Aalto University
School of Science

MS-E2114 Investment Science

Lecture 1: Cash flow analysis

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December 11, 2017

Overview

- Investment science
- Investment types
- Cash flow models of investments
- Time value of money
- Interest
- Inflation
- Present value and future value
- Ideal bank and the market
- Internal rate of return
- Examples

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What is investment science?

- **Investment** = Commitment of current resources with the aim of receiving later benefits
- Examples
  - Study this course to learn and to earn credits
  - Construct new factory to achieve profitable growth
  - Purchase shares of company X to get dividends and to profit from possibly increasing share prices
- **Investor** = A person or an organization who is responsible for making the investment decision
  - Resources need not be investor's own
- **Investment science** = Consists of scientific principles, theory, methods and tools which guide investors in making investments

Financial investments

- **Financial asset** = Contracts, rights, equipment, or other tangible or intangible resources that can create economic value
  - Economic value of financial assets is measured in monetary terms (cash)
- **Financial investment** = Purchase of an asset which either
  1. generates savings or income or
  2. can appreciate in value (so that it can possibly be sold at a profit later on)
- This course considers financial investments only

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Examples of investments

1. Bank deposit
   - Deposit 80 000 € for one year at the 1.5% interest rate
2. Pension insurance
   - Pay an extra 200 € per month for 15 years
   - Get an extra 300 € addition to your monthly pension starting at the age of 65
3. Stock investment
   - Buy shares of stock at the current price 2000 €
   - Sell once the stock has appreciated to 4000 €
   - Make a profit 2000 € per share

Investment types

- Single asset – Multiple assets
  - Buy a single stock vs. a portfolio of many stocks
  - Cf. actively managed mutual funds vs. passive index funds
- Deterministic value – Random value
  - Cf. fixed interest rate investments vs. investments with uncertain future value
- Liquid market – Illiquid market
  - Exchange traded asset vs. illiquid items (e.g., rare works of art)
- Single period – Multiperiod - Continuous
  - Either
    (i) decide now whether to invest or not
    vs.
    (ii) make an initial investment now and reserve an option to invest more either at 1) fixed time points or 2) in continuous time

Investing in markets

- Comparison principle
  - The attractiveness of an investment is determined by comparing it with other comparable investments
- Arbitrage principle
  - There is no such thing as “a free lunch”
  - Such “lunches” will spotted by investors and will disappear
- Price dynamics
  - The price of an asset is not a static number but a dynamically evolving (stochastic) process
- Risk attitudes
  - Different investors have different risk attitudes
  - Many private persons are risk averse whilst large institutional investors are less so

Important types of investment problems

1. Pricing
   - What is the “correct” price of a financial asset in the light of all available information?
2. Hedging
   - If the investor is exposed to financial risks, how should she invest to reduce these risks?
   - Cf. determination of premiums in insurance
3. Portfolio optimization
   - Which portfolio consisting of available (financial) investment opportunities best matches the investor’s risk-return preferences?

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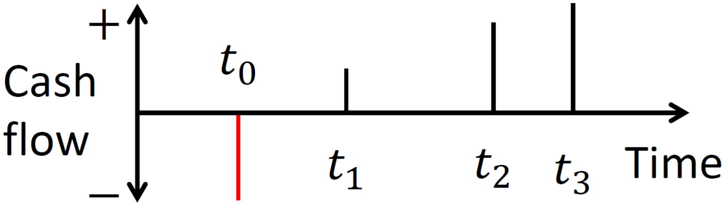
Cash flow models of investments

- The investment yields a **cash flow** $x = (x_0, x_1, x_2, \ldots, x_n)$
  - x_i = cash flow at time t_i, $t_i < t_j$ for all $i < j$
  - Negative cash flows $x_i < 0$ are outflows
    - E.g., expenses for buying equipment investment
  - Positive cash flows $x_i > 0$ are inflows
    - E.g., revenues from selling assets



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Time value of money

- It is widely held that the earlier cash is available, the better
  - Cash can be used for consumption
  - New opportunities may become available in mid-course
    - Cf. normal savings accounts vs. fixed-term deposits: From the savings account one usually withdraw cash for investments at any time
- Consider alternatives A and B

  A. Receive $A = 100$€ now

  B. Receive $B = (100 + x)$€ after one year

  - The smallest x for which these alternatives are equivalent to you is your <u>time value for money</u> . . . or is it?

Time value of money

- Time value of money can be difficult to measure
  - Inflation
  - Decreasing marginal utility
    - Preferences depend on the current level of wealth
  - Risk aversion
- Investor's preferences for market traded assets should not depend on her time value!
  - This follows from the main theorem on present value (presented later)

Further reading: Frederick, S., Loewenstein, G., & O'Donoghue, T. (2002). Time discounting and time preference: A critical review. *Journal of Economic Literature*, 351-401.

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Interest

- A simple loan for a single period:
  - Investor lends a **principal** A_0 € for a single period at the end of which the investor receives $A_1 = A_0 + x$ €
- **Interest** x compensates the lost time value of money to the investor
  - The quantity $r = x/A_0$ is the **interest rate**
- The interest rate of a loan with principal 100 € and interest 10 € is $r = 10$ €$/100$ € $= 10\%$

Compound interest

- Let A_n be the value of a $n = 0, 1, \ldots$ period loan (= how much the investor receives at the start of period n)
- **Simple interest** is paid only on the principal

$$A_n = A_0 + rnA_0 = (1 + rn)A_0$$

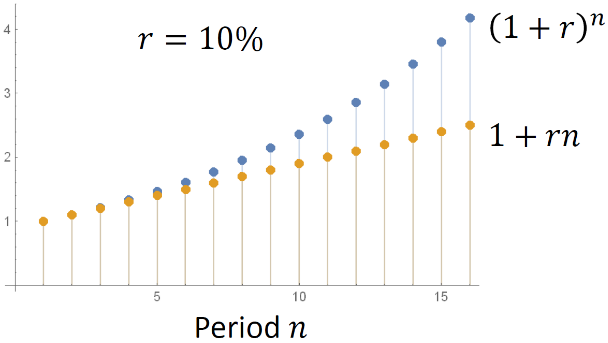
- **Compound interest** pays interest on interest as well

$$A_n = (1 + r)A_{n-1} = (1 + r)^2 A_{n-2} = \cdots$$
$$\Rightarrow A_n = (1 + r)^n A_0$$

  - The term comes from the fact that interest is *compounded* (=added to the principal) at the end of each period

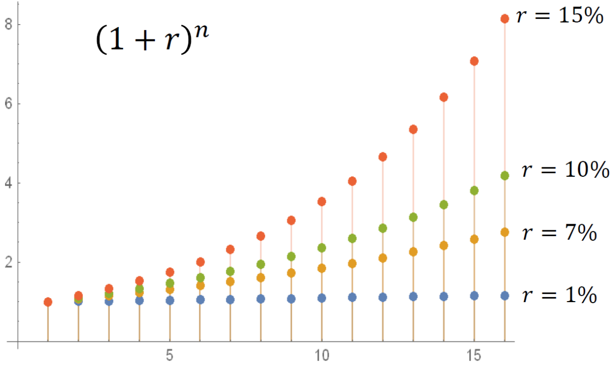
Compound interest

- Simple interest ⇒ Linear growth
- Compound interest ⇒ Geometric growth



Compound interest

- Rule of thumb: the value of the investment doubles in
  - about 10 periods when the interest rate is $r = 7\%$
  - about 7 periods when the ineterest rate is $r = 10\%$



Interest rates with varying length periods

- Interest rates are usually annualized
- $r' =$ interest rate for a single period of length $t = k/m$ years $m, k \in \mathbb{N}$
  - In k years, the interest at rate is r' compounded m times
  - In k years, the one-year interest rate r is compounded k times
- These interests are years equal when

$$(1 + r')^m = (1 + r)^k$$

  - The rate $r > 0$ which solves this equation is the **annualized rate** of r', computed as the principal k-th root

$$r = (1 + r')^{m/k} - 1$$

Compounding frequency

- Interest is often expressed in terms of
  - Nominal interest rate r (per annum, p.a.)
  - Compounding frequency m times per year
- The one-period interest rate is r/m
- **Effective rate** = annualized r/m =

$$r_e = (1 + r/m)^m - 1$$

- Example

$$r = 6\% \quad \text{with monthly compounding}$$
$$\Rightarrow m = 12, \quad \frac{r}{m} = \frac{6\%}{12} = 0.5\%$$
$$\Rightarrow r_e = (1 + 0.005)^{12} - 1 \approx 6.17\%$$

Continuous compounding

- Continuous compounding $\equiv$ compounding frequency $m \to \infty$
- The **effective rate** for continuous compounding is

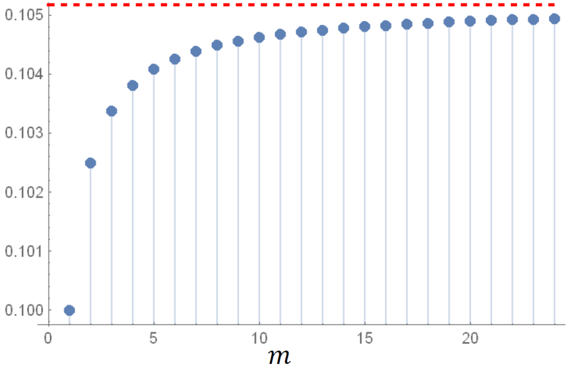
$$r_e = \lim_{m\to\infty} (1 + r/m)^m - 1$$
$$\Rightarrow r_e = e^r - 1$$

- **Example**: If the nominal rate with continuous compounding is 6%, the effective rate is

$$r_e = e^{0.06} - 1 \approx 6.18\%$$

Impact of compounding frequency

- Effective rate $r_e = (1 + r/m)^m$ with nominal rate $r = 10\%$ p.a. for various compounding frequencies
- Limit when $m \to \infty$ is $e^r - 1 \approx 0.105$



Growth of value with compound interest

- Nominal rate r, principal A_0
- Value growth when compounding m times per year
  - After k periods, the value is

$$A_k = \left(1 + \frac{r}{m}\right)^k A_0$$

- Value growth with continuous compounding
  - At time $t = k/m$, the value is $(1 + r/m)^{mt} A_0$, hence

$$A_t = \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{mt} A_0 = e^{rt} A_0$$

Overview

Inflation

- **Inflation** = Reduction of purchasing power due to increased prices
- **Inflation rate** f = %-annual price increase (for a given year)
  - $f = 2\% \Rightarrow$ an item that costs 1 € in 2017 will cost 1.02 € in 2018
  - Hence the **real value** of 1 € in 2018 is the same as 1/1.02 = 0.98 € in 2017
  - If you get a 5% interest on your investment, your purchasing power grows by the **real interest rate** r_0

$$1 + r_0 = \frac{1+r}{1+f}$$

$$\Rightarrow r_0 = \frac{r-f}{1+f} = \frac{3\%}{1.02} \approx 2.9\%$$

Overview

Present value and future value

- **Future value**: Value of principal after compounding

$$FV = (1 + r)A_0$$

- **Present value**: Value of a principal whose value after compounding is A_1

$$PV = \frac{1}{1+r}A_1 = dA_1$$

  - Discount factor $d = 1/(1 + r)$
  - Let $r = 8\%$, then $d = 1/1.08 \approx 0.93$
  - Let $A_1 = 100$ €, then $PV = dA_1 \approx 93$ €
  - Let $A_0 = 100$ €, then $FV = A_0/d \approx 108$ €

Present value and future value

- Cash flow $x = (x_1, \ldots, x_n)$ with flows at periods $1, 2, \ldots, n$
- **Present value of cash flow** x is

$$
\begin{aligned}
PV &= x_0 + \frac{x_1}{1+r} + \cdots + \frac{x_n}{(1+r)^n} \\
&= x_0 + dx_1 + \cdots + d^n x_n
\end{aligned}
$$

  - $r =$ interest rate of a single period
  - $d =$ discount factor of a single period
- **Net present value** (NPV) accounts for both positive and negative cash flows

Present value and future value

- **Future value of a cash flow** at period n is

$$FV = (1+r)^n x_0 + (1+r)^{n-1} x_1 + \cdots + x_n$$

- Example:

$$x = (-2, 1, 1, 1), \quad \text{period 1 year, annual interest rate } r = 10\%$$

$$FV = -2 \cdot 1.1^3 + 1 \cdot 1.1^2 + 1 \cdot 1.1 + 1 = 0.648$$

$$PV = -2 + \frac{1}{1.1} + \frac{1}{1.1^2} + \frac{1}{1.1^2} = 0.487$$

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Market price

- Assets are traded in the market
  - Freely = Transaction costs and taxes are negligible
- Market price determined by supply and demand
  - **Bid price** = highest price that a buyer is willing to pay for an asset
  - **Ask price** = lowest price for which a seller is willing to sell an asset
- Usually no **arbitrage**
  - Bid price $\leq$ Ask price
- Comparison principle: If two assets have the same cash flows, then they have the same price

Ideal bank and the market

- The ideal bank ...
  - ... offers the same interest rate when lending/borrowing
  - ... is capable of unlimited lending and borrowing
  - ... charges no transaction costs
- The money market is an approximation of an ideal bank
  - Bid price $\approx$ Ask price
  - Transaction costs are practically negligible (as a fraction of investments)
  - Short selling (i.e., selling assets that one does not own) is permitted
  - Cf. `http://www.investopedia.com/terms/m/moneymarket.asp`

Main theorem on present value

Theorem

*(**Main theorem on present value**) The cash flow streams $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$ are equivalent for a constant ideal bank with interest rate r if and only if the present values of the two streams, evaluated at the bank's interest rate, are equal.*

Proof outline: x equivalent to $(PV_x, 0, \ldots, 0)$ and y equivalent to $(PV_y, 0, \ldots, 0)$. These equal only if $PV_x = PV_y$

- Implication: Investor needs only to compute PV, the timing of cash flows can be adjusted using the ideal bank

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Internal rate of return

- **Internal rate of return (IRR)** of the cash flow $x = (x_0, \ldots, x_n)$ is the interest rate r such that PV x is 0
- This r solves the equation

$$0 = x_0 + \frac{x_1}{1+r} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n}$$

- Equivalently, IRR is a number r satisfying $1/(1+r) = c$, where c satisfies the polynomial equation

$$0 = x_0 + x_1 c + x_2 c^2 + \cdots x_n c^n$$

Main theorem of internal rate of return

Theorem

*(**Main theorem of internal rate of return**) Suppose the cash flow stream $(x_0, x_1, \ldots, x_n)$ is such that $x_0 < 0$ and $x_k \geq 0, k = 1, 2, \ldots, n$ with strict inequality for some $k \geq 1$. Then there is a unique positive root to the equation*

$$0 = x_0 + x_1 c + x_2 c^2 + \cdots + x_n c^n$$

Furthermore, if $\sum_{k=0}^{n} x_k > 0$, then the corresponding internal rate of return $r = (1/c) - 1$ is positive.

Proof: Let $f(c) = x_0 + x_1 c + \cdots + x_n c^n$. By assumption $f(0) = x_0 < 0$. Since $x_k \geq 0 \forall k \geq 1$ with strict inequality for some $k \geq 1$, it follows that $f'(c) > 0, c > 0$ and $f(c^*) = 0$ has exactly one solution c^*. If also $\sum_{k=0}^{n} x_k > 0$, then $f(1) > 0 \Rightarrow 0 < c^* = \frac{1}{1+r} < 1 \Rightarrow r > 0$ □

NPV vs IRR

	Selection	Pros	Cons
NPV	Higher NPV preferred to lower NPV	Easy to compute Aims to maximize value	Interest rate must be defined (e.g. weighted average cost of capital; WACC) Insensitive to the size of the investment (an investment of 10 k€ and 10 M€ can have the same NPV)
IRR	Higher IRR preferred to lower IRR	Ranks by productivity	Computation of IRR non-linear (may cause problems) IRR for a portfolio of investments is not a linear function of the IRRs of the individual investments' IRRs

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Ostridge farm

- A farmer herds ostridges for meat
  - His interest rate is $r = 0.1 = 10\%$
  - He can slaughter after one year for double the initial investment $\Rightarrow A = (-1, 2)$
  - Alternatively, he can slaughter after 2 years in return for triple the investment $\Rightarrow B = (-1, 0, 3)$

$$NPV_A = -1 + \frac{2}{1.1} = 0.82$$
$$NPV_B = -1 + \frac{3}{1.1^2} = 1.48 > NPV_A$$
$$IRR_A = 1.0, \text{ because } -1 + \frac{2}{1 + 1.0} = 0$$
$$IRR_B = 0.7, \text{ because } -1 + \frac{3}{(1 + 0.7)^2} = 0$$

- $NPV_A < NPV_B, IRR_A > IRR_B \Rightarrow$ Recommendations differ!

Ostridge farm

- The possibility of reinvesting profits for alternative A after one year was not considered in the IRR calculation
- If this profit can indeed be reinvested, a more meaningful evaluation can be made by replicating the investments to achieve equal time horizons

Year	0	1	2	3	4
A	-1	2			
		-2	4		
			-4	8	
				-8	16
	-1				**16**

Year	0	1	2	3	4
B	-1		3		
			-3		9
	-1				**9**

Production machine

- A production facility uses a machine whose
  - procurement price is 10 k€
  - operating expenses are 2k€ in the first year and then 1k€ higher in each consecutive year
  - salvage value is 0 €
- Interest rate $r = 0.1 = 10\%$
- How often should a new machine be procured?
- This is an example of a **cycle problem** of comparing recurring investments with different periods

Production machine

- Let $x = (x_1, \ldots, x_k)$ be the cash flows of a cycle
  - k is the length of cycle
  - Present value of one cycle is PV_k
- After one cycle, the identical decision recurs at time k

$$
\begin{aligned}
PV &= PV_k + \frac{1}{(1+r)^k} PV \\
\Rightarrow PV &= \frac{(1+r)^k}{(1+r)^k - 1} PV_k \\
&= C_k PV_k, \text{ where } C_k = \frac{(1+r)^k}{(1+r)^k - 1}
\end{aligned}
$$

Production machine

- Cash flows for different cycles length k shown below
  - PVs computed using `SUMPRODUCT`-routine of Excel
- It is optimal strategy to replace the machine every 5 years

k	0	1	2	3	4	5	6	7	8	9	C_k	PV_k	PV
1	-10	-2									11	-12	-130
2	-10	-2	-3								6	-14	-82
3	-10	-2	-3	-4							4	-17	-70
4	-10	-2	-3	-4	-5						3	-21	-65
5	-10	-2	-3	-4	-5	-6					3	-24	-64
6	-10	-2	-3	-4	-5	-6	-7				2	-28	-65
7	-10	-2	-3	-4	-5	-6	-7	-8			2	-32	-67
8	-10	-2	-3	-4	-5	-6	-7	-8	-9		2	-37	-69
9	-10	-2	-3	-4	-5	-6	-7	-8	-9	-10	2	-41	-71

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