Spectra (Chapman & Hall Ltd., London 1976) p. 394)
electron whose energy is not quantized; in free–free transitions, both initial and final states are unquantized. Thus the emission line can have any frequency whatsoever. Similarly, ionizations and free–free transitions can give rise to a continuous absorption spectrum.

Each spectrum contains a continuous component, or continuum, and spectral lines. Sometimes, however, the lines are so closely packed and so broad that they seem to form a nearly continuous spectrum.

When the pressure of hot gas is increased, the spectral lines begin to broaden. At high pressure, atoms bump into each other more frequently, and the close neighbors disturb the energy levels. When the pressure is high enough, the lines begin to overlap. Thus the spectrum of hot gas at high pressure is continuous. Electric fields also broaden spectral lines (the Stark effect).

In liquids and solids the atoms are more densely packed than in gaseous substances. Their mutual perturbations broaden the energy levels, producing a continuous spectrum.

5.7 Blackbody Radiation

A blackbody is defined as an object that does not reflect or scatter radiation shining upon it, but absorbs and re-emits the radiation completely. A blackbody is a kind of an ideal radiator, which cannot exist in the real world. Yet many objects behave very much as if they were blackbodies.

The radiation of a blackbody depends only on its temperature, being perfectly independent of its shape, material and internal constitution. The wavelength distribution of the radiation follows Planck’s law, which is a function of temperature only. The intensity at a frequency \( \nu \) of a blackbody at temperature \( T \) is

\[
B_\nu(T) = B(\nu; T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{hv/(kT)} - 1},
\]

where

\[
\begin{align*}
h &= \text{the Planck constant} = 6.63 \times 10^{-34} \text{ Js,} \\
c &= \text{the speed of light} \approx 3 \times 10^8 \text{ m s}^{-1}, \\
k &= \text{the Boltzmann constant} = 1.38 \times 10^{-23} \text{ J K}^{-1}.
\end{align*}
\]

By definition of the intensity, the dimension of \( B_\nu \) is \( \text{W m}^{-2} \text{Hz}^{-1} \text{ sterad}^{-1} \).

Blackbody radiation can be produced in a closed cavity whose walls absorb all radiation incident upon them (and coming from inside the cavity). The walls and the radiation in the cavity are in equilibrium; both are at the same temperature, and the walls emit all the energy they receive. Since radiation energy is constantly transformed into thermal energy of the atoms of the walls and back to radiation, the blackbody radiation is also called thermal radiation.

The spectrum of a blackbody given by Planck’s law (5.12) is continuous. This is true if the size of the radiator is very large compared with the dominant wavelengths. In the case of the cavity, this can be understood by considering the radiation as standing waves trapped in the cavity. The number of different wavelengths is larger, the shorter the wavelengths are compared with the size of the cavity. We already mentioned that spectra of solid bodies are continuous; very often such spectra can be quite well approximated by Planck’s law.

We can also write Planck’s law as a function of the wavelength. We require that \( B_\lambda \, d\lambda = -B_\nu \, d\nu \). The wavelength decreases with increasing frequency; hence the minus sign. Since \( \nu = c/\lambda \), we have

\[
\frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2},
\]

whence

\[
B_\nu = -B_\lambda \frac{d\nu}{d\lambda} = -B_\lambda \frac{c}{\lambda^2},
\]

or

\[
B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/(\lambda kT)} - 1},
\]

\[
\{B_\lambda\} = \text{W m}^{-2} \text{ m}^{-1} \text{ sterad}^{-1}.
\]

The functions \( B_\nu \) and \( B_\lambda \) are defined in such a way that the total intensity can be obtained in the same way using either of them:

\[
B(T) = \int_0^{\infty} B_\nu \, d\nu = \int_0^{\infty} B_\lambda \, d\lambda.
\]

Let us now try to find the total intensity using the first of these integrals:

\[
B(T) = \int_0^{\infty} B_\nu(T) \, d\nu = \frac{2hc}{c^2} \int_0^{\infty} \frac{\nu^3 \, d\nu}{e^{hv/(kT)} - 1}.
\]
We now change the integration variable to $x = hν/(kT)$, whence $\text{d}ν = (kT/h)\text{d}x$:

$$B(T) = \frac{2h}{c^2} k^4 T^4 \int_0^\infty \frac{x^3 \text{d}x}{e^x - 1}.$$  

The definite integral in this expression is just a real number, independent of the temperature. Thus we find that

$$B(T) = AT^4,$$  

where the constant $A$ has the value

$$A = \frac{2k^4 \pi^4}{c^3 h^3 15}.$$  

(In order to get the value of $A$ we have to evaluate the integral. There is no elementary way to do that. We can tell those who are familiar with all the exotic functions so beloved by theoretical physicists, that the integral can rather easily be expressed as $\Gamma(4)\zeta(4)$, where $\zeta$ is the Riemann zeta function and $\Gamma$ is the gamma function. For integral values, $\Gamma(n)$ is simply the factorial $(n-1)!$. The difficult part is showing that $\zeta(4) = \pi^4/90$. This can be done by expanding $x^4 - x^2$ as a Fourier-series and evaluating the series at $x = \pi$.)

The flux density $F$ for isotropic radiation of intensity $B$ is (Sect. 4.1):

$$F = \pi B$$

or

$$F = \sigma T^4.$$  

This is the Stefan-Boltzmann law, and the constant $\sigma$ ($= \pi A$) is the Stefan-Boltzmann constant, $\sigma = 5.67 \times 10^{-8}$ W m$^{-2}$ K$^{-4}$.

From the Stefan-Boltzmann law we get a relation between the luminosity and temperature of a star. If the radius of the star is $R$, its surface area is $4\pi R^2$, and if the flux density on the surface is $F$, we have

$$L = 4\pi R^2 F.$$  

If the star is assumed to radiate like a blackbody, we have $F = \sigma T^4$, which gives

$$L = 4\pi \sigma R^2 T^4.$$  

In fact this defines the effective temperature of the star, discussed in more detail in the next section.

The luminosity, radius and temperature of a star are interdependent quantities, as we can see from (5.19). They are also related to the absolute bolometric magnitude of the star. Equation (4.13) gives the difference of the absolute bolometric magnitude of the star and the Sun:

$$M_{\text{bol}} - M_{\text{bol,\odot}} = -2.5 \lg \frac{L}{L_{\odot}}.$$  

But we can now use (5.19) to express the luminosities in terms of the radii and temperatures:

$$M_{\text{bol}} - M_{\text{bol,\odot}} = -5 \lg \frac{R}{R_{\odot}} - 10 \lg \frac{T}{T_{\odot}}.$$  

As we can see in Fig. 5.10, the wavelength of the maximum intensity decreases with increasing total intensity (equal to the area below the curve). We can find the wavelength $\lambda_{\text{max}}$ corresponding to the maximum intensity by differentiating Planck’s function $B_\lambda(T)$ with
respect to \( \lambda \) and finding zero of the derivative. The result is the **Wien displacement law**:

\[
\lambda_{\text{max}} T = b = \text{const}, \tag{5.22}
\]

where the **Wien displacement constant** \( b \) is

\[
b = 0.0028978 \text{ K m}. \tag{5.23}
\]

We can use the same procedure to find the maximum of \( B_\nu \). But the frequency \( \nu_{\text{max}} \) thus obtained is different from \( \nu_{\text{max}} = c/\lambda_{\text{max}} \) given by (5.22). The reason for this is the fact that the intensities are given per unit frequency or unit wavelength, and the dependence of frequency on wavelength is nonlinear.

When the wavelength is near the maximum or much longer than \( \lambda_{\text{max}} \) Planck’s function can be approximated by simpler expressions. When \( \lambda \approx \lambda_{\text{max}} \) (or \( hc/(\lambda kT) \gg 1 \)), we have

\[
e^{hc/(\lambda kT)} \approx 1. \tag{5.24}
\]

In this case we get the **Wien approximation**

\[
B_\nu(T) \approx \frac{2hc^2}{\lambda^5} e^{-hc/(\lambda kT)}. \tag{5.25}
\]

When \( hc/(\lambda kT) \ll 1 (\lambda \gg \lambda_{\text{max}}) \), we have

\[
e^{hc/(\lambda kT)} \approx 1 + hc/(\lambda kT),
\]

which gives the **Rayleigh–Jeans approximation**

\[
B_\nu(T) \approx \frac{2hc^2}{\lambda^3} \frac{\lambda}{hc} = \frac{2ckT}{\lambda^4}. \tag{5.26}
\]

This is particularly useful in radio astronomy.

Classical physics predicted only the Rayleigh–Jeans approximation. Were (5.24) true for all wavelengths, the intensity would grow beyond all limits when the wavelength approaches zero, contrary to observations. This contradiction was known as the ultraviolet catastrophe.

### 5.8 Temperatures

Temperatures of astronomical objects range from almost absolute zero to millions of degrees. Temperature can be defined in a variety of ways, and its numerical value depends on the specific definition used. All these different temperatures are needed to describe different physical phenomena, and often there is no unique ‘true’ temperature.

Often the temperature is determined by comparing the object, a star for instance, with a blackbody. Although real stars do not radiate exactly like blackbodies, their spectra can usually be approximated by blackbody spectra after the effect of spectral lines has been eliminated. The resulting temperature depends on the exact criterion used to fit Planck’s function to observations.

The most important quantity describing the surface temperature of a star is the **effective temperature** \( T_e \). It is defined as the temperature of a blackbody which radiates with the same total flux density as the star. Since the effective temperature depends only on the total radiation power integrated over all frequencies, it is well defined for all energy distributions even if they deviate far from Planck’s law.

In the previous section we derived the Stefan-Boltzmann law, which gives the total flux density as a function of the temperature. If we now find a value \( T_e \) of the temperature such that the Stefan-Boltzmann law gives the correct flux density \( F \) on the surface of the star, we have found the effective temperature. The flux density on the surface is

\[
F = \sigma T_e^4. \tag{5.27}
\]

The total flux is \( L = 4\pi R^2 F \), where \( R \) is the radius of the star, and the flux density at a distance \( r \) is

\[
F' = \frac{L}{4\pi r^2} = \frac{R^2}{r^2} F = \left( \frac{\alpha}{2} \right)^2 \alpha T_e^4, \tag{5.28}
\]

where \( \alpha = 2R/r \) is the observed angular diameter of the star. For direct determination of the effective temperature, we have to measure the total flux density and the angular diameter of the star. This is possible only in the few cases in which the diameter has been found by interferometry.

If we assume that at some wavelength \( \lambda \) the flux density \( F_\lambda \) on the surface of the star is obtained from Planck’s law, we get the **brightness temperature** \( T_\lambda \). In the isotropic case we have then \( F_\lambda = \pi B_\lambda(T_\lambda) \). If the radius of the star is \( R \) and distance from the Earth \( r \), the observed flux density is

\[
F_\lambda' = \frac{R^2}{r^2} F_\lambda. \tag{5.29}
\]
Again $F_{\nu}$ can be determined only if the angular diameter $\alpha$ is known. The brightness temperature $T_b$ can then be solved from

$$F_{\nu} = \left(\frac{\alpha}{2}\right)^2 \pi B_{\nu}(T_b).$$

(5.27)

Since the star does not radiate like a blackbody, its brightness temperature depends on the particular wavelength used in (5.27).

In radio astronomy, brightness temperature is used to express the intensity (or surface brightness) of the source. If the intensity at frequency $\nu$ is $I_{\nu}$, the brightness temperature is obtained from

$$I_{\nu} = B_{\nu}(T_b).$$

$T_b$ gives the temperature of a blackbody with the same surface brightness as the observed source.

Since radio wavelengths are very long, the condition $h\nu \ll kT$ of the Rayleigh–Jeans approximation is usually satisfied (except for millimetre and submillimetre bands), and we can write Planck's law as

$$B_{\nu}(T_b) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/(kT_b)} - 1}$$

$$\approx \frac{2h\nu^3}{c^2} \frac{1}{1 + h\nu/(kT_b) + \ldots - 1}$$

$$\approx \frac{2h\nu^3}{c^2} \frac{1}{kT_b}.$$  

Thus we get the following expression for the radio astronomical brightness temperature:

$$T_b = \frac{c^2}{2k\nu^2} I_{\nu} = \frac{\lambda^2}{2k} I_{\nu}.$$  

(5.28)

A measure of the signal registered by a radio telescope is the antenna temperature $T_A$. After the antenna temperature is measured, we get the brightness temperature from

$$T_A = \eta T_b,$$  

(5.29)

where $\eta$ is the beam efficiency of the antenna (typically $0.4 \lesssim \eta \lesssim 0.8$). Equation (5.29) holds if the source is wide enough to cover the whole beam, i.e. the solid angle $\Omega_s$ from which the antenna receives radiation. If the solid angle subtended by the source, $\Omega_s$, is smaller than $\Omega_A$, the observed antenna temperature is

$$T_A = \eta \frac{\Omega_s}{\Omega_A} T_b, \quad (\Omega_s < \Omega_A).$$  

(5.30)

The colour temperature $T_c$ can be determined even if the angular diameter of the source is unknown (Fig. 5.11). We only have to know the relative energy distribution in some wavelength range $[\lambda_1, \lambda_2]$; the absolute value of the flux is not needed. The observed flux density as a function of wavelength is compared with Planck’s function at different temperatures. The temperature giving the best fit is the colour temperature in the interval $[\lambda_1, \lambda_2]$. The colour temperature is usually different for different wavelength intervals, since the shape of the observed energy distribution may be quite different from the blackbody spectrum.

A simple method for finding a colour temperature is the following. We measure the flux density $F_{\nu}'$ at two wavelengths $\lambda_1$ and $\lambda_2$. If we assume that the intensity distribution follows Planck’s law, the ratio of these flux densities must be the same as the ratio obtained from Planck’s law:

$$\frac{F_{\nu_1}(T)}{F_{\nu_2}(T)} = \frac{B_{\lambda_1}(T)}{B_{\lambda_2}(T)} = \frac{\lambda_2^5 e^{h\nu_1/(\lambda_2 kT)} - 1}{\lambda_1^5 e^{h\nu_2/(\lambda_1 kT)} - 1}.$$  

(5.31)

The temperature $T$ solved from this equation is a colour temperature.

The observed flux densities correspond to certain magnitudes $m_{\lambda_1}$ and $m_{\lambda_2}$. The definition of magnitudes gives

$$m_{\lambda_1} - m_{\lambda_2} = -2.5 \log \frac{F_{\nu_1}}{F_{\nu_2}} + \text{const},$$

where the constant term is a consequence of the different zero points of the magnitude scales. If the temperature
is not too high, we can use the Wien approximation in the optical part of the spectrum:

\[ m_{\lambda_1} - m_{\lambda_2} = -2.5 \log \left( \frac{B_{\lambda_1}}{B_{\lambda_2}} \right)^{5} + 2.5 \frac{hc}{kT} \left( \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) \log e + \text{const}. \]

This can be written as

\[ m_{\lambda_1} - m_{\lambda_2} = a + b/T, \quad (5.32) \]

where \( a \) and \( b \) are constants. This shows that there is a simple relationship between the difference of two magnitudes and the colour temperature.

Strictly speaking, the magnitudes in (5.32) are monochromatic, but the same relation can be also used with broadband magnitudes like \( B \) and \( V \). In that case, the two wavelengths are essentially the effective wavelengths of the \( B \) and \( V \) bands. The constant is chosen so that \( B - V = 0 \) for stars of the spectral type A0 (see Chap. 8). Thus the colour index \( B - V \) also gives a colour temperature.

The kinetic temperature \( T_k \), is related to the average speed of gas molecules. The kinetic energy of an ideal gas molecule as a function of temperature follows from the kinetic gas theory:

\[ \text{Kinetic energy} = \frac{1}{2} m v^2 = \frac{3}{2} kT_k. \]

Solving for \( T_k \) we get

\[ T_k = \frac{mv^2}{3k}, \quad (5.33) \]

where \( m \) is the mass of the molecule, \( v \) its average velocity (or rather its r.m.s velocity, which means that \( v^2 \) is the average of the squared velocities), and \( k \), the Boltzmann constant. For ideal gases the pressure is directly proportional to the kinetic temperature (c.f. *Gas Pressure and Radiation Pressure, p. 238):

\[ P = nkT_k, \quad (5.34) \]

where \( n \) is the number density of the molecules (molecules per unit volume). We previously defined the excitation temperature \( T_{\text{exc}} \) as a temperature which, if substituted into the Boltzmann distribution (5.11), gives the observed population numbers. If the distribution of atoms in different levels is a result of mutual collisions of the atoms only, the excitation temperature equals the kinetic temperature, \( T_{\text{exc}} = T_k \).

The ionization temperature \( T_i \) is found by comparing the number of atoms in different states of ionization. Since stars are not exactly blackbodies, the values of excitation and ionization temperatures usually vary, depending on the element whose spectral lines were used for temperature determination.

In thermodynamic equilibrium all these various temperatures are equal.

5.9 Other Radiation Mechanisms

The radiation of a gas in thermodynamic equilibrium depends on the temperature and density only. In astrophysical objects deviations from thermodynamic equilibrium are, however, quite common. Some examples of non-thermal radiation arising under such conditions are mentioned in the following.

Maser and Laser (Fig. 5.12). The Boltzmann distribution (5.11) shows that usually there are fewer atoms in excited states than in the ground state. There are, however, means to produce a population inversion, an excited state containing more atoms than the ground state. This inversion is essential for both the maser and the laser (Microwave/Light Amplification by Stimulated Emission of Radiation). If the excited atoms are now illuminated with photons having energies equal to the excitation energy, the radiation will induce downward transitions. The number of photons emitted greatly exceeds the number of absorbed photons, and radiation is amplified. Typically the excited state is a metastable state, a state with a very long average lifetime, which means that the contribution of spontaneous emission is negligible. Therefore the resulting radiation is coherent and monochromatic. Several maser sources have been found in interstellar molecular clouds and dust envelopes around stars.

Synchrotron Radiation. A free charge in accelerated motion will emit electromagnetic radiation. Charged particles moving in a magnetic field follow helices around the field lines. As seen from the direction of
Radiation Mechanisms

Fig. 5.12. The operational principle of the maser and the laser. A metastable state (a state with a relatively long average lifetime) stores atoms where they accumulate; there are more atoms in the metastable state than in the ground state. This population inversion is maintained by radiatively exciting atoms to a higher excitation state (“pumping”), from which they spontaneously jump down to the metastable state. When the atoms are illuminated by photons with energies equal to the excitation energy of the metastable state, the photons will induce more radiation of the same wavelength, and the radiation is amplified in geometric progression.

5.10 Radiative Transfer

Propagation of radiation in a medium, also called radiative transfer, is one of the basic problems of astrophysics. The subject is too complicated to be discussed here in any detail. The fundamental equation of radiative transfer is, however, easily derived.

Assume we have a small cylinder, the bottom of which has an area \(dA\) and the length of which is \(dr\). Let \(I_v\) be the intensity of radiation perpendicular to the bottom surface going into a solid angle \(d\omega\) (\([I_v] = \text{W m}^{-2}\text{ Hz}^{-1}\text{ sterad}^{-1}\)). If the intensity changes by an amount \(dI_v\) in the distance \(dr\), the energy changes by

\[
dE = dI_v \, dA \, d\nu \, d\omega \, dt
\]

in the cylinder in time \(dt\). This equals the emission minus absorption in the cylinder. The absorbed energy is (c.f. (4.14))

\[
dE_{\text{abs}} = \alpha_v I_v \, dr \, dA \, d\nu \, d\omega \, dt,
\]

where \(\alpha_v\) is the opacity of the medium at frequency \(v\). Let the amount of energy emitted per hertz at frequency \(v\) into unit solid angle from unit volume and per unit time be \(j_v\) (\([j_v] = \text{W m}^{-3}\text{ Hz}^{-1}\text{ sterad}^{-1}\)). This is called the emission coefficient of the medium. The energy emitted into solid angle \(d\omega\) from the cylinder is then

\[
dE_{\text{em}} = j_v \, dr \, dA \, d\nu \, d\omega \, dt.
\]

The equation

\[
dE = -dE_{\text{abs}} + dE_{\text{em}}
\]

gives then

\[
dI_v = -\alpha_v I_v \, dr + j_v \, dr
\]

or

\[
\frac{dI_v}{\alpha_v \, dr} = -I_v + \frac{j_v}{\alpha_v}.
\]

We shall denote the ratio of the emission coefficient \(j_v\) to the absorption coefficient or opacity \(\alpha_v\) by \(S_v\):

\[
S_v = \frac{j_v}{\alpha_v}.
\]

\(S_v\) is called the source function. Because \(\alpha_v \, dr = d\tau_v\), where \(\tau_v\) is the optical thickness at frequency \(v\), (5.37) can be written as

\[
\frac{dI_v}{d\tau_v} = -I_v + S_v.
\]

Equation (5.39) is the basic equation of radiative transfer. Without solving the equation, we see that if \(I_v < S_v\), then \(dI_v/d\tau_v > 0\), and the intensity tends to increase in the direction of propagation. And, if \(I_v > S_v\), then \(dI_v/d\tau_v < 0\), and \(I_v\) will decrease. In an equilibrium the emitted and absorbed energies are equal, in which case we find from (5.35) and (5.36)

\[
I_v = j_v/\alpha_v = S_v.
\]
Substituting this into (5.39), we see that \(dI_\nu/d\tau_\nu = 0\). In thermodynamic equilibrium the radiation of the medium is blackbody radiation, and the source function is given by Planck’s law:

\[
S_\nu = B_\nu(T) = \frac{2\hbar^3 c^2}{e^{\hbar\nu/(kT)} - 1}.
\]

Even if the system is not in thermodynamic equilibrium, it may be possible to find an excitation temperature \(T_{\text{exc}}\) such that \(B_\nu(T_{\text{exc}}) = S_\nu\). This temperature may depend on frequency.

A formal solution of (5.39) is

\[
I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - t)} S_\nu(t) \, dt.
\] (5.41)

Here \(I_\nu(0)\) is the intensity of the background radiation, coming through the medium (e.g. an interstellar cloud) and decaying exponentially in the medium. The second term gives the emission in the medium. The solution is only formal, since in general, the source function \(S_\nu\) is unknown and must be solved simultaneously with the intensity. If \(S_\nu(\tau_\nu)\) is constant in the cloud and the background radiation is ignored, we get

\[
I_\nu(\tau_\nu) = S_\nu \int_0^{\tau_\nu} e^{-(\tau_\nu - t)} \, dt = S_\nu (1 - e^{-\tau_\nu}).
\] (5.42)

If the cloud is optically thick (\(\tau_\nu \gg 1\)), we have

\[
I_\nu = S_\nu,
\] (5.43)

i.e. the intensity equals the source function, and the emission and absorption processes are in equilibrium.

An important field of application of the theory of radiative transfer is in the study of planetary and stellar atmospheres. In this case, to a good approximation, the properties of the medium only vary in one direction, say along the \(z\) axis. The intensity will then depend only on \(z\) and \(\theta\), where \(\theta\) is the angle between the \(z\) axis and the direction of propagation of the radiation.

In applications to atmospheres it is customary to define the optical depth \(\tau_\nu\) in the vertical direction as

\[
d\tau_\nu = -\alpha_\nu \, dz.
\]

Conventionally \(z\) increases upwards and the optical depth inwards in the atmosphere. The vertical line element \(dz\) is related to that along the light ray, \(dr\), according to

\[
dz = dr \cos \theta.
\]

With these notational conventions, (5.39) now yields

\[
\cos \theta \frac{dI_\nu(z, \theta)}{d\tau_\nu} = I_\nu - S_\nu.
\] (5.44)

This is the form of the equation of radiative transfer usually encountered in the study of stellar and planetary atmospheres.

A formal expression for the intensity emerging from an atmosphere can be obtained by integrating (5.44) from \(\tau_\nu = \infty\) (we assume that the bottom of the atmosphere is at infinite optical depth) to \(\tau_\nu = 0\) (corresponding to the top of the atmosphere). This yields

\[
I_\nu(0, \theta) = \int_0^{\infty} S_\nu e^{-\tau_\nu \sec \theta} \sec \theta \, d\tau_\nu.
\] (5.45)

This expression will be used later in Chap. 8 on the interpretation of stellar spectra.

## 5.11 Examples

**Example 5.1** Find the wavelength of the photon emitted in the transition of a hydrogen atom from \(n_2 = 110\) to \(n_1 = 109\).

Equation (5.8) gives

\[
\frac{1}{\lambda} = R \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right)
\]

\[
= 1.097 \times 10^7 \text{ m}^{-1} \left( \frac{1}{109^2} - \frac{1}{110^2} \right)
\]

\[
= 16.71 \text{ m}^{-1},
\]

whence

\[
\lambda = 0.060 \text{ m}.
\]

This is in the radio band. Such radiation was observed for the first time in 1965 by an NRAO radio telescope.

**Example 5.2** The effective temperature of a star is 12,000 K and the absolute bolometric magnitude 0.0. Find the radius of the star, when the effective temperature of the Sun is 5000 K and the absolute bolometric magnitude 4.7.
We can apply (5.21):

\[ M_{\text{bol}} - M_{\text{bol, } \odot} = -5 \log \frac{R}{R_{\odot}} - 10 \log \frac{T}{T_{\odot}} \]

\[ \Rightarrow \frac{R}{R_{\odot}} = \left( \frac{T_{\odot}}{T} \right)^{2} 10^{-0.2(-M_{\text{bol, } \odot})} \]

\[ = \left( \frac{5800}{12000} \right)^{2} 10^{-0.2(0.0 - 4.7)} \]

\[ = 2.0. \]

Thus the radius is twice the Solar radius.

**Example 5.3** Derive the Wien displacement laws.

Let us denote \( x = h\nu/(\lambda kT) \). Planck’s law then becomes

\[ B_{\nu}(T) = \frac{2\kappa^{3}T^{5}}{\hbar^{2}c^{3}} \frac{x^{5}}{e^{x} - 1}. \]

For a given temperature, the first factor is constant. Thus, it is sufficient to find the maximum of the function

\[ f(x) = \frac{x^{5}}{e^{x} - 1}. \]

First we must evaluate the derivative of \( f \):

\[ f'(x) = \frac{5x^{4}(e^{x} - 1) - x^{5}e^{x}}{(e^{x} - 1)^{2}} \]

\[ = \frac{x^{4}e^{x}}{(e^{x} - 1)^{2}} (5 - 5e^{-x} - x). \]

By definition, \( x \) is always strictly positive. Hence \( f'(x) \) can be zero only if the factor \( 5 - 5e^{-x} - x \) is zero. This equation cannot be solved analytically. Instead we write the equation as \( x = 5 - 5e^{-x} \) and solve it by iteration:

\[ x_{0} = 5, \quad \text{(this is just a guess)} \]

\[ x_{1} = 5 - 5e^{-x_{0}} = 4.96631, \]

\[ \vdots \]

\[ x_{5} = 4.96511. \]

Thus the result is \( x = 4.965 \). The Wien displacement law is then

\[ \lambda_{\text{max}}T = \frac{hc}{k} = b = 2.898 \times 10^{-3} \text{ K m}. \]

In terms of frequency Planck’s law is

\[ B_{\nu}(T) = \frac{2\kappa^{3}h^{3}}{c^{2}} \frac{1}{e^{h\nu/(kT)} - 1}. \]

Substituting \( x = h\nu/(kT) \) we get

\[ B_{\nu}(T) = \frac{2\kappa^{3}h^{3}}{c^{2}} \frac{x^{3}}{e^{x} - 1}. \]

Now we study the function \( f(x) = x^{3}/(e^{x} - 1) \):

\[ f'(x) = \frac{3x^{2}(e^{x} - 1) - x^{3}e^{x}}{(e^{x} - 1)^{2}} \]

\[ = \frac{x^{2}e^{x}}{(e^{x} - 1)^{2}} (3 - 3e^{-x} - x). \]

This vanishes, when \( 3 - 3e^{-x} - x = 0 \). The solution of this equation is \( x = 2.821 \). Hence

\[ \frac{cT}{v_{\text{max}}} = \frac{hc}{k} = b' = 5.100 \times 10^{-3} \text{ K m} \]

or

\[ \frac{T}{v_{\text{max}}} = 1.701 \times 10^{-11} \text{ K s}. \]

Note that the wavelength corresponding to \( v_{\text{max}} \) is different from \( \lambda_{\text{max}} \). The reason is that we have used two different forms of Planck’s function, one giving the intensity per unit wavelength, the other per unit frequency.

**Example 5.4** a) Find the fraction of radiation that a blackbody emits in the range \([\lambda_{1}, \lambda_{2}]\), where \( \lambda_{1} \) and \( \lambda_{2} \gg \lambda_{\text{max}} \). b) How much energy does a 100 W incandescent light bulb radiate in the radio wavelengths, \( \lambda \geq 1 \text{ cm} \)? Assume the temperature is 2500 K.

Since the wavelengths are much longer than \( \lambda_{\text{max}} \) we can use the Rayleigh–Jeans approximation \( B_{\nu}(T) \approx 2\kappa T/\lambda^{4} \). Then

\[ B' = \int_{\lambda_{1}}^{\lambda_{2}} B_{\nu}(T) d\lambda \approx 2\kappa T \int_{\lambda_{1}}^{\lambda_{2}} \frac{d\lambda}{\lambda^{4}} \]

\[ = \frac{2\kappa T}{3} \left( \frac{1}{\lambda_{1}^{3}} - \frac{1}{\lambda_{2}^{3}} \right), \]

and hence

\[ \frac{B'}{B_{\text{tot}}} = \frac{5\kappa^{3}h^{3}}{2\pi^{4}T^{3}} \left( \frac{1}{\lambda_{1}^{3}} - \frac{1}{\lambda_{2}^{3}} \right). \]

Now the temperature is \( T = 2500 \text{ K} \) and the wavelength range \([0.01 \text{ m}, \infty)\), and so

\[ B' = 100 \text{ W} \times 1.529 \times 10^{-7} \frac{1}{2500^{3} \times 0.01^{3}} \]

\[ = 9.8 \times 10^{-10} \text{ W}. \]
It is quite difficult to listen to the radio emission of a light bulb with an ordinary radio receiver.

**Example 5.5  Determination of Effective Temperature**

The observed flux density of Arcturus is

\[ F' = 4.5 \times 10^{-8} \text{ W m}^{-2} \text{ sr}^{-1}. \]

Interferometric measurements give an angular diameter of \( \alpha = 0.020' \). Thus, \( \alpha/2 = 4.85 \times 10^{-8} \text{ radians.} \) From (5.26) we get

\[ T_c = \left( \frac{4.5 \times 10^{-8}}{(4.85 \times 10^{-8})^2 \times 5.669 \times 10^{-8}} \right)^{1/4} \text{ K} \]

\[ = 4300 \text{ K}. \]

**Example 5.6**  Flux densities at the wavelengths 440 nm and 550 nm are 1.30 and 1.00 W m\(^{-2}\) m\(^{-1}\), respectively. Find the colour temperature.

If the flux densities at the wavelengths \( \lambda_1 \) and \( \lambda_2 \) are \( F_1 \) and \( F_2 \), respectively, the colour temperature can be solved from the equation

\[ \frac{F_1}{F_2} = \frac{B_{\lambda_1}(T_c)}{B_{\lambda_2}(T_c)} = \left( \frac{\lambda_2}{\lambda_1} \right)^5 \frac{e^{hc/(\lambda_2 k T_c)} - 1}{e^{hc/(\lambda_1 k T_c)} - 1}. \]

If we denote

\[ A = \frac{F_1}{F_2} \left( \frac{\lambda_1}{\lambda_2} \right)^5, \]

\[ B_1 = \frac{hc}{\lambda_1 k}, \]

\[ B_2 = \frac{hc}{\lambda_2 k}, \]

we get the equation

\[ A = \frac{e^{B_2/T_c} - 1}{e^{B_1/T_c} - 1} \]

for the colour temperature \( T_c \). This equation must be solved numerically.

In our example the constants have the following values:

\[ A = \frac{1.00}{1.30} \left( \frac{550}{440} \right)^5 = 2.348, \]

\[ B_1 = 32,700 \text{ K}, \quad B_2 = 26,160 \text{ K}. \]

By substituting different values for \( T_c \), we find that \( T_c = 7545 \text{ K} \) satisfies our equation.

**5.12 Exercises**

**Exercise 5.1**  Show that in the Wien approximation the relative error of \( B_\lambda \) is

\[ \frac{\Delta B_\lambda}{B_\lambda} = -e^{hc/(\lambda k T)} \]

**Exercise 5.2**  If the transition of the hydrogen atom \( n + 1 \rightarrow n \) were to correspond to the wavelength 21.05 cm, what would the quantum number \( n \) be? The interstellar medium emits strong radiation at this wavelength. Can this radiation be due to such transitions?

**Exercise 5.3**  The space is filled with background radiation, remnant of the early age of the universe. Currently the distribution of this radiation is similar to the radiation of a blackbody at the temperature of 2.7 K. What is \( \lambda_{\text{max}} \) corresponding to this radiation? What is its total intensity? Compare the intensity of the background radiation to the intensity of the Sun at the visual wavelengths.

**Exercise 5.4**  The temperature of a red giant is \( T = 2500 \text{ K} \) and radius 100 times the solar radius.

a) Find the total luminosity of the star, and the luminosity in the visual band 400 nm \( \leq \lambda \leq 700 \text{ nm.} \)

b) Compare the star with a 100 W lamp that radiates 5% of its energy in the visual band. What is the distance of the lamp if it looks as bright as the star?

**Exercise 5.5**  The effective temperature of Sirius is 10,000 K, apparent visual magnitude -1.5, distance 2.67 kpc and bolometric correction 0.5. What is the radius of Sirius?

**Exercise 5.6**  The observed flux density of the Sun at \( \lambda = 300 \text{ nm} \) is 0.59 W m\(^{-2}\) nm\(^{-1}\). Find the brightness temperature of the Sun at this wavelength.

**Exercise 5.7**  The colour temperature can be determined from two magnitudes corresponding to two
different wavelengths. Show that

\[ T_c = \frac{7000 \text{ K}}{(B - V) + 0.47}. \]

The wavelengths of the \( B \) and \( V \) bands are 440 nm and 548 nm, respectively, and we assume that \( B = V \) for stars of the spectral class A0, the colour temperature of which is about 15,000 K.

Exercise 5.8 The kinetic temperature of the plasma in the solar corona can reach \( 10^6 \) K. Find the average speed of the electrons in such a plasma.