Exercise 1. Consider a finite interval $I = [a, b] \subset \mathbb{R}$. Show that the continuity points of an increasing function $f : I \to \mathbb{R}$ are dense in I.

Solution: Let

$$g(x) = \lim_{t \downarrow x} f(x) - \lim_{t \uparrow x} f(x).$$

Since f is increasing $g \geq 0$ and f is continuous at x iff $g(x) = 0$. Hence f is discontinuous at x iff $g(x) > 0$. For any $a < x < b$

$$g(x) \leq f(b) - f(a)$$

and for any finite set of points $a < x_1 < \cdots < x_n < b$

$$\sum_{i=1}^{n} g(x_i) \leq f(b) - f(a).$$

Since $f(b) - f(a)$ is finite, the sets

$$S_n = \left\{ x \in I \,\middle|\, g(x) \geq \frac{1}{n} \right\}$$

are finite for all n. Thus the set $S = \cup_n S_n$ is countable. The discontinuity points of f coincide with S and the continuity points $I \setminus S$ of f need to be dense in I.

Exercise 2. Modes of convergence:

- Construct a sequence of random variables X_n so that $X_n \to X$ in probability, but not almost surely.
- Construct a sequence of random variables Y_n so that $Y_n \to Y$ in distribution, but not in probability.

Solution: Simple random walk projected on a circle. Consider $\phi : \mathbb{R} \to \mathbb{S}^1$ defined via $x \mapsto \left(\cos\left(\pi - \pi\frac{1}{1+|x|}\right), \operatorname{sgn}(x)\sin\left(\pi - \pi\frac{1}{1+|x|}\right)\right)$. Now intuitively $\phi(0) = (1, 0)$ and $\phi(\infty) = \phi(-\infty) = (-1, 0)$. Since for any $M > 0$, the probability of a simple random walk S_t being in $[-M, M]$ converges to zero, $\phi(S_t) \to_P (-1, 0)$. However, by Pólya's recurrence theorem, S_t will return to 0 with probability 1. Hence the convergence is not almost sure. (Notice how extreme this example is. $X_n \to_P X$ but $\mathbb{P}[\lim X_n = X] = 0$.)

Let Y_n be -1 or 1 with equal probabilities (the same random variable for each n). Let $Y = -Y_n$.

Exercise 3. Show that if $k_n/n \to 0$, as $n \to \infty$ then $X_{(n-k_n,n)} \to \infty$ almost surely for i.i.d. X_i with an unbounded distribution.

Proof. Let $k_n/n \to 0$ as $n \to \infty$, we consider the complement of the event. Assume that for some $r \in \mathbb{R}$

$$\frac{k_n}{n} = \frac{1}{n}\sum_{i=1}^{n} 1\left(X_i \geq X_{(n-k_n,n)}\right) \geq \frac{1}{n}\sum_{i=1}^{n} 1\left(X_i \geq r\right)$$

infinitely often (this is exactly the event $X_{(n-k_n,n)} < r$). The left-hand side converges to zero while the right hand side converges to $\mathbb{P}\left[X > r\right] > 0$ almost surely. Hence, the above estimate may only hold infinitely often in a subset of Ω that is of measure zero. Thus $\mathbb{P}\left[X_{(n-k_n,n)} \to \infty\right] = 1$ □