- Capital budgeting problems can be solved based on, for example, the benefit-cost ratio (that is, present value of benefits per present value of the costs) or the net present value (the present value of benefits - present value of the costs). According to the benefit-cost ratio, a project is beneficial if the ratio is greater than one, i.e., the present value of the benefits is greater than that of the costs. Similarly, when evaluating projects based on net present value, a project is worth carrying out if NPV>0.

- If there are more projects available than there is capital to fund them, an approximate solution to the capital budgeting problem is obtained by ordering the projects by their benefit-cost ratio and selecting them one-by-one until the budget limit is reached.

- If the benefits a_j of the projects are independent, the capital budgeting can be written as

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} a_i x_i \\ s.t. \quad & x_i \in \{0,1\}, \; i = 1, \ldots, m \\ & \mathbf{x} \in \mathcal{X}_F, \end{aligned}$$

  where $\mathbf{x} = [x_1 \; x_2 \ldots \; x_m]$ represents the project portfolio such that $x_i = 1$ if project i will be implemented and $x_i = 0$ if not. $\mathcal{X}_F$ is the feasible set of the projects, which is defined by, for example, budget limits and project interdependencies. If the project interdependencies affect the benefits of the projects, the object function of the optimization problem has to be modified correspondingly.

- For example, if the only constraint is the budget C and the cost of each project i is c_i, the feasible set is defined as $\mathcal{X}_F = \{\mathbf{x} \in \{0,1\}^m \mid \sum_{i=1}^{m} c_i x_i \leq C\}$, and if projects i and j are mutually exclusive (that is, only one of them can be implemented), a constraint $x_i + x_j \leq 1$ has to be subjected.

- A firm can be evaluated by, for example, based on the paid dividends. Using a constant-growth dividend model, the value of a firm can be defined as the present value of the dividend stream. Suppose a constant growth rate g and interest rate r and the first dividend D_1 paid at the end of first period. The present value of the dividends is

$$V_0 = \frac{D_1}{1+r} + \frac{D_1(1+g)}{(1+r)^2} + \frac{D_1(1+q)^2}{(1+r)^3} + \cdots = D_1 \sum_{k=1}^{\infty} \frac{(1+g)^{k-1}}{(1+r)^k} \Leftrightarrow V_0 = \frac{D_1}{r-g},$$

  where the last equation is the Gordon formula.

4.1 (L5.1) (Capital budgeting) A firm is considering funding several proposed projects that have the financial properties shown in Table 1. The available budget is 600 000 €. What set of projects would be recommended by the approximate method based on benefit-cost ratios? What is the optimal set of projects (using net present value)?

Table 1: Financial properties of the proposed projects.

	Outlay	Present value of benefits
Project	(1 000 €)	(1 000 €)
1	100	200
2	300	500
3	200	300
4	150	200
5	150	250

Solution:
The projects can be ranked based on their benefit-cost ratio ϕ or their net present value, defined as
$\phi = \dfrac{\text{Present value of benefits}}{\text{Investment cost}}$, NPV=Present value of benefits - Investment cost

The projects ranked by their benefit-cost ratios are presented in Table 2 below. Based on the benefit-cost

Table 2: Projects ranked by the benefit-cost ratios ϕ.

	Outlay	Present value of benefits	ϕ
Project	(1 000 €)	(1 000 €)	
1	100	200	2.00
2	300	500	1.67
5	150	250	1.67
3	200	300	1.50
4	150	200	1.33

ratios, projects 1,2 and 5 are selected, having total investment cost of 550 000 < 600 000 €. The total net present value of the projects of this approximate solution is NPV=400 000 €.

Those projects that create the greatest total net present value of the project portfolio comprise the optimal project portfolio. The project selection can be formulated as an optimization problem as follows:

$$\begin{aligned} \max \quad & \sum_{i=1}^{5} \text{NPV}_i x_i \\ s.t. \quad & \sum_{i=1}^{5} c_i x_i \leq 600000\, e, \\ & x_i \in \{0,1\}\,,\ i = 1,\ldots,5, \end{aligned}$$

where $\mathbf{x} = [x_1\ x_2 \ldots\ x_m]$ ($x_i = 1$ if i selected and 0, otherwise) presents the selections of projects in the project portfolio, NPV_i is the net present value and c_i the investment cost of project i. This optimization problem can be solved with, for example, Solver of Excel. We find the solution to this problem to be the projects 1,2 and 5. Thus, the approximate solution using benefit-cost ratios provided the optimal solution in this case.

4.2 (L5.3) (Two-period budget) A company has identified a number of promising projects, as indicated in Table 3. The cash flows for the first 2 years are shown (they are all negative).

Table 3: A list of projects.

	Cash flow (1 000 €)		
Project	year 1	year 2	NPV (1 000 €)
1	-90	-58	150
2	-80	-80	200
3	-50	-100	100
4	-20	-64	100
5	-40	-50	120
6	-80	-20	150
7	-80	-100	240

The cash flows in later years are positive, and net present value of each project is shown. The company managers have decided that they can allocate up to 250 000 € in each the first 2 years to fund these projects. If less than 250 000 € is used the first year, the balance can be invested at 10% and used to augment the next year's budget. Which projects should be funded? Formulate the problem as an optimization problem.

Solution:
We define binary variables $x_i,\ i = 1,\dots,7$ so that $x_i = 1$, if project i is selected and $x_i = 0$, otherwise. The project selection problem can be formulated as and optimization problem. The objective of the optimization problem is

$$\max f(x) = \sum_{i=1}^{7} \mathrm{NPV}_i x_i = 150x_1 + 200x_2 + 100x_3 + 100x_4 + 120x_5 + 150x_6 + 240x_7.$$

where NPV_i is the net present value of project i, and the unit is 1000 €.

In general, inequalities can be written as equations by introducing slack variables. For example, we can write $x \leq C$ as $C - x - s^+ = 0$, where $s^+ \geq 0$. Using this method, the budget constraints for the first two years can be set using slack variables s_i^+ $(i = 1, 2)$, which define the amount of budget remaining in each years. Hence, we write the budget constraint for the first year as

$$250 - 90x_1 - 80x_2 - 50x_3 - 20x_4 - 40x_5 - 80x_6 - 80x_7 - s_1^+ = 0.$$

The remaining balance s_1^+ from the first year can be invested at 10% interest to be added to the budget 250 000 € of the second year. The budget for the second year is then $250 + 1.1s_1^+$ (1000 €). We can then write the budget constraint for the second year as

$$250 + 1.1s_1^+ - 58x_1 - 80x_2 - 100x_3 - 64x_4 - 50x_5 - 20x_6 - 100x_7 - s_2^+ = 0.$$

We formulate the optimization problem of the project selection as follows:

$$\begin{aligned}
\max_x \quad & f(x) \\
& \text{s.t.} \\
250 - 90x_1 - 80x_2 - 50x_3 - 20x_4 - 40x_5 - 80x_6 - 80x_7 - s_1^+ &= 0 \\
250 + 1.1s_1^+ - 58x_1 - 80x_2 - 100x_3 - 64x_4 - 50x_5 - 20x_6 - 100x_7 - s_2^+ &= 0 \\
x_i &\in \{0,1\} \quad i = 1, \ldots, 7 \\
s_1^+, s_2^+ &\geq 0.
\end{aligned}$$

s_1^+ is the remaining budget from the total budget 250 000 € after the expenses of the first year, and which can be invested at 10% interest. s_2^+ is the excess capital that remains unused after two years.

We solve this optimization problem using Solver of Excel. There are two solutions,

$$x_4 = x_5 = x_6 = x_7 = 1, \text{ and } x_1 = x_2 = x_3 = 0,$$

at a cost of 220 in the first year and 234 in the second year,and

$$x_1 = x_4 = x_5 = x_7 = 1, \text{ and } x_2 = x_3 = x_6 = 0,$$

at a cost of 230 in the first year and 272 in the second year. Both these have a total net present value of 610 and are under the budget, but the first costs less.

4.3 (L5.4) (Bond matrix) Suppose that we face a known sequence of future monetary obligations. In cash matching problem, we design a portfolio that will provide the necessary cash as required for the obligations. We formulate this optimization problem in matrix form as follows. Let the number of bonds be m and the time horizon be n. The cash flow stream of bond j can be denoted as $\mathbf{c}_j \in \mathbb{R}^{n\times 1}$ and the yearly obligations as $\mathbf{b} \in \mathbb{R}^{n\times 1}$. We denote the bond matrix that has columns of the cash flows $\mathbf{c}_j$ as $\mathbf{C} \in \mathbb{R}^{n\times m}$. Furthermore, the prices of the bonds can be denoted as $\mathbf{p} \in \mathbb{R}^{m\times 1}$ and the numbers of the bonds in the portfolio as $\mathbf{x} \in \mathbb{R}^{m\times 1}$. The cash matching problem can be expressed as

$$\begin{aligned} \min \quad & \mathbf{p}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{C}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

a) The cash flow structure of a cash flow matching problem is presented in Table 3. Define $\mathbf{C}$, $\mathbf{b}$, $\mathbf{p}$ and $\mathbf{x}$.
b) Suppose the bonds are priced according to a conventional spot rate curve. The price vector $\mathbf{p}$ can be then written as

$$\mathbf{C}^T\mathbf{v} = \mathbf{p},$$

where $\mathbf{v} \in \mathbb{R}^{n\times 1}$ is a vector of the discount rates. Moreover, if the portfolio $\mathbf{x}^*$ matches the obligations exactly, we have

$$\mathbf{C}\mathbf{x}^* = \mathbf{b}.$$

Show that the price $\mathbf{p}^T\mathbf{x}^*$ of the portfolio is $\mathbf{v}^T\mathbf{b}$ and interpret this.
c) The optimization problem presented above seeks a solution that matches the obligations each year exactly. If the cash flows cannot be matched exactly, the present value of the portfolio is greater than the present value of the obligations. How does this model differ from immunization of a portfolio? What factor of portfolio immunization is neglected in this approach? Which approach is better?

Table 4: Bonds of exercise 3.

	Bonds											
Year	1	2	3	4	5	6	7	8	9	10	Required	Actual
1	10	7	8	6	7	5	10	8	7	100	100	171.74
2	10	7	8	6	7	5	10	8	107		200	200.00
3	10	7	8	6	7	5	110	108			800	800.00
4	10	7	8	6	7	105					100	119.34
5	10	7	8	106	107						800	800.00
6	110	107	108								1200	1200.00
$\mathbf{p}$	109	94.8	99.5	93.1	97.2	92.9	110	104	102	95.2	2381.14	
$\mathbf{x}$	0	11.215	0	6.807	0	0	0	6.302	0.283	0	Cost	

Solution:

a) The bond matrix $\mathbf{C}$ and vectors of obligations $\mathbf{b}$, bond prices $\mathbf{p}$ and numbers of bonds in the portfolio $\mathbf{x}$ can be directly read from Table 5:

$$\mathbf{C} = \begin{bmatrix} 10 & 7 & 8 & 6 & 7 & 5 & 10 & 8 & 7 & 100 \\ 10 & 7 & 8 & 6 & 7 & 5 & 10 & 8 & 107 & 0 \\ 10 & 7 & 8 & 6 & 7 & 5 & 110 & 108 & 0 & 0 \\ 10 & 7 & 8 & 6 & 7 & 105 & 0 & 0 & 0 & 0 \\ 10 & 7 & 8 & 106 & 107 & 0 & 0 & 0 & 0 & 0 \\ 110 & 107 & 108 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} 100 \\ 200 \\ 800 \\ 100 \\ 800 \\ 1200 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 109 \\ 94.8 \\ 99.5 \\ 93.1 \\ 97.2 \\ 92.9 \\ 110 \\ 104 \\ 102 \\ 95.2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0 \\ 11.2 \\ 0 \\ 6.81 \\ 0 \\ 0 \\ 0 \\ 6.3 \\ 0.28 \\ 0 \end{bmatrix}$$

b) We have $\mathbf{C}^T\mathbf{v} = \mathbf{p}$ and $\mathbf{C}\mathbf{x}^* = \mathbf{b}$, where elements of $\mathbf{v}$ are $[\mathbf{v}]_k = 1/(1+s_k)^k$

Because $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$, we can write $\mathbf{C}^T\mathbf{v} = \mathbf{p} \Leftrightarrow \mathbf{p}^T = \mathbf{v}^T\mathbf{C}$. Hence, the price of the portfolio can be written as:
$\mathbf{p}^T\mathbf{x}^* = \mathbf{v}^T\mathbf{C}\mathbf{x}^* = \mathbf{v}^T\mathbf{b}$.
Interpretation: If the cash flows of the portfolio match the obligations exactly, the present value of the project portfolio (which equals the price of the portfolio) matches the present value of the obligations.

c) In immunization the present value and first order derivative (quasi-modified duration) of the portfolio is matched with those of the obligation stream. The cash flows of an immunized portfolio do not necessarily have to match those of the obligations, and instead the assets in the portfolio are sold when needed to pay the obligations.

In cash flow matching, the positive cash flows from the bonds always suffice to pay the obligations, regardless of the interest rates. However, the present value of the portfolio is not matched to that of the obligations. From part b) of this exercise can be seen that if $\mathbf{Cx} > \mathbf{b}$, then $\mathbf{p}^T\mathbf{x} = \mathbf{v}^T\mathbf{Cx} > \mathbf{v}^T\mathbf{b}$, and hence the present value (price) of the portfolio exceeds that of the obligation stream when the cash flows received from the bonds exceed the obligations.

This problem of cash flow matching can be diminished by, for example, introducing artificial bonds that are consistent with the forward rates, or by allowing extra cash to be put ”under the matress”.

4.4 (L5.12) (Two-stage growth) When pricing financial instruments, the dividend discount model can be extended by taking more growth phases into account. Consider Nokia Corp. that paid 1439M € of dividends in year 2003. Suppose that the dividends grow at a constant rate $G = 1.3$ in the first five years (that is, during years 2004-2008), and the dividends grow at rate $g = 1.05$ from year 2009 onwards.
a) Formulate the general formula for two-stage dividend discount model for valuing a publicly traded company. The growth rate is constant G for k years and then g from year $k+1$ onwards. The dividend of the first year D_0 is paid immediately.
b) What is the market value of Nokia Corp., if it is valued solely based on the shared dividends? Assume a constant interest rate $r = 0.1$ and that first dividend is paid immediately.

Solution:
a) Growth rate is G for the first k years, the first dividends are D_0 and the discount factor is $1/R = 1/(1+r)$. Hence, the present value of the dividends in the first k years is

$$PV_1 = D_0 + D_0\frac{G}{R} + \cdots + D_0\left(\frac{G}{R}\right)^k = D_0\sum_{i=0}^{k}\left(\frac{G}{R}\right)^i \tag{1}$$

After k years, the growth rate changes to g. The present value of the paid dividends from year $k+1$ onwards is then

$$PV_2 = D_0\left(\frac{G}{R}\right)^k\frac{g}{R} + D_0\left(\frac{G}{R}\right)^k\left(\frac{g}{R}\right)^2 + \cdots. \tag{2}$$

Combining equations (1) and (2) yields the present value of the whole stream as

$$PV = D_0 + D_0\frac{G}{R} + \cdots + D_0\left(\frac{G}{R}\right)^{k-1} + D_0\left(\frac{G}{r}\right)^k\left[1 + \frac{g}{R} + \left(\frac{g}{R}\right)^2 + \cdots\right]. \tag{3}$$

The formula for finite geometric sum is $a + ar + \cdots + ar^{n-1} = a(1-r^n)/(1-r)$, and the formula for infinite ($n \to \infty$) geometric series is $a + ar + ar^2 + \cdots = a/(1-r)$. The first formula applies for $r \neq 1$ and second for $r < 1$. Using these formulas (assuming $G \neq R$ and $g < R$), the present value of the dividend stream becomes

$$PV = D_0\left[\frac{1-\left(\frac{G}{R}\right)^k}{1-\frac{G}{R}} + \left(\frac{G}{R}\right)^k\frac{1}{1-\frac{g}{R}}\right] \tag{4}$$

b) We substitute the values $G = 1.30, g = 1.05, R = 1.10, D_0 = 1439$, and $k = 5$ into (4) and get the market value of Nokia Corp. as 83 317 M€. The price per share (the present value divided by the number of shares) is then 17.37 €. For comparison, the average price of Nokia stocks was 14.12 € in 2003.