Aalto University
School of Science

MS-E2114 Investment Science
Lecture 7: Arbitrage pricing theory

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Overview

Single factor model

Multifactor models

Arbitrage Pricing Theory (APT)

Parameter estimation

Utility theory and risk aversion

Previous lecture

- The last lecture covered CAPM
- CAPM can be challenging to apply
  - The number of parameters is high
    - $\Rightarrow$ For n assets, there are n expected returns, variances and $n(n-1)/2$ covariances
    - $\Rightarrow$ There are $2n + n(n-1)/2$ parameters be estimated
    - $\Rightarrow$ Thus for $n = 500$ assets, there are 125750 parameters
  - Data may not be readily available
    - Time series may not be long enough
    - How old data is relevant and usable? (e.g., if the asset is company stock, does the appointment of a new CEO change asset characteristics?)
    - Covariances can depend on economic cycles and other external conditions

This lecture

- This lecture: Factor models and arbitrage pricing theory
- Investors may care about factors other than expected return and variance
  - E.g., preferences for geographical location, sectoral focus, ethical conduct of companies
  - Hence CAPM does not account for all factors
- Factor models help reduce the number of parameters to be estimated
- In addition, Von Neumann & Morgenstern utility theory visited briefly

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Single factor model

- Explain asset returns with an external factor
  - E.g. GDP (gross domestic product) or stock index
  - Return of asset i estimated as

$$r_i = a_i + b_i f + e_i,$$

    where a_i and b_i are constants, f is the random explanatory variable and e_i is the random error term
- Assumptions:
  - $\mathbb{E}[e_i] = 0$
  - e_i is not correlated with f

$$\Rightarrow \mathbb{E}\left[(f - \bar{f})(e_i - \bar{e}_i)\right] = \mathbb{E}\left[(f - \bar{f})e_i\right] = 0$$

  - Error terms of the assets are uncorrelated

$$\Rightarrow \mathbb{E}\left[(e - \bar{e}_i)(e_j - \bar{e}_j)\right] = \mathbb{E}\left[e_i e_j\right] = 0, i \neq j$$

  - Variances of error terms are known (estimated)

$$\Rightarrow \mathbb{E}\left[e_i^2\right] = \sigma_{e_i}^2$$

Single factor model

- With these assumptions

$$\bar{r}_i = \mathbb{E}[r_i] = a_i + b_i\,\mathbb{E}[f] + \mathbb{E}[e_i]$$

$$\Rightarrow \bar{r}_i = a_i + b_i\bar{f}$$

$$\sigma_i^2 = \text{Var}[r_i] = \mathbb{E}\left[(r_i - \bar{r}_i)^2\right] = \mathbb{E}\left[(a_i + b_i f + e_i - a_i - b_i\bar{f})^2\right]$$

$$= \mathbb{E}\left[\left(b_i(f - \bar{f}) + e_i\right)^2\right] = \mathbb{E}\left[b_i^2(f - \bar{f})^2 + 2b_i(f - \bar{f})e_i + e_i^2\right]$$

$$\Rightarrow \sigma_i^2 = b_i^2\sigma_f^2 + \sigma_{e_i}^2$$

$$\sigma_{ij} = \text{Cov}[r_i, r_j] = \mathbb{E}\left[(r_i - \bar{r}_i)(r_j - \bar{r}_j)\right]$$

$$= \mathbb{E}\left[\left(b_i(f - \bar{f}) + e_i\right)\left(b_j(f - \bar{f}) + e_j\right)\right]$$

$$= \mathbb{E}\left[b_i b_j(f - \bar{f})^2 + (b_j e_i + b_i e_j)(f - \bar{f}) + e_i e_j\right]$$

$$\Rightarrow \sigma_{ij} = b_i b_j \sigma_f^2, i \neq j$$

Single factor model

- It follows that

$$
\begin{aligned}
\text{Cov}[r_i, f] &= \mathbb{E}\left[(r_i - \bar{r}_i)(f - \bar{f})\right] \\
&= \mathbb{E}\left[\left(b_i(f - \bar{f}) + e_i\right)(f - \bar{f})\right] \\
\Rightarrow \text{Cov}[r_i, f] &= b_i \sigma_f^2 \\
\Rightarrow b_i &= \frac{\text{Cov}[r_i, f]}{\sigma_f^2}
\end{aligned}
$$

- A total of $3n + 2$ parameters to be estimated
  - $\bar{f}, \sigma_f^2, a_i, b_i$, and $\sigma_{e_i}^2$, for $i = 1, 2, \ldots, n$
  - CAPM has $2n + n(n-1)/2$ parameters

Estimating a_i and b_i

- Parameters a_i and b_i can be estimated from the time series of r_i and f
  - Estimates differ depending on the selected time span
  - Averaging and other statistical methods can be used to improve accuracy
- Standard statistical estimators

$$\hat{\bar{r}}_i = \frac{1}{n} \sum_{k=1}^{n} r_i^k$$

$$\hat{\sigma}_i^2 = \frac{1}{n-1} \sum_{k=1}^{n} \left(r_i^k - \hat{\bar{r}}_i \right)^2$$

$$\widehat{\text{Cov}}[r_i, f] = \frac{1}{n-1} \sum_{k=1}^{n} \left(r_i^k - \hat{\bar{r}}_i \right) \left(f^k - \hat{\bar{f}} \right),$$

where superscript k denotes the number of sample

Estimating a_i and b_i

- The model parameters can be calculated using the standard estimates

$$b_i = \frac{\widehat{\mathrm{Cov}}[r_i, f]}{\hat{\sigma}_f^2}$$

$$a_i = \hat{\bar{r}}_i - b_i \hat{\bar{f}}$$

- The variance of error terms become

$$\sigma_i^2 = b_i^2 \sigma_f^2 + \sigma_{e_i}^2$$

$$\Rightarrow \hat{\sigma}_{e_i}^2 = \hat{\sigma}_i^2 - b_i^2 \hat{\sigma}_f^2$$

Portfolios in the single factor model

- Form a portfolio of n assets
  - Asset i has weight w_i
- Returns of the assets follow the factor model

$$r_i = a_i + b_i f + e_i$$

- Return of the portfolio

$$r = \sum_{w=1}^{n} w_i r_i = \sum_{i=1}^{n} r_i a_i + \left(\sum_{i=1}^{n} w_i b_i \right) f + \sum_{i=1}^{n} w_i e_i = a + bf + e,$$

where

$$a = \sum_{i=1}^{n} w_i a_i, \quad b = \sum_{i=1}^{n} w_i b_i, \quad e = \sum_{i=1}^{n} w_i e_i$$

Portfolios in the single factor model

- For the error term of the portfolio return, we have

$$\mathbb{E}[e] = \mathbb{E}\left[\sum_{i=1}^{n} w_i e_i\right] = \sum_{i=1}^{n} w_i\, \mathbb{E}[e_i]$$

$$\Rightarrow \mathbb{E}[e] = 0$$

$$\text{Cov}[f, e] = \mathbb{E}\left[(f - \bar{f}) \sum_{i=1}^{n} w_i e_i\right] = \sum_{i=1}^{n} w_i\, \mathbb{E}[(f - \bar{f}) e_i]$$

$$\Rightarrow \text{Cov}[f, e] = 0$$

$$\text{Var}[e] = \mathbb{E}\left[\left(\sum_{i=1}^{n} w_i e_i\right)\left(\sum_{j=1}^{n} w_j e_j\right)\right] = \sum_{i=1}^{n} w_i^2\, \mathbb{E}[e_i^2]$$

$$\Rightarrow \text{Var}[e] = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2$$

Portfolios in the single factor model

- Assume that assets have equal weights and the variance of error terms is $\sigma_{e_i}^2 = s^2$. Then the variance of the error term of the portfolio is

$$\sigma_e^2 = \mathrm{Var}[e] = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2 = \sum_{i=1}^{n} \frac{1}{n} s^2 = \frac{1}{n} s^2,$$

  and the variance of the portfolio return is

$$\sigma^2 = \mathrm{Var}[r] = b^2 \sigma_f^2 + \sigma_e^2,$$

  where $\sigma_e^2 \to 0$ as $n \to \infty$
- Variance related to the error terms e_i can be diversified
- Variance related to terms $b_i f$ cannot be diversified

Single factor model and CAPM

- CAPM is a special case of the single factor model
  - Factor = return of the market portfolio

$$r_i - r_f = \alpha_i + \beta_i(r_M - r_f) + e_i$$

  - This formss the **characteristic line** (set $e_i = 0$)
- Taking the expectation gives

$$\Rightarrow \bar{r}_i - r_f = \alpha_i + \beta_i(\bar{r}_M - r_f)$$

- In CAPM, the intercept $\alpha_i = 0$
  - See Lecture 6, Jensen Index
- Covariance of $r_i - r_f$ with r_M is

$$\sigma_{iM} = \mathrm{Cov}[r_i - r_f, r_M] = \mathrm{Cov}[\alpha_i + \beta_i(r_M - r_f) + e_i, r_M] = \beta_i \sigma_M^2$$

$$\Rightarrow \beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

Overview

Single factor model

Multifactor models

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Parameter estimation

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Multifactor models

- The return can be explained with more than one factor
- With two factors

$$r_i = a_i + b_{i1} f_i + b_{2i} f_2 + e_i,$$

  where a_i is the intercept and b_{1i}, b_{2i} are factor loadings
- Assumptions
  - Expected error $\mathbb{E}[e_i] = 0$
  - Error terms are uncorrelated with factors and each other
  - **Factors can correlate with each other**

Multifactor models

- Expected return in the two factor model

$$\bar{r}_i = \mathbb{E}[r_i] = a_i + b_{1i}\bar{f} + b_{2i}\bar{f}_2$$

- Covariance

$$\begin{aligned}
\mathrm{Cov}[r_i, r_j] &= \mathbb{E}\Big[\big(b_{1i}(f_1 - \bar{f}_1) + b_{2i}(f_2 - \bar{f}_2) + e_i\big) \\
&\qquad \big(b_{1j}(f_1 - \bar{f}_1) + b_{2j}(f_2 - \bar{f}_2) + e_j\big) \Big] \\
&= \begin{cases} b_{1i}b_{1j}\sigma_{f_1}^2 + (b_{1i}b_{2j} + b_{2i}b_{1j})\sigma_{f_1,f_2} + b_{2i}b_{2j}\sigma_{f_2}^2, & i \neq j \\ b_{1i}^2\sigma_{f_1}^2 + 2b_{1i}b_{2j}\sigma_{f_1,f_2} + b_{2i}^2\sigma_{f_2}^s + \sigma_{e_i}^2, & i = j \end{cases}
\end{aligned}$$

Estimating the factor model parameters

- Loadings b_{1i}, b_{2i} can be estimated from the covariance matrix

$$\begin{aligned}\mathrm{Cov}[r_i, f_1] &= \mathbb{E}\left[\left(b_{1i}(f_1 - \bar{f}_1) + b_{2i}(f_2 - \bar{f}_2) + e_i\right)(f_1 - \bar{f}_1)\right] \\ &= b_{1i}\sigma_{f_1}^2 + b_{2i}\sigma_{f_1,f_2} \\ \mathrm{Cov}[r_i, f_2] &= b_{2i}\sigma_{f_2}^2 + b_{1i}\sigma_{f_1,f_2}\end{aligned}$$

  - Solve these equations for b_{1i} and b_{2i}
- The use of multiple factors is warranted if a single factor model has a large error term variance
  - If the error term variance is nearly as high as the variance of returns, the factor model does not explain much
  - Too many factors leads to overfitting $\Rightarrow$ Poor predictions power
- Factor models are easier to estimate than CAPM
  - There are fewer parameters
  - New factors can be introduced as necessary

Selection of factors

- No unambiguous answer - depends on what the key factors are believed to be
- External factors, such as
  - Gross National Product (GNP)
  - Consumer price indices
  - Unemployment rate
- Extracted factors, such as
  - Market portfolio return
  - Average return of companies in one industry
  - Days since the last market peak
- Firm characteristics, such as
  - Price-earnings ratio
  - Dividend payout ratio
  - Earnings forecast

Overview

Arbitrage Pricing Theory (APT)

- APT is a pricing theory (complementary to CAPM)
- Key assumptions
  - Investors prefer higher returns
  - There is a large number of assets
  - The investors need not optimize with respect to mean-variance (unlike what CAPM assumes)
- Idea: There is a factor price for each factor
  - Price of asset follows from linearity of pricing
  - Factor prices derived based on the assumption of no arbitrage
- A single factor with two assets and no error term

$$r_i = a_i + b_i f,$$
$$r_j = a_j + b_j f$$

Arbitrage Pricing Theory (APT)

- Invest in assets i (weight $w_i = w$) and j ($w_j = 1 - w$) that follow single factor model
- Portfolio return

$$
\begin{aligned}
r &= w(a_i + b_i f) + (1 - w)(a_j + b_j f) \\
&= wa_i + (1 - w)a_j + \left(wb_i + (1 - w)b_j\right) f
\end{aligned}
$$

- Select the weight w so that the coefficient of factor f is 0

$$
\begin{aligned}
wb_i + (1 - 1)b_j &= 0 \\
\Rightarrow w &= \frac{b_j}{b_j - b_i}
\end{aligned}
$$

Arbitrage Pricing Theory (APT)

- The portfolio with coefficient 0 for factor f is risk free (no variance), hence its return must be $r_f = \lambda_0$

$$
\begin{aligned}
r &= \frac{b_j}{b_j - b_i} a_i + \left(1 - \frac{b_j}{b_j - b_i}\right) a_j \\
&= \frac{b_j}{b_j - b_i} a_i - \frac{b_i}{b_j - b_i} a_j = \lambda_0 \\
\Rightarrow & b_j a_i - b_i a_j = \lambda_0 (b_j - b_i) \\
\Rightarrow & b_j (a_i - \lambda_0) = b_i (a_j - \lambda_0) \\
\Rightarrow & \frac{a_i - \lambda_0}{b_i} = \frac{a_j - \lambda_0}{b_j}
\end{aligned}
$$

Arbitrage Pricing Theory (APT)

- Thus, for every asset i, ratio $(a_i - \lambda_0)/b_i$ must be equal to some constant c

$$\Rightarrow \quad \frac{a_i - \lambda_0}{b_i} = c$$
$$\Leftrightarrow \quad a_i = \lambda_0 + b_i c$$

- Thus

$$\bar{r}_i = a_i + b_i \bar{f} = \lambda_0 + b_i c + b_i \bar{f}$$
$$= \lambda_0 + b_i (c + \bar{f}) = \lambda_0 + b_i \lambda_1,$$

which can be generalized to several factors

Arbitrage Pricing Theory (APT)

Definition

(**Simple APT**) Suppose that there are n assets whose rates of return are governed by $m < n$ factors according to the equation

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} f_j$$

for $i = 1, 2, \ldots, n$. Then there are constants $\lambda_0, \lambda_1, \ldots, \lambda_m$ such that

$$\bar{r}_i = \lambda_0 + \sum_{j=1}^{m} b_{ij} \lambda_j$$

for $j = 1, 2, \ldots, m$.

- λ_j=price of risk associated with factor j (i.e., factor price)
- b_{ij}= factor loading j of asset i

Well-diversified portfolio

- Adding the error term e_i to the simple APT gives

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} f_j + e_i$$

- Variance of error terms (slide 12)

$$\sigma_e^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2$$

- Assume that all variances are bounded, that is

$$\sigma_{e_i}^2 \leq S^2$$

  for some S, and assume that all assets have similar weights (i.e., we have $w_i \leq W/n$ for some $W \approx 1$)
  - This means that the portfolio is well diversified

Well-diversified portfolio

- With the assumptions of similar and bounded weights, we have

$$\sigma_e^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2 \leq \sum_{i=1}^{n} \frac{W^2}{n^2} S^2 = \frac{1}{n} W^2 S^2$$

$$\Rightarrow \lim_{n \to \infty} \sigma_e^2 = 0$$

- Hence, a well-diversified portfolio has practically no non-diversifiable risk
- As a result, the return of well diversified portfolios are fully explained by the factor model (because the error terms tend to cancel out)

$$r = a + \sum_{j=1}^{m} b_j f_j,$$

General APT

- By simple APT, the expectation of the return of well a diversified portfolio is

$$\bar{r} = \lambda_0 + \sum_{j=1}^{m} b_j \lambda_j,$$

- Because different well-diversified portfolios can be formed with weights that differ on only a small number of basic assets, it follow that these individual assets must also satisfy

$$\bar{r}_i = \lambda_0 + \sum_{j=1}^{m} b_{ij} \lambda_j,$$

  - This pricing equation is referred to as the general APT (this argumentation is not technically rigorous, but the above argument is the key idea)

APT and CAPM

- Consider the two factor model

$$r_i = a_i + b_{1i} f_1 + b_{2i} f_2 + e_i$$

$$\begin{aligned}\text{Cov}[r_M, r_i] &= \mathbb{E}\left[(r_M - \bar{r}_M)\left(b_{1i}(f_1 - \bar{f}_1) + b_{2i}(f_2 - \bar{f}_2) + e_i\right)\right] \\ &= b_{1i}\,\text{Cov}[r_M, f_1] + b_{2i}\,\text{Cov}[r_M, f_2] + \text{Cov}[r_M, e_i]\end{aligned}$$

- Because in APT n is large (market portfolio well diversified), we have $\text{Cov}[r_M, e_i] \approx 0$
- Dividing by σ_M^2 gives the beta of an asset

$$\begin{aligned}\beta_i &= b_{1i}\frac{\text{Cov}[f_1, r_M]}{\sigma_M^2} + b_{2i}\frac{\text{Cov}[f_2, r_M]}{\sigma_M^2} \\ &= b_{1i}\beta_{f_1} + b_{2i}\beta_{f_2}\end{aligned}$$

- Difference between CAPM and APT
  - In CAPM, the β of an asset is the sum of the factors' betas
  - Factors have β:s and assets have loading factors only (i.e., they correlate with market through the factors)

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Parameter estimation

- Annual return r_y formed from monthly returns $r_1, r_2, \ldots, r_{12}$

$$1 + r_y = (1 + r_1)(1 + r_2) \cdots (1 + r_{12})$$

- Assume that monthly returns are small:

$$1 + r_y \approx 1 + r_1 + r_2 + \cdots + r_{12}$$
$$\Rightarrow r_y = r_1 + r_2 + \cdots + r_{12}$$

- Monthly returns are equally distributed and uncorrelated

$$\bar{r}_y = 12\bar{r}$$
$$\sigma_y^2 = \mathbb{E}\left[\left(\sum_{i=1}^{12}(r_i - \bar{r})\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^{12}(r_i - \bar{r})^2\right] = 12\sigma^2$$

Parameter estimation

- Even periods of other length are possible
- If there are p periods in a year, then

$$\bar{r}_p = p\bar{r}_y$$
$$\sigma_p = \sigma_y\sqrt{p}$$

- When p becomes smaller, the ratio of standard deviation and expected value increases
  - $\Rightarrow$ Finding short term estimators becomes more difficult
  - If the yearly parameters are $\mathbb{E}[r_y] = 12\%$ and $\sigma_y = 15\%$, the monthly parameters $p = 1/12$ are $\mathbb{E}[r] = 1\%$ and $\sigma = 1/\sqrt{12} \cdot 15\% = 4.33\%$
  - The one month return is within the interval $1 \pm 4.33\%$ with a 68% probability, which is a rather wide confidence interval
- Thus, single period expected returns are hard to estimate reliably even if the time series are long

Parameter estimation

- Let the time series consist of n months with monthly expected return $\mathbb{E}[r]$ and standard deviation σ
- Unbiased estimator

$$\hat{\bar{r}} = \frac{1}{n}\sum_{i=1}^{n} r_i$$

$$\Rightarrow \mathbb{E}\left[\hat{\bar{r}}\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[r_i] = \frac{1}{n}\sum_{i=1}^{n}\bar{r} = \bar{r}$$

- Variance and standard deviation of the estimator are

$$\sigma^2_{\hat{\bar{r}}} = \mathbb{E}\left[(\hat{\bar{r}} - \bar{r})^2\right] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(r_i - \bar{r})\right]^2 = \frac{1}{n}\sigma^2$$

$$\Rightarrow \sigma_{\hat{\bar{r}}} = \frac{1}{\sqrt{n}}\sigma$$

Parameter estimation

- Standard deviation $\sigma_{\hat{\hat{r}}}$ of estimator $\hat{\hat{r}}$ decreases slowly with n, because $\sqrt{n}$ is in its denominator
  - Let monthly $\mathbb{E}[r] = 1\%$ and $\sigma = 4.33\%$ and consider $n = 12$ monthly time series

$$\sigma_{\hat{\hat{r}}} = \frac{1}{\sqrt{12}} 4.33\% = 1.25\%$$

  - Should we want to estimate the standard deviation which is within 10% of the expected returns ($0.1 \cdot 1\% = 0.10\%$), then we would need a time series of 156 years and 3 months

$$\sigma_{\hat{\hat{r}}} = \frac{1}{\sqrt{12}} 4.33\% = 0.10\%$$
$$\Rightarrow n = 1875 = 12 \cdot 156.25$$

Overview

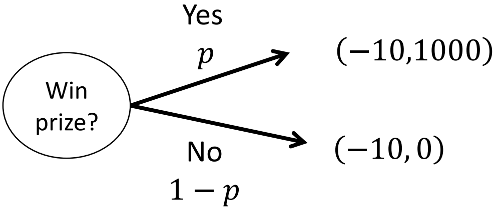
Investor's risk preferences

- We have discussed the formation of efficient portfolios
  - Mean-variance portfolio theory
  - CAPM
- Which one out of the many efficient portfolios should an investor select?
- Specifically, portfolios are characterized by the expected return and risk (standard deviation), that is, by tuple (r, σ)

  $\Rightarrow$ Which combination of these parameters should be selected?

Uncertain cash flows

- Cash flows typically uncertain
  - E.g. price of stock at a future date
- Modeled with random cash flows
  - $x \sim D$, where D is the distribution of the random vector x
  - E.g., consider a lottery ticket which costs 10 € and gives a 1000 € prize with probability p



Expected utility theory

(Von Neumann & Morgenstern 1944)

- Rational investors' preferences under uncertainty are consistent with a utility function $U : R \to R$
  - Wealth level W_1 preferred to wealth level W_2 if and only if

$$\mathbb{E}\left[U(W_1)\right] > \mathbb{E}\left[U(W_2)\right]$$

- Utility functions are
  - Non-decreasing and continuous
  - Unique up to positive affine transformations

    $\Rightarrow$ $U(W)$ and $V(W)$ yield same preferences if and only if

$$U(W) = aV(W) + b,$$

where $a > 0$

Utility theory example

- Investor invests in either
  - A: bank account for a profit of 6 k€
  - B: Stock that yields a profit of
    - 10 k€ (probability 0.2)
    - 5 k€ (probability 0.4)
    - 1 k€ (probability 0.4)
- Her utility function is $U(x) = \sqrt{x}$, where unit of x is k€

$$\mathbb{E}[U(A)] = U(6) = 2.45$$
$$\mathbb{E}[U(B)] = 0.2U(10) + 0.4U(5) + 0.4U(1) = 1.93$$

$\Rightarrow$ A is preferred to B, because
$\mathbb{E}[U(A)] = 2.45 > 1.93 = \mathbb{E}[U(B)]$

Widely used utility functions

- Linear

$$U(W) = W$$

- Exponential ($a > 0$)

$$U(W) = -e^{-aW}$$

- Logarithmic

$$U(W) = \ln W$$

- Power ($b \leq 1, b \neq 0$)

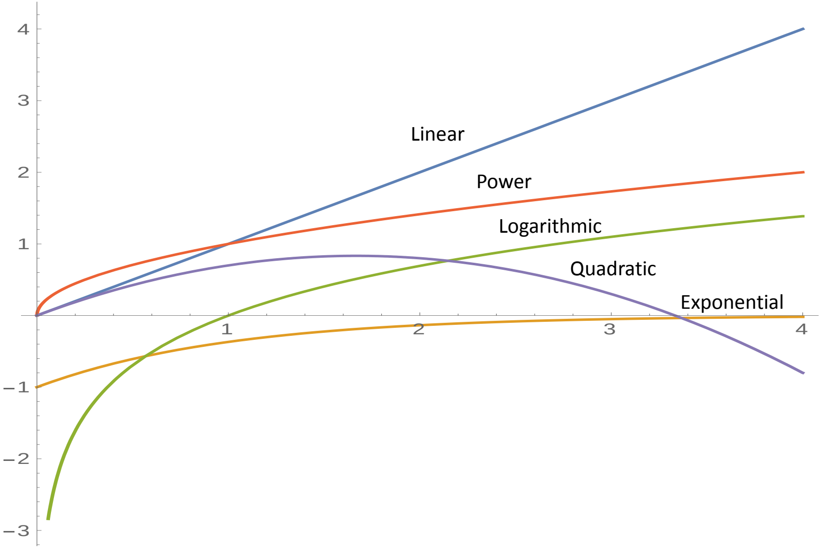
$$U(W) = bW^b$$

- Quadratic

$$U(W) = W - bW^2$$

(increasing for $W < 1/(2b)$)

Widely used utility functions



Risk aversion

- Investor with utility function U is:
  - Risk neutral, if U is linear
  - Risk averse, if U is concave

$$U(\lambda x + (1-\lambda)y) \geq \lambda U(x) + (1-\lambda)U(y)$$

  for all $0 \leq \lambda \leq 1$
- The certainty equivalent of a random wealth X is the certain wealth C for which

$$U(C) = \mathbb{E}\left[U(x)\right]$$

  - E.g, for a 50% chance to win 100 € and 50% chance of winning nothing, the certainty equivalent could be $C = 40$€
  - Certainty equivalent can be calculated as

$$C = U^{-1}\left(\mathbb{E}\left[U(x)\right]\right)$$

Risk aversion coefficient

- Arrow-Pratt risk aversion coefficient

$$a(x) = -\frac{U''(x)}{U'(x)}$$

  - Measures degree of risk aversion with the normalized concavity
- For the exponential utility function, risk aversion coefficient is constant

$$U(x) = -e^{-bx}$$
$$\Rightarrow a(x) = b$$

- For logarithmic utility function, risk aversion decreases with wealth

$$U(x) = \ln x$$
$$\Rightarrow a(x) = \frac{1}{x}$$

Elicitation of utility functions

- The utility function may help the investor choose investments that suit her
- Elicitation methods
  - Ask for certainty equivalents to get the value of U for different x
  - Select the functional form of utility function, fix some parameters to 1, proceed by carrying out more utility assessments
  - Questionnaires (Luenberger p. 238)

Utility function and the mean-variance criterion

- Risk aversion is related to the mean-variance criterion
- Example: Assume quadratic utility

$$U(x) = ax - \frac{1}{2}bx^2, \quad \text{where } a > 0, b \geq 0$$

  - This is increasing for $x \leq a/b$
  - Assume that the initial wealth level is 0 (the result can be extended for positive wealth levels)
  - Portfolio with random wealth y has

$$\mathbb{E}\left[U(y)\right] = \mathbb{E}\left[ay - \frac{1}{2}by^2\right] = a\,\mathbb{E}[y] - \frac{1}{2}b\,\mathbb{E}[y^2]$$

$$\Rightarrow \mathbb{E}\left[U(y)\right] = a\,\mathbb{E}[y] - \frac{1}{2}b\,\mathbb{E}[y]^2 - \frac{1}{2}b\,\mathrm{Var}[y]$$

$\Rightarrow$ If one seeks to attain a given expected return $\mathbb{E}[y]$, the portfolio is selected by minimizing variance subject to this requirement

Overview

Single factor model

Multifactor models

Arbitrage Pricing Theory (APT)

Parameter estimation

Utility theory and risk aversion