Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}$ be any nondecreasing function and

$$f^{\leftarrow}(y) = \inf\{x \mid f(x) \geq y\} \qquad f^{\rightarrow}(y) = \inf\{x \mid f(x) > y\}$$

($f^{\leftarrow}$ is the left-continuous inverse of f and $f^{\rightarrow}$ is the right-continuous inverse of f.) Check that

1. $(f^{\leftarrow})^{\leftarrow} = f^{-}$ with f^{-} the left-continuous version of f (i.e. $f^{-}(x) = \lim\limits_{t \uparrow x} f(t)$)

2. $(f^{\rightarrow})^{\rightarrow} = f^{+}$ with f^{+} the right-continuous version of f (i.e. $f^{+}(x) = \lim\limits_{t \downarrow x} f(t)$)

3. $f^{-}(f^{\leftarrow}(t)) \leq t \leq f^{+}(f^{\leftarrow}(t))$

Definition 1. A measurable function $f : \mathbb{R}_{+} \to \mathbb{R}$ that is positive after $M > 0$ ($f(x) > 0$ for $x > M$), is **regularly varying** with index α if

$$\lim_{t \to \infty} \frac{f(tx)}{f(t)} = x^{\alpha}.$$

If $\alpha = 0$, f is said to be **slowly varying**.

Exercise 2. Find an integer-valued random variable X whose distribution F has a regularly varying tail with index $\alpha < 0$ (i.e. $1 - F$ is regularly varying with index $\alpha < 0$).

Exercise 3. Find examples of the following:

1. A regularly varying function f with $\alpha > 0$ that's discontinuous for every $x > M$ for some $M \in \mathbb{R}$.

2. A slowly varying function satisfying $f \to \infty$ as $x \to \infty$.

3. A bounded function whose derivative converges to 0 as $x \to \infty$ that is not regularly varying for any α.