Mohr-Coulomb based models

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Content

1. Mohr-Coulomb failure criterion
2. Perfect plasticity
3. Mohr-Coulomb model
4. Hardening soil model (HS, Plaxis)
5. Hardening Mohr-Coulomb (HMC, Optum)
To learn

1. Mohr-Coulomb model (review & revision from Advanced Soil Mechanics Course)
2. Hardening Soil Model (Plaxis)
3. Hardening Mohr-Coulomb (Optum)

In red – new information!
Application of those models

Main use:
- finding out limit state / strength of soil
- computing failure load
- assessing forces on construction due to soil failure

Secondary use:
- may be used for assessment of deformation of granular materials, especially when deformations are due to soil failure (shearing etc)

Should not be used:
- calculation of settlements in cohesive materials
- use in soft soil with care...
- special care needed in undrained conditions / short term stability of soft soils
Application of those models

Doherty & Muir Wood (2013)
Mohr Coulomb Model
Elasto-plasticity: Mohr-Coulomb

\[ F = (\sigma_1' - \sigma_3') - (\sigma_1' + \sigma_3') \sin \phi' - 2c' \cos \phi' = 0 \]
\[ \tau_f = c' + \sigma_n' \tan \phi' \]

\[ F = (\sigma_1' - \sigma_3') - (\sigma_1' + \sigma_3') \sin \phi' - 2c' \cos \phi' = 0 \]
Mohr-Coulomb in Principal Stress Space

- Mohr – Coulomb failure surface is a irregular hexagon in the principal stress space

\[
\sigma'_1 = \sigma'_2 = \sigma'_3
\]
Mohr-Coulomb in Principal Stress Space

- It has corners that may sometimes create problems in computations.
Flow Rule for Mohr-Coulomb

For Mohr-Coulomb flow rule is defined through the ‘dilatancy angle’ of the soil.

\[ G(\sigma') = \tau - \sigma'_n \tan \psi' - \text{const.} = 0 \]

where \( \psi' \) is the dilatancy angle and \( \psi' \leq \phi' \).
Limitations of MC model (1)
Limitations of MC model (2)

possible **overestimation** of the safety factor in slope stability calculations!
Warning for dense sands

Post-peak softening (dense sand)

Do not rely on high $\varphi$!

Post-peak softening

$\sim$ progressive failure

Be conservative: $\varphi \leq 35^\circ$

Dilatancy: $\psi \approx \varphi - 30^\circ$ (\(\psi \geq 0\))
Tips for fine-grained soils

Cohesive soils (clay)

experimental observation: $c$ is proportional to $w$
($c$ is proportional to OCR)

$w \approx w_l \ldots c$ is very small
$w \approx w_p \ldots c$ is significant

Small tensile strength can be considered in special cases
(needed for rock and concrete!)
• Mohr-Coulomb failure criterion is well proven through experiments for most geomaterials, but data for clays is still contradictory!

• An associated flow rule implies continuous dilation at a constant rate upon shearing; this is unrealistic and leads to negative pore pressures in undrained conditions. In an non-associated flow rule with $\psi' < \phi'$, the rate of dilation is less, but it is still constant. If $\psi' = 0$ then the rate of dilation is zero.

• Care must be taken in applying the model for undrained loading.
Drawbacks of MC

- Soils on shearing exhibit **variable volume change characteristics** depending on **pre-consolidation pressure** which cannot be accounted for with MC.

- In **soft soils** **volumetric plastic strains** on shearing are **compressive** (negative dilation) whilst **Mohr-Coulomb model** will predict continuous dilation.
To summarize the limitations of MC are:

• bi-linearity (const. E’)

• unlimited dilation

• isotropy

• elastic response far from the limit state

• ...

→ more advanced approximation of soil behavior:
   Hardening Soil Model (sand), Soft Soil Model (clay), critical state soil models...
Mohr Coulomb in Plaxis & Optum
Elasto-plasticity: Mohr-Coulomb in Plaxis

\[ |\sigma_1 - \sigma_3| \]

\[ E_0 \]

\[ E_{50} \]

strain - \[ \varepsilon_1 \]
Elasto-plasticity: Mohr-Coulomb in Plaxis

\[ |\sigma_1 - \sigma_3| \]

\[ E_0 \quad E_{50} \]

\[ \text{strain} - \varepsilon_1 \]
Elasto-plasticity: Mohr-Coulomb in Plaxis

*Increase of stiffness* \((E_{inc})\)

In real soils, the stiffness depends significantly on the stress level, which means that the stiffness generally increases with depth. When using the Mohr-Coulomb model, the stiffness is a constant value. In order to account for the increase of the stiffness with depth the \(E_{inc}\)-value may be used, which is the increase of the Young’s modulus per unit of depth (expressed in the unit of stress per unit depth). At the level given by the \(y_{ref}\) parameter, and above, the stiffness is equal to the reference Young’s modulus, \(E_{ref}\), as entered in the *Parameters* tab sheet. Below, the stiffness is given by:

\[
E(y) = E_{ref} + (y_{ref} - y)E_{inc} \quad (y < y_{ref})
\]  

(3.14)

**Note:** \(y\) decreases with depth!
Elasto-plasticity: Mohr-Coulomb in Plaxis

Similarly – cohesion:

\[ c(y) = c_{\text{ref}} + (y_{\text{ref}} - y)c_{\text{inc}} \quad (y < y_{\text{ref}}) \quad (3.15a) \]

\[ s_u(y) = s_{u,\text{ref}} + (y_{\text{ref}} - y)s_{u,\text{inc}} \quad (y < y_{\text{ref}}) \quad (3.15b) \]

Note: \( y \) decreases with depth!

Tension cut-off: means tension is not allowed... Default tensile strength is zero
Elasto-plasticity: Mohr-Coulomb in Optum

Non-associated flow with dilation cap:
Elasto-plasticity: Mohr-Coulomb in Optum

Tension cut-off:
Elasto-plasticity: Mohr-Coulomb in Optum

Hardening:

Fissures can also be modelled
Hardening Soil Model
## HS input parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{50}^{\text{ref}}$</td>
<td>Stiffness modulus for primary loading in drained triaxial test</td>
</tr>
<tr>
<td>$E_{\text{oed}}^{\text{ref}}$</td>
<td>Stiffness modulus for primary loading in oedometer test</td>
</tr>
<tr>
<td>$E_{ur}^{\text{ref}}$</td>
<td>Stiffness modulus for unloading/reloading in drained triaxial test</td>
</tr>
<tr>
<td>$m$</td>
<td>Modulus exponent for stress dependency</td>
</tr>
<tr>
<td>$\nu_{ur}$</td>
<td>Poisson’s ratio for loading/unloading</td>
</tr>
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<td>$c'$</td>
<td>Effective cohesion at failure</td>
</tr>
<tr>
<td>$\phi'$</td>
<td>Effective friction angle at failure</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Dilatancy angle at failure</td>
</tr>
<tr>
<td>POP: $(\sigma_p - \sigma_{vo}')$</td>
<td>Initial preconsolidation stress, $\sigma_p$</td>
</tr>
<tr>
<td>OCR: $\sigma_p/\sigma_{vo}'$</td>
<td>Earth pressure coefficients at rest</td>
</tr>
<tr>
<td>$K_{o}^{nc} = 1 - \sin\phi'$</td>
<td>Earth pressure coefficients at rest</td>
</tr>
</tbody>
</table>
Model characteristics:

- Hyperbolic stress-strain relationship in axial compression
- Plastic strain in mobilizing friction (shear hardening)
- Plastic strain in primary compression (volumetric hardening)
- Stress-dependent stiffness according to a power law
- Elastic unloading/reloading compared to virgin loading
- Memory of pre-consolidation stress
- Mohr-Coulomb (MC) failure criterion
- Dilatancy below MC line
- Small strain stiffness (HS-small only)
Hardening soil model (Plaxis)

Representation of total yield contour of the Hardening-Soil model in principal stress space for cohesionless soil.

Hyperbolic stress-strain relation in primary loading for a standard drained triaxial test.

Definition of $E_{oed}^{ref}$ in oedometer test results.
Hardening soil model (Plaxis)

Representation of total yield contour of the Hardening-Soil model in principal stress space for cohesionless soil.
Hardening soil model (Plaxis)

Created and validated for sand only! Is it suitable for clay?

- plastic volumetric strain zero / small when compared to the axial strain

\[ \gamma^p := \varepsilon_1^p - \varepsilon_2^p - \varepsilon_3^p = 2\varepsilon_1^p - \varepsilon_v^p \approx 2\varepsilon_1^p \]

\[ q_f = (c \cot \varphi - \sigma'_3) \frac{2 \sin \varphi}{1 - \sin \varphi} \]

\[ q_a = \frac{q_f}{R_f} \]
Hardening soil model (Plaxis)

See the Schanz et al. (2000) Brinkgreave 1994

The stress strain behaviour for primary loading is highly nonlinear. The parameter $E_{50}$ is the confining stress dependent stiffness modulus for primary loading. $E_{50}$ is used instead of the initial modulus $E_I$ for small strain which, as a tangent modulus, is more difficult to determine experimentally. It is given by the equation:

$$E_{50} = E_{50}^{ref} \left( \frac{\sigma_3 + c \cot \varphi_p}{\sigma^{ref} + c \cot \varphi_p} \right)^m.$$

(3)

$E_{50}^{ref}$ is a reference stiffness modulus corresponding to the reference stress $p^{ref}$. The actual stiffness depends on the minor principal stress, $\sigma'_3$, which is the effective confining pressure in a triaxial test. The amount of stress dependency is given by the power $m$. In order to simulate a logarithmic stress dependency, as observed for soft clays, the power should be taken equal to 1.0. As a secant modulus $E_{50}^{ref}$ is determined from a triaxial stress-strain-curve for a mobilization of 50% of the maximum shear strength $q_f$. 
Hardening Soil Model

• Volumetric hardening is complemented by deviatoric hardening
• Deviatoric hardening surface evolves with deviatoric strains until failure is reached
Hyperbolic stress-strain relationship

Duncan-Chang or hyperbolic model in (tri)axial loading:

For $q < q_f$: \[ \varepsilon_1 = \varepsilon_{50} \frac{q}{q_a - q} \]

where \[ q_f = \frac{2 \sin \varphi}{1 - \sin \varphi} (\sigma'_3 + c \cot \varphi) \]

and \[ q_a = \frac{q_f}{R_f} \geq q_f \]

$R_f$ = ‘failure ratio’ (standard value: 0.9)

The hyperbolic model with Mohr-Coulomb failure criterion constitutes the basis for the HS and HS-small models. In contrast to the Duncan-Chang model, the HS models are elastoplastic models.
Shear hardening

Yield function (cone): \[ f^s = \frac{q_a}{E_{so}} \frac{q}{q_a - q} - \frac{2q}{E_{ur}} - \gamma^{ps} \]

where \( \gamma^{ps} \) is a state parameter that is tracking the opening of the cone:

Evolution law for \( \gamma^{ps} \): \( d\gamma^{ps} = d\lambda^s \) where \( d\lambda^s \) is the plastic multiplier for the cone type yield surface of the model.
Volumetric (density) hardening

Yield function (cap): \( f^c = \frac{\tilde{q}^2}{\alpha^2} - p'^2 - p_p^2 \)

where \( p_p \) is a state parameter that remembers the position of the cap:

Volumetric or vertical strain,

\( \sigma'_c \)

\( \varepsilon_1 \)
Volumetric (density) hardening

Yield function (cap): \( f^c = \frac{q^2}{\alpha^2} - p'^2 - p_p^2 \)

where \( p_p \) is a state parameter that remembers the position of the cap:
Volumetric (density) hardening

Modified Cam Clay

\[ \alpha \]

no control of \( K_0^{NC} \)

Hardening Soil Model

\[ K_0^{NC} \text{ is controlled by } \alpha. \text{ By inputting a realistic value of } K_0^{NC} \text{ as a result } \alpha-\text{value in large and the yield cap is relatively steep.} \]

Volumetric-strain hardening is dominant for NC-clays and very loose sands.

Only for \( m=1 \) we recover Modified-Cam-Clay-like volumetric hardening on cap. Only option when you specify oedometric moduli
Shear hardening

Deviatoric stress
\[ |\sigma_1 - \sigma_3| \]

Mohr-Coulomb failure line

\[ \varepsilon_v^p = \sin \varphi_m \dot{\gamma}^p \]

Mean effective stress

\[ \tilde{q} = \sigma_1 + (\alpha - 1)\sigma_2' - \alpha \sigma_3' \]

with
\[ \alpha = \frac{3 + \sin \varphi}{3 - \sin \varphi} \]

\[ \sin \varphi_m = \frac{\sigma_1' - \sigma_3'}{\sigma_1' + \sigma_3' - 2c \cot \varphi} \]

\( \varphi_{cv} \) is the critical state friction angle. \( \varphi_m \) is the mobilised friction angle.
Shear hardening

Deviatoric stress
\[ |\sigma_1 - \sigma_3| \]

Mohr-Coulomb failure line

Mean effective stress

\[ \dot{\varepsilon}_V^p = \sin \varphi_m \dot{\gamma}^p \]

For \( \sin \varphi_m < \frac{3}{4} \sin \varphi \):
\[ \psi_m = 0 \]
\[ \sin \psi_m = \max \left( \frac{\sin \varphi_m - \sin \varphi_{cv}}{1 - \sin \varphi_m \sin \varphi_{cv}}, 0 \right) \]

For \( \sin \varphi_m \geq \frac{3}{4} \sin \varphi \) and \( \psi > 0 \)
\[ \psi_m = \psi \]
\[ \psi_m = 0 \]

For \( \sin \varphi_m \geq \frac{3}{4} \sin \varphi \) and \( \psi \leq 0 \)
\[ \psi_m = \psi \]
\[ \psi_m = 0 \]

\( \varphi_{cv} \) is the critical state friction angle.
\( \varphi_m \) is the mobilised friction angle.
Cap is closing the MC cone in principal stress space

Therefore: \( \tilde{q} = f(\sigma_1, \sigma_2, \sigma_3, \varphi) \)

Evolution law: \( dp_p = \frac{K_s K_c}{K_s - K_c} \left( \frac{\sigma_3 + a}{p_{ref} + a} \right)^m d\varepsilon^p_v \)

where \( K_s = \frac{E_{ur}^{ref}}{3(1-2\nu)} \) and the bulk stiffness of the cap \( K_c \) is determined from \( E_{oed} \) and \( K_0^{nc} \)}
Stress dependent moduli

- All stiffness moduli are updated according to current stress level.
- Input stiffness are values at reference stress, e.g. $p_a' = 100 \text{ kPa}$.

\[ E_{50} = \frac{E_{50}^{ref}}{\left(\frac{\sigma_3' + a}{p_a' + a}\right)^m} \]
(Secant modulus)

\[ E_{oed} = \frac{E_{oed}^{ref}}{\left(\frac{\sigma_3' / K_0^{\text{nc}} + a}{p_a' + a}\right)^m} \]
(Tangent modulus)
Effect of modulus exponent $m$

Shear strain contours in $p$-$q$-plane for $c' = 0$

Only for $m=1$ we recover Modified-Cam-Clay like volumetric hardening. Only option when you specify oedometric moduli.
Elastic unloading/ reloading

Unloading, reloading by:

\[ E_{ur} = E_{ur}^{ref} \left( \frac{\sigma_3' + a}{p_a' + a} \right)^m \]

\[ \nu_{ur} = \text{low value} \]

Inside cone and cap:

<table>
<thead>
<tr>
<th>Bulk modulus</th>
<th>Shear modulus</th>
<th>Oedometer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_{ur} = \frac{E_{ur}}{3(1-2\nu_{ur})} )</td>
<td>( G_{ur} = \frac{E_{ur}}{2(1+\nu_{ur})} )</td>
<td>( E_{oed} = \frac{E_{ur}(1-\nu_{ur})}{(1-2\nu_{ur})(1+\nu_{ur})} )</td>
</tr>
</tbody>
</table>
Initial conditions for the HS model

Preconsolidation is entered by OCR or POP relative to initial vertical stress and is then converted to $p_p$.

$$\sigma'_c = OCR \cdot \sigma_{y0}'$$

$$\sigma'_c = \sigma_{y0}' + POP$$

Hardening Soil model and Hardening Soil model with small-strain stiffness:

$$p_p^{eq} = \sqrt{(p')^2 + \frac{q^2}{\alpha^2}}$$

(where $\alpha$ is an internal model parameter)
Initial stresses:

Initial conditions for the HS model

Output:
\[ 'OCR' = OCR_{iso} = \frac{p_c}{p_{eq}} \]
\[ p_{eq} = \sqrt{(p')^2 + q^2 / \alpha^2} \]
Four stiffness zones

- $E_{50}$ & $E_{oed}$: combined hardening
- $E_{50}$: shear hardening
- $E_{ur}$: cap hardening
Dilatancy formulation:
Rowe (1962) modified

\[
\sin \varphi_{cv} = \frac{\sin \varphi' - \sin \psi}{1 - \sin \varphi' \sin \psi}
\]

\[
\sin \varphi_m = \frac{\sigma'_1 - \sigma'_3}{\sigma'_1 + \sigma'_3 - 2c' \cot \varphi'}
\]

\[
\sin \psi_m = \frac{\sin \varphi_m - \sin \varphi_{cv}}{1 - \sin \varphi_m \sin \varphi_{cv}}
\]
Dilatancy from cone (most important part):

Nonassociated cone flow:
Increasing dilatancy $\psi_m$ from zero at $\phi_{cv}$ to input value $\psi_{input}$ on MC line (Rowe).

Dilatancy set to zero for $\sin \phi_m < \frac{3}{4} \cdot \sin \phi$, see Material Model Manual.

Contractancy from cap during shearing (less dominating):

Associated cap flow:
Increasing contractancy from zero to a maximum value at MC line, but only when cap moves!
## HS input parameters

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<tr>
<td>$\phi'$</td>
<td>Effective friction angle at failure</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Dilatancy angle at failure</td>
</tr>
</tbody>
</table>

**POP:** $(\sigma_p - \sigma_{vo'})$  
**OCR:** $\sigma_p/\sigma_{vo'}$  
$K_n^c = 1 - \sin\phi'$  
Earth pressure coefficients at rest
Example: Settlement of strip footing

Monotonic loading
Example: Vertical displacement of a retaining wall

Wall pulled up

Typical vertical displacements behind a retaining wall
(sheet pile wall in clay)

- Mohr Coulomb
- Hardening Soil

vertical displacements [mm]
distance from wall [m]
Limitation: No small strain stiffness

Shear modulus $G/G_0 [-]$

- Very small strains
- Small strains
- Larger strains

Shear strain $\gamma [-]$

10^6 10^5 10^4 10^3 10^2 10^1

Dynamic methods

Retaining walls
Foundations
Tunnels
Conventional soil testing

Local gauges
Limitation: Unloading/reloading stiffness is the same

\[ q = \sigma_1 - \sigma_3 \text{ [kPa]} \]

- Truly elastic behaviour on for very small loops
- At small strains stiffness increases
- Hysteresis increases with increasing strains
Unloading – reloading in standard drained tests on sand

\[ \sigma_3' = 200 \, \text{kPa}, \, e_0 = 0.81 \]

\[ \sigma_3' = 400 \, \text{kPa}, \, e_0 = 0.56 \]

loose: \[ E_{ur} = (3 - 5) \cdot E_{50} \]
dense: \[ E_{ur} = (2 - 3) \cdot E_{50} \]

Default in Plaxis code: \[ E_{ur} = 3 \cdot E_{50} \]
Limitation: Non-monotonic loading in heavily OC clays

Need to use artificially low POP/OCR value to trigger plasticity within ‘yield surface’ in order to represent different stiffness for loading/unloading for non-monotonic loading.

However, the stress path may still be wrong when approaching to failure.
Recommended procedure for application

MC model: for simple estimates and for safety factors (stability)
Advanced soil models: for more accurate deformation predictions

Hardening Soil model:
• Use previous experience from lab, field and case records for strength and stiffness ($E_{50}$ etc)
• Simulate an oedometer or/and a triaxial test to calibrate your soil parameter set
• Run your design problem
• Check the results and compare to hand calculations or other estimates / experience
Comparison HS models and MC model

Isotropic compression test:

Illustration by Brinkgreve, R.B.J.
Comparison HS models and MC model

Drained triaxial test:

Illustration by Brinkgreve, R.B.J.
Comparison HS models and MC model

Drained triaxial test:
Comparison HS models and MC model

One-dimensional compression test (oedometer):
## Plaxis: which model in which situation?

<table>
<thead>
<tr>
<th>Model</th>
<th>Soft soil (NC-clay, peat)</th>
<th>Hard soils (OC-clay, sand, gravel)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Primary load.</strong> (surcharge)</td>
<td>Soft Soil (Crp), HS / HSSmall</td>
<td>HS / HSSmall</td>
</tr>
<tr>
<td><strong>Unloading + deviatoric load</strong> (excavation)</td>
<td>HS / HSSmall</td>
<td>HS / HSSmall</td>
</tr>
<tr>
<td><strong>Deviatoric loading</strong></td>
<td>Soft Soil (Crp), HS / HSSmall</td>
<td>HS / HSSmall</td>
</tr>
<tr>
<td><strong>Secondary compression</strong></td>
<td>Soft Soil Creep</td>
<td>n/a</td>
</tr>
</tbody>
</table>

**NGI –ADP model** / only available as VIP package...
Hardening Mohr-Coulomb Model
Hardening Mohr Coulomb (Optum)

Based on the Doherty Muir-Wood (2013)

- pressure dependence of stiffness moduli
- models compaction, dilation & critical state
- well suited for granular materials

Figure 16.6: HMC fits to drained triaxial test data for loose, medium and dense Erksak sand (test data from Yu 2006).
Hardening Mohr Coulomb (Optum)

Based on the Doherty Muir-Wood (2013)

- pressure dependence of stiffness moduli
- models compaction, dilation & critical state
- well suited for granular materials

Figure 16.7: HMC fits to drained triaxial test data for dense Lund sand (test data from Ahadi and Krenk 2000). The model is fitted for a confining pressure of 160 kPa while the response at higher and lower pressures are predictions.
Hardening Mohr Coulomb (Optum)

The yield surface

The EMC model uses a Mohr–Coulomb yield surface given by

\[ F(\sigma, a, \phi_y) = (p' - a) \sin \phi_y + JK(\theta, \phi_y) = 0 \] (4)

where

\[ c' = a \tan \phi_y \]

\[ K(\theta, \phi_y) = \cos \theta - \frac{\sin \theta \sin \phi_y}{\sqrt{3}} \] (5)

Lode’s angle dependent (see Mohr-Coulomb formulated in p-q space)
Hardening Mohr Coulomb (Optum)

\[ \frac{\partial \phi_y}{\partial \varepsilon^p_j} = \frac{(\phi_p - \phi_y)^2}{\beta \phi_p} \]

Plastic potentials
\[ Q(\sigma, a, \phi_{cv}) = 0 \]

Yield loci
\[ F(\sigma, a, \phi_y) = 0 \text{ with } \phi_y > \phi_{cv} \]

\[ F(\sigma, a, \phi_y) = 0, \text{ with } \phi_y = \phi_{cv} \]

\[ F(\sigma, a, \phi_y) = 0, \text{ with } \phi_y < \phi_{cv} \]

Taylor dilatancy:
\[ \frac{\delta \varepsilon^p}{\partial \varepsilon^p_j} = -\frac{J}{p' - a} - \frac{\sin \phi_{cv}}{K(\theta, \phi_{cv})} \]

Deviatoric plastic strain
\[ \varepsilon^p_j = \frac{\beta \phi_y}{\phi_p - \phi_y} - \frac{\beta \phi_{y0}}{\phi_p - \phi_{y0}} \]

Max. Mobilised friction angle

Hardening rule...
# Hardening Mohr Coulomb (Optum)

<table>
<thead>
<tr>
<th>Parameter</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( G )</td>
<td>Elastic shear modulus: kPa</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Poisson’s ratio</td>
</tr>
<tr>
<td>( \phi_p )</td>
<td>Peak friction angle: degrees</td>
</tr>
<tr>
<td>( \phi_{cv} )</td>
<td>Constant-volume friction angle: degrees</td>
</tr>
<tr>
<td>( a )</td>
<td>Attraction: kPa</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Hardening parameter</td>
</tr>
</tbody>
</table>

+ preconsolidation stress \( p_0 \)
+ maximum previously mobilised friction angle \( \phi_y^0 \)
Hardening Mohr Coulomb (Optum)

Taylor dilatancy:

\[ \frac{\dot{\varepsilon}_p}{\dot{\varepsilon}_v} = M - N - \frac{q}{p} \]

\[ \frac{\dot{\varepsilon}_p}{\dot{\varepsilon}_s} = \frac{6 \sin \phi}{3 - \sin \phi} - \frac{6 \sin \psi}{3 - \sin \psi} - \frac{q}{p} \]

Figure 13.3: Hardening, compaction and dilation in the EMC model.
Hardening Mohr Coulomb (Optum)

Triaxial compression:

\[
N = \frac{3 \sin \psi}{\sqrt{3 \cos \theta + \sin \theta \sin \psi}}
\]

\[
\frac{\dot{\varepsilon}_V}{\dot{\varepsilon}_S} = \frac{6 \sin \phi}{3 - \sin \phi} - \frac{6 \sin \psi}{3 - \sin \psi} - \frac{q}{p}
\]

If constant dilation, we recover Mohr-Coulomb:

\[
\frac{\dot{\varepsilon}_V}{\dot{\varepsilon}_S} = -\frac{6 \sin \psi}{3 - \sin \psi}
\]

Figure 13.3: Hardening, compaction and dilation in the EMC model.
Figure 13.7: Response of EMC under undrained conditions for three materials with $\psi = -5^\circ$, $0^\circ$, and $5^\circ$. 
Hardening Mohr Coulomb (Optum)

Influence of elastic parameters...

Figure 16.8: Model response (Medium sand in Table 16.1) as function of $E_{ur,ref}$ and $E_{50,ref}$. 
Hardening Mohr Coulomb (Optum)

Note that in Optum:
- elastic moduli are pressure dependent, not defined as in the Doherty & Muir Wood (2013)

\[
E_{ur} = E_{ur,\text{ref}} \Pi(\sigma_3) \\
G_{ur} = G_{ur,\text{ref}} \Pi(\sigma_3) \\
K_{ur} = K_{ur,\text{ref}} \Pi(\sigma_3) \\
E_{50} = E_{50,\text{ref}} \Pi(\sigma_3) \\
G_{50} = G_{50,\text{ref}} \Pi(\sigma_3) \\
\Pi(\sigma_3) = \left( \frac{\sigma_3 + c/\tan \phi}{\rho_{\text{ref}} + c/\tan \phi} \right)^m
\]
Hardening Mohr Coulomb (Optum)

Note that in Optum:
- hardening parameter beta changes, so it reproduces specified secant elastic moduli (E_{50} or G_{50})

\[
\beta = \begin{cases} 
\frac{3}{2}(p + c/\tan \phi) \frac{9 - M}{9 - (M - N)M \ln 2 - 3N} \frac{1 - E_{50}/E_{ur}}{E_{50}}, & \text{for Flow Rule = Taylor} \\
\frac{3}{2}(p + c/\tan \phi) \frac{9 - M}{9 - (3 - N)M \ln 2 - 3N} \frac{1 - E_{50}/E_{ur}}{E_{50}}, & \text{for Flow Rule = Constant Dilation}
\end{cases}
\]

while the expression consistent with a user specified G_{50,ref} is:

\[
\beta = \frac{1}{6}(p + c/\tan \phi) \frac{3 - M}{3 - M \ln 2} \frac{1 - G_{50}/G_{ur}}{G_{50}}, \quad \text{independent of Flow Rule,}
\]

(16.21)
Hardening Mohr Coulomb (Optum)

- if m=1, no pressure dependency, beta is constant and Doherty & Muir Wood (2013) model is recovered exactly.

\[
\begin{align*}
E_{ur} &= E_{ur,\text{ref}} \Pi(\sigma_3) \\
G_{ur} &= G_{ur,\text{ref}} \Pi(\sigma_3) \\
K_{ur} &= K_{ur,\text{ref}} \Pi(\sigma_3) \\
E_{50} &= E_{50,\text{ref}} \Pi(\sigma_3) \\
G_{50} &= G_{50,\text{ref}} \Pi(\sigma_3) \\
\Pi(\sigma_3) &= \left( \frac{\sigma_3 + c/\tan \phi}{\rho_{\text{ref}} + c/\tan \phi} \right)^m
\end{align*}
\]
Hardening soil model with small strain stiffness HS Small
Hardening soil model – small (Plaxis)

Representation of total yield contour of the Hardening-Soil model in principal stress space for cohesionless soil.
Introduction

The aim is to discuss the extension of the HS model to account for small strain stiffness.

The topics to be covered include:

- Small strain stiffness
- HSsmall model formulation
- Limitations of HSsmall model
Parameters of the HS(small) model

Parameters:

- $E_{50}^{\text{ref}}$: Secant stiffness from triaxial test at reference pressure
- $E_{\text{oed}}^{\text{ref}}$: Tangent stiffness from oedometer test at $p^{\text{ref}}$ Reference
- $E_{ur}^{\text{ref}}$: Stiffness in unloading / reloading
- $G_0^{\text{ref}}$: Reference shear stiffness at small strains (HSsmall only)
- $\gamma_{0.7}$: Shear strain at which $G$ has reduced to 70% (HSsmall only)
- $m$: Rate of stress dependency in stiffness behaviour
- $p^{\text{ref}}$: Reference pressure (100 kPa)
- $\nu_{ur}$: Poisson’s ratio in unloading / reloading
- $c'$: Cohesion
- $\varphi'$: Friction angle
- $\psi$: Dilatancy angle
- $R_f$: Failure ratio $q_f/q_a$ like in Duncan-Chang model (0.9)
- $K_0^{nc}$: Stress ratio $\sigma_{xx}'/\sigma_{yy}'$ in 1D primary compression
Soil Stiffness at Small Strains

- **INITIAL STATE OF STRESS**
- **LINEAR ELASTIC;** \( \frac{\epsilon_1^p}{\epsilon_1} = 0 \)
- **NON-LINEAR ELASTIC;** \( \frac{\epsilon_1^p}{\epsilon_1} \neq 0 \)
- **ELASTO-PLASTIC;** \( 0 < \frac{\epsilon_1^p}{\epsilon_1} < 1 \)
- **ON STATE BOUNDARY SURFACE;** \( \epsilon_1 = \epsilon_1^p \)

Characteristic segments of stress-strain curve. Elastic regions highly kinematic time and stress history dependent.
Soil Stiffness at Small Strains

\[ q = \sigma_1 - \sigma_3 \text{[kPa]} \]

\[ \varepsilon_1 \text{[\%]} \]
Why is small-strain stiffness important?

Not accounting for small strain stiffness in geotechnical analyses may potentially result in overestimating foundation settlements and retaining wall deflections consequently underestimating stresses. The gradient of settlement troughs behind retaining walls or above tunnels may be underestimated. Piles or anchors within the working load range may show a too soft response.

Analysis results are also less sensitive to the choice of proper boundary conditions. Large meshes no longer cause extensive accumulation of displacements, because marginally strained mesh parts are very stiff.
Experimental evidence and data for small-strain stiffness

True elastic stiffness was first observed in soil dynamics. Back then, the apparent higher soil stiffness in dynamic loading applications was attributed to the nature of loading, e.g. inertia forces and strain rate effects. Nowadays, static small-strain measurements are available as well. These show only little differences to dynamic measurements. Still, the term dynamic soil stiffness is sometimes used when true elastic or small-strain stiffness is meant.

\[ v_p = \sqrt{\frac{\lambda + 2G}{\rho}} \]
\[ v_s = \sqrt{\frac{G}{\rho}} \]
\[ \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \]
Soil Stiffness at Small Strains

Shear modulus \( G/G_0 [-] \)

- Retaining walls
- Foundations
- Tunnels
- Conventional soil testing

Very small strains
Small strains
Larger strains

Shear strain \( \gamma [-] \)

Dynamic methods
Local gauges

Shear modulus vs. Shear strain graph with different applications and strain ranges.
Experimental evidence and data for small-strain stiffness

Empirical relationships between $G_0$ or $E_0 = 2(1+\nu_{ur})G_0$ and void ratio $e$:

Most relationships are of the form:

$$G_0 = G_0^{\text{ref}} \left( \frac{p'}{p_{\text{ref}}} \right)^m$$

with $G_0^{\text{ref}} = \text{function}(e) \cdot \text{OCR}^k$

A simple relationship for soils with $w_1 < 50\%$ is proposed by Biarez & Hicher*:

$$E_0 = E_0^{\text{ref}} \sqrt{\frac{p'}{p_{\text{ref}}}}$$

with $E_0^{\text{ref}} = \frac{140 \text{ MPa}}{e}$

Experimental data & empirical relationships ($E_0$)

The relationship between $E_0$ and $E_{ur}$ can be estimated from the chart by Alpan* assuming $E_{\text{dynamic}}/E_{\text{static}} \approx E_0/E_{ur}$ (10 kg/m²=1 MPa):

\[
\frac{E_{\text{dynamic}}}{E_{\text{static}}} \approx \frac{E_0}{E_{ur}}
\]

\[E_{\text{static}}[\text{kg/cm}^2] \approx E_{ur}\]

Experimental data & empirical relationships

Stiffness reduction curves according to Seed & Idris* (left) and Vucetic & Dobry** (right)

Empirical relationship for $\gamma_{0.7}$

Based on statistical evaluation of test data, Darandeli* proposed correlations for a hyperbolic stiffness reduction model, similar to the one used inside the HSS model. Correlations are given for different plasticity indices.

Based on Darendeli’s work, $\gamma_{0.7}$ can be estimated to:

$IP = 0$: $\gamma_{0.7} = 0.00015 \sqrt{\frac{p'}{p_{ref}}}$

$IP = 30$: $\gamma_{0.7} = 0.00026 \sqrt{\frac{p'}{p_{ref}}}$

$IP = 100$: $\gamma_{0.7} = 0.00055 \sqrt{\frac{p'}{p_{ref}}}$

Note: The indicated stress dependency of $\gamma_{0.7}$ is not implemented in the commercial HSS model. If needed, the stress dependency of $\gamma_{0.7}$ can be incorporated into boundary value problems through definition of sub-layers.

HS-Small model

![Graph showing the HS-Small model with the x-axis labeled $\gamma/\gamma_0$ and the y-axis labeled $G/G_0$. The graph includes several data points and a smooth curve. The values range from $10^{-3}$ to $10^1$ on the x-axis and from $0$ to $1$ on the y-axis. There are two horizontal lines at $G/G_0 = 0.8$ and $G/G_0 = 1$. The data points are connected by a smooth curve, and there are markers at specific points. The axes labels are $10^{-3}$, $10^{-2}$, $10^{-1}$, $10^0$, $10^1$, $10^2$, $10^3$. The graph also includes a legend with symbols and markers. The y-axis label is $G/G_0$. The x-axis label is $\gamma/\gamma_0$. The graph shows a decrease in $G/G_0$ as $\gamma/\gamma_0$ increases.]

The graph illustrates the behavior of the HS-Small model under different stress conditions, demonstrating how the modulus $G$ changes with respect to the normalized stress $\gamma/\gamma_0$. The lines and markers indicate the relationship between these variables, with the horizontal lines at $G/G_0 = 0.8$ and $G/G_0 = 1$ providing reference points for the model's performance.
Small-strain stiffness in the HS model (HSsmall)

Strain(path)-dependent elastic overlay model:

\[ G_s = \frac{G_0}{1 + 0.385 \gamma / \gamma_{0.7}} \]

\[ G_t = \frac{G_0}{\left(1 + 0.385 \gamma / \gamma_{0.7}\right)^2} \geq G_{ur} \]

G starts again at \( G_0 \) after full strain reversal
Small-strain stiffness in the HS model (HSsmall)

Cyclic loading leads to 

Energy dissipation

Damping

CiTG, Geo-engineering,
http://geo.citg.tudelft.nl
Small-strain stiffness in the HS model (HSsmall)
The 1-dimensional model by Hardin & Drnevich*:

\[ \frac{G}{G_0} = \frac{1}{1 + \gamma / \gamma_r} \]

Modified HS-Small:

\[ \frac{G}{G_0} = \frac{1}{1 + (3\gamma)/(7\gamma_{0.7})} \]

Note: \( \gamma_r \) in the original approach by Hardin & Drnevich relates to the failure shear stress \( \tau_f \).

HS-small model – stiffness reduction

Left: Secant modulus reduction $\rightarrow$ Parameter input

Right: Tangent modulus reduction $\rightarrow$ Stiffness reduction cut-off

If the small-strain stiffness relationship that is implemented in the HS-Small model predicts a tangent stiffness lower than $G_{ur}^{ref}$, the model's elastic stiffness is set constant as then hardening plasticity accounts for further stiffness reduction.
Model response in a standard triaxial test. Here: Dense Hostun sand

Parameter: $E_{ur}^{ref} = 90$ MPa, $E_0^{ref} = 270$ MPa, $m = 0.55$, $\gamma = 2 \times 10^{-4}$

The stress-strain curves of the Hardening Soil model and the HS-Small model are almost identical (Figure left-hand side). However, in zooming into the first part of the curve, the difference in the two models can be observed (Figure right-hand side).
Excavation example

Limburg excavation: Settlement trough

A comparison:
MC (E\textsubscript{50}): MC calculation with E = E\textsubscript{50}
MC (E\textsubscript{ur}): MC calculation with E = E\textsubscript{ur}
HS: HS calculation with E\textsubscript{oed} = E\textsubscript{50}
HSS: Same as HS but with small strain stiffness
Excavation example

Limburg excavation: Horizontal wall displacement

- MC-Model (E₅₀)
- MC Model (Eₐ₀)
- Hardening Soil Model
- Hardening Soil Small

Depth below surface [m]
Excavation example

Limburg excavation: Bending moments in [kNm/m]
Tunnel example

Steinhaldenfeld - NATM

Distance to tunnel axis [m]

Settlement [m]

Measurement

HS (original)

HS-Small

Fill

Upper Keuper Marl

Lower Keuper Marl

Lacustrine Limestone

Distance to tunnel axis [m]

Settlement [m]
HS-Small model

\[ G/G_0 [-] \]

\[ \gamma/\gamma_0 [-] \]
Parameters of the HS(small) model

Parameters:

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- $\psi$: Dilatancy angle
- $R_f$: Failure ratio $q_f/q_a$ like in Duncan-Chang model (0.9)
- $K_0^{nc}$: Stress ratio $\sigma'_{xx}/\sigma'_{yy}$ in 1D primary compression
Selected references

• Brinkgreve, R.B.J. et al (20xx): Users manual for PLAXIS 2D.


HS-small model application

Elastic stiffness properties of the HS-Small model can be visualized in state variable 10.

The dark (blue) area is the strain area where $G = G_{ur}$. The light gray (yellow) area is the very-small-strain area with $G \approx G_0$. In between $G_{ur}$ and $G_0$ is the area where shear strains are small but not very small according to the definition by Atkinson.
Limitation: Heavily OC clays

Need to use artificially low POP/OCR value to trigger plasticity within ‘yield surface’ in order to represent different stiffness for loading/unloading for non-monotonic loading.

However the stress path may still be wrong when approaching to failure.
Limitation: Heavily OC clays

By default, the initial stiffness is set to $G_0$.

Care needs to be taken when the geologic loading history of a soil is modeled. If, for example, a vertical surcharge was applied and removed in order to model OCR, the model remembers the vertical heave upon unloading including its decreased small-strain stiffness. The initial stiffness at the onset of loading the footing might then look as the one shown at the left-hand side of the above figure. Here, the material should be exchanged or a reverse load step applied.
Thank you