Aalto University
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MS-E2114 Investment Science
Lecture 8: Derivative securities: forwards, futures and swaps

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Overview

Forward contracts

Swaps

Futures

Hedging

This lecture

- In earlier lectures, we have considered fixed cash flow streams and single-period random cash flow streams
  - These assets that have intrinsic value (e.g., bonds with coupons and payment of principal
- In this and next lectures, we consider derivative securities
- **Derivative security** = A financial asset whose payoff is determined by an external variable, e.g.:
  - Betting on the result of a football match
  - Weather (insurance for outdoor event organizers)
- We consider primarily derivative securities in which this variable is the price of the underlying asset

Examples of derivatives

- Forward contract
  - Commitment to buy 2000 kg sugar in 6 weeks for 0.30 €/kg
  - Value of this contract depends on what the price of sugar p is in 6 weeks
  - If price $p > 0.30$ €/kg, the contract has positive value
  - If price $p < 0.30$ €/kg, the commitment results in a loss
- Option
  - The right (which does not oblige) to buy 100 shares of company A in 6 months for 25 €
  - If the share price rises to $P > 25$ €, the value in 6 months is $(P - 25) \cdot 100$€
  - If price $P \leq 25$ €, the option is worthless, because these shares can be bought from the market at a lower price

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Forward contracts

- Forward contract: A commitment to buy or sell a predetermined amount of a commodity at a specific price and time
  - E.g. commitment to sell 1000 barrels of oil after 9 months for 100 €/barrel
  - The buyer of commodity has a long position
    - If the commodity price were to rise to $P > 100$ €/barrel, the buyer has secured the purchase at the lower predetermined price of 100 €
  - The supplier of commodity has a short position
    - If the commodity price were to fall to $P < 100$ €/barrel, the supplier has secured the sale at the higher predetermined price of 100 €/barrel
  - Forward-price F is the price paid at time of delivery
    - Usually F is determined so that the value of the contract is zero at the time of making the contract
    - Value of commodity determined on the spot market, where the commodity is traded

Forward price formula

Definition

(**Forward price formula**) Suppose that the asset whose spot price at time $t = 0$ is S can be stored at zero cost and that short selling is possible. Then the theoretical forward price F for delivery at T is

$$F = \frac{S}{d(0, T)},$$

where $d(0, T)$ is the risk-free discount factor for the time period $[0, T]$.

Forward price formula

Proof: If there are no storage costs, the forward price is implied by forward rates through the following arbitrage arguments:

- Take a short position (commitment to sell)
- Buy the commodity immediately from the market at price S and store it
- Supply it at the forward price F at time T
- If the contract has zero value, the NPV of the cash flow $(-S, F)$ must be zero (otherwise there is an arbitrage opportunity)

$$-S + d(0,T)F = 0$$

$$\Rightarrow F = \frac{S}{d(0,T)}$$

Forward price formula with carrying costs

- In practice, there may be costs due to storage
  - Cost of renting warehouses, insurance,...

Definition

(**Forward price formula with carrying costs**) Suppose that the holding costs for an asset are $c(k)$ per unit in period k, and that it can be sold short. Suppose the initial price is S. Then the theoretical forward price is

$$F = \frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)},$$

where $d(k,M)$ is the risk-free discount factor from k to M.

Forward price formula with carrying costs

Proof: The forward price can again be derived based on no arbitrage assumption

- Assume that the commodity is delivered after M periods
- Let $c(k)$ be the unit cost of storage from period k to $k+1$
- In a short position, one can purchase the commodity from the spot market at time 0 for price S, pay storage costs and deliver the commodity at end of period M
- Cash flow $\left(-S-c(0), \quad -c(1), \quad \cdots, \quad -c(M-1), \quad F\right)$
- Setting NPV$=0$ gives

$$0 = -S - \sum_{k=0}^{M-1} d(0,k)c(k) + d(0,M)F$$

$$\Rightarrow F = \frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)}$$

Value of forward contract

- Example
  - Price of sugar is 0.17€/kg. What is the forward price for a sugar delivery in 5 months if the interest rate is 9% and if storage costs 0.01€/ month?
  - Monthly interest rate is 9%/12 = 0.75%

$$d(k, M) = \frac{1}{1.0075^{M-k}}$$

$$F = 0.17 \cdot 1.0075^5 + \sum_{k=0}^{4} 0.01 \cdot 1.0075^{5-k}$$

$$\Rightarrow F = 0.23 \text{ €/kg}$$

- The value of the contract is zero at the time when it is made; but as the price of the (underlying) commodity changes, so does the value of the contract

Value of forward contract

Definition

(**The value of a forward**) Suppose a forward contract for delivery at time T in the future has a delivery price F_0 and a current forward price F_t. The value of the contract at time t is

$$f_t = (F_t - F_0)d(t, T),$$

where $d(t, T)$ is the discount factor over the period from t to T.

Proof: Consider a portfolio in which the investor takes a short position at forward price F_0 at time 0 (no cash flow) and a long position at forward price F_t at time t. The cash flow at time t is f_t. At time, T the investor receives the guaranteed cash flow $F_0 - F_t$. This cash flow is certain and thus its NPV must be 0:

$$f_t + (F_0 - F_t)d(t, T) = 0$$
$$\Rightarrow f_t = (F_t - F_0)d(t, T)$$

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Swaps

- A swap is an agreement to transform one type of cash flow to another
  - E.g. a variable interest rate loan to a fixed interest rate loan
- Plain vanilla swap
  - Fixed cash flow to variable cash flows
- Swaps are typically tailored for specific purposes
- The size of swap markets is hundreds of billions of euros
- **Example**: Plain vanilla interest rate swap
  - Party A pays a fixed semiannual interest rate to B
  - Party B pays a floating rate of interest to A
    - E.g. 6 month Euribor
  - If B has given a loan to C at the same floating rate, the swap with A eliminates the interest rate risk which B has as a result of having given a loan to C

Value of a commodity swap

- Consider a swap with M periods such that in each period party A
  1. receives the payment S_i which is equal to the value of N units of commodity at the spot price in period i
  2. pays a fixed price X for each commodity unit
- Let F_i be the unit forward price for delivery at the end of period i
  - When the contract is written, its value is 0
  - $S_i = d(0,i)F_i$
- This yields the NPV of the contract (for A) as

$$V = \sum_{i=1}^{M} d(0,i)(F_i - X)N$$

- Often X is defined so that the initial value of the contract is zero

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Futures

- Forwards cannot be easily traded in exchange markets
  - It is not easy to standardize delivery prices
  - Forwards with otherwise identical terms can have different delivery prices

    $\Rightarrow$ Challenges in book keeping
- Market are organized by exchanges

  $\Rightarrow$ No need to look for counterparts

  $\Rightarrow$ Delivery dates, quantities and place are standardized

  $\Rightarrow$ No need to assess counterparty risk

Futures

- In the futures market, changes in the future delivery price are accounted each day by a clearing house (**marking to market**)
  - Futures prices are determined by supply and demand
  - If the futures price increases:
    - Those in long position receive to their **margin account** a deposit which corresponds to the increase in value
    - This deposit is taken from those who have a short position
  - If the futures price decreases:
    - A deposit is taken from those in long position
    - This deposit is received by those in short position
  - The margin account needs an initial deposit for possible compensations (margin)
    - Typically about 5-10% of the price of the contract
    - If the margin becomes too small, a margin call is issued
    - The account (and the futures position) will be closed if the owner does not make an additional deposit

Example: Futures contract

- A company takes a long position for 5000 bushels of corn in March for 6.10 €/bushel
- The clearing house requires a margin that is initially 800 € and must not fall below 600 €
- The margin is recalculated on a daily basis
- If the futures price drops the next day to 6.07 €, the clearing house withdraws $5000 \cdot 0.03$ € $=$ 150 €
  - $\Rightarrow$ The balance of margin account becomes 800 - 150 = 650 €
- If the futures price drops the next day additional 0.02 €, the clearing house withdraws $5000 \cdot 0.02$ € $=$ 100 €
  - $\Rightarrow$ The balance becomes 550 €
- Because 550 € $<$ 600 € the clearing house makes a margin call to the company for an additional margin payment

Futures-forward equivalence

- Difference: Futures are marked-to-market on a daily basis, but with forwards, cash flows occur only when the contract is terminated

Theorem

*(**Futures-forward equivalence**) Suppose that interest rates are known to follow expectations dynamics. Then the theoretical futures and forward prices of respective contracts are identical.*

Futures-forward equivalence

Proof: Let F_0 = futures price and G_0 = forward price at time $t = 0$. Let there be T periods and let the discount rate from period j to $k > j$ be $d(j, k)$.

Consider the following two strategies.

A Take the following positions in futures

0: Go long in $d(1, T)$ units of futures

1: Increase position to $d(2, T)$

⋮

k: Increase position to $d(k + 1, T)$

⋮

$T - 1$: Increase position to 1

- In period $k + 1$, the profit from previous period is $(F_{k+1} - F_k)d(k + 1, T)$.
- Invest these at the risk free rate until T

Futures-forward equivalence

A At time T, the value of the investment made at the end of period k is

$$V_k = \frac{(F_{k+1} - F_k)d(k+1, T)}{d(k+1, T)} = F_{k+1} - F_k$$

This no cost strategy yields the profit

$$\sum_{k=0}^{T-1} (F_{k+1} - F_k) = F_T - F_0 = S_T - F_0$$

B Take a long position with a single forward contract

- No initial cash flow
- At the end the profit is $S_T - G_0$

- If the strategy is to buy A and sell B then
  - There is no commitment but the profit will be $G_0 - F_0$
  - It follows that $G_0 = F_0$ (or else there would be arbitrage) □

Expected spot price

- How does the futures price F of a commodity relate to the expected spot price at time T?
  - If $F < \mathbb{E}[S_T]$, a speculator buys a long position and sells the delivery on the spot market
  - If $F > \mathbb{E}[S_T]$, a speculator takes a short position and buys the commodity on the spot market to fulfil delivery
  - The speculator seeks to make a profit (not arbitrage)
- Parties can be interested in hedging rather than making profits through speculation
  - If there are more hedgers in the short position, then speculators balance the market by taking long position for a price such that $F < \mathbb{E}[S_T]$ (normal backwardation)
    - E.g., farmers wish to guarantee a minimum price for their crops
  - If there are more hedgers in the long position, then speculators balance the market for a price such that $F > \mathbb{E}[S_T]$ (contango)
    - E.g., uncertain supply of critical raw material for production

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Perfect hedge

- In the perfect hedge, one takes a position that is equal and opposite to the commitment that one wishes to hedge against
  - ⇒ Price risk is eliminated
  - Equivalent to making the deal of an anticipated future transaction today
  - Perfect hedging may not be possible if futures with suitable terms are not available
- Example:
  - A Finnish company delivers goods in 6 months to an American company and receives \$ 2 million upon delivery
  - The company can protect against exchange rate risk due to the possible increase in the value of € by taking a short position for \$2 million in the euro futures market
  - ⇒ Exchange rate risk is completely eliminated (but the counterparty risk of the American company defaulting still remains)

Minimum variance hedge

- Perfect hedging is not always possible
  - Commodity may not have a suitable futures market
  - The available contracts may not have suitable terms (e.g. delivery date)
  - The supply of futures contracts is insufficient
  - Markets are not liquid enough
- **Implication** = Mismatch between spot price and futures price
- Minimum variance hedge: Build a futures contract such that the variance of the future cash flow minimized
- Consider a commitment to sell W units of commodity at time T
  - At the spot price S_T, the delivery is worth $x = WS_T$
  - Hedge this position with h units worth of futures contracts

  $\Rightarrow$ At time T, cash flow is

$$y = x + (F_T - F_0)h$$

Minimum variance hedge

- Variance of cash flow $y = x + (F_T - F_0)h$ is

$$\begin{aligned}\mathrm{Var}[y] &= \mathbb{E}\left[\left(x - \bar{x} + (F_T - \bar{F}_T)h\right)^2\right] \\ &= \mathrm{Var}[x] + 2h\,\mathrm{Cov}[x, F_T] + h^2\,\mathrm{Var}[F_T]\end{aligned}$$

- Choose $h = h^*$ to minimize variance $\mathrm{Var}[y]$
  - Set the derivative with respect to h at $h = h^*$ to 0

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}h}\mathrm{Var}[y]\Big|_{h=h^*} &= 2\,\mathrm{Cov}[x, F_T] + 2h^*\,\mathrm{Var}[F_T] = 0 \\ \Rightarrow h^* &= -\frac{\mathrm{Cov}[x, F_T]}{\mathrm{Var}[F_T]}\end{aligned}$$

$\Rightarrow$ Minimum variance

$$\mathrm{Var}[y]\Big|_{h=h^*} = \mathrm{Var}[x] - \frac{\mathrm{Cov}[x, F_T]^2}{\mathrm{Var}[F_T]}$$

Minimum variance hedge

- When $x = WS_T$, the hedge is

$$h^* = -\frac{\text{Cov}[S_T, F_T]}{\text{Var}[F_T]} W \equiv -\beta W$$

- **Special case**: If the markets for commodities are identical so that $F_T = S_T$, we have $h^* = -W$ and thus

$$\begin{aligned}
\text{Var}[y]\big|_{h=h^*} &= \text{Var}[WS_T] - \frac{\text{Cov}[WS_T, F_T]^2}{\text{Var}[F_T]} \\
&= W^2\,\text{Var}[S_T] - \frac{W^2\,\text{Cov}[F_T, F_T]^2}{\text{Var}[F_T]} \\
\Rightarrow \text{Var}[y]\big|_{h=h^*} &= 0
\end{aligned}$$

Optimal hedging

- Minimum variance hedge does not account for risk preferences
  - One can use utility functions

$$\max_h \mathbb{E}\left[U(y)\right] = \max_h \mathbb{E}\left[U(x + (F_T - F_0)h)\right]$$

- Consider the quadratic utility function

$$U(y) = y - \frac{b}{2}y^2, \quad b > 0$$

$\Rightarrow$ Maximizing the expectation $\mathbb{E}[U(y)]$ gives

$$\begin{aligned}\max_h \mathbb{E}\left[U(y)\right] &= \max_h \mathbb{E}\left[y - \frac{b}{2}y^2\right] \\ &= \max_h \mathbb{E}[y] - \frac{b}{2}\mathbb{E}\left[(y - \bar{y} + \bar{y})^2\right]\end{aligned}$$

$$\Rightarrow \max_h \mathbb{E}\left[U(y)\right] = \max_h \mathbb{E}[y] - \frac{b}{2}\left(\text{Var}[y] + \mathbb{E}[y]^2\right)$$

Optimal hedging

- Optimum depends only on 1. and 2. moments (expectation and variance, respectively)
  - Attained in the border of the region defined by feasible combinations of $(\mathbb{E}[x], \mathrm{Var}[x])$
- Thus the problem is equivalent to maximizing

$$V(y) = \mathbb{E}[y] - r\,\mathrm{Var}[y], \qquad \text{for some } r > 0$$

  - r represents risk preferences
  - Reasonable choice, e.g., $r \approx 1/\left(2\,\mathbb{E}[x]\right)$
- The hedging problem becomes

$$\max_h V(y) = \max_h \{\mathbb{E}\left[x + (F_T - F_0)h\right] - r\,\mathrm{Var}[x + hF_T]\}$$

$$= \max_h \left\{\bar{x} + (\bar{F}_T - F_0)h - r\,\mathrm{Var}[x] - 2rh\,\mathrm{Cov}[x, F_T] - rh^2\,\mathrm{Var}[F_T]\right\}$$

Optimal hedging

- The problem can be solved by setting the derivative of $V(y)$ with respect to h at $h = h^*$ to 0

$$(\bar{F}_T - F_0) - 2r\,\mathrm{Cov}[x, F_T] - 2rh^*\,\mathrm{Var}[F_T] = 0$$

$$\Rightarrow h^* = \frac{\bar{F}_T - F_0}{2r\,\mathrm{Var}[F_T]} - \frac{\mathrm{Cov}[x, F_T]}{\mathrm{Var}[F_T]}$$

- Second term = minimum variance hedge
- First term = utility from the future as an investment
  - Reliable estimates for $\bar{F}_T = \mathbb{E}[F_T]$ are usually not available
  - Often the minimum variance hedge is used

Example: Optimal hedging

- Company gets $W = 500\,000$ barrels of oil in 3 months and seeks to hedge its long position from price risk with a short position in the futures market for oil
- The current price of oil futures with delivery in 3 months is $F_0 = 20$ €/barrel
- Company estimates that price of oil increases 5% from its current futures price in 3 months, that is,

$$\mathbb{E}\left[\frac{S_T}{F_0}\right] = \mathbb{E}\left[\frac{F_T}{F_0}\right] = \frac{\bar{F}_T}{F_0} = 1.05$$

- The standard deviation of this estimate is assessed to be 15%, that is, $\mathrm{Std}[F_T/F_0] = 0.15$

Example: Optimal hedging

- The optimal hedge is

$$h^* = \frac{\bar{F}_T - F_0}{2r\,\mathrm{Var}[F_T]} - \frac{\mathrm{Cov}[x, F_T]}{\mathrm{Var}[F_T]} = \frac{\bar{F}_T - F_0}{2rF_0^2\,\mathrm{Var}[F_T/F_0]} - \frac{\mathrm{Cov}[WF_T, F_T]}{\mathrm{Var}[F_T]}$$

$$= \frac{1}{2rF_0\,\mathrm{Var}[F_T/F_0]}\left(\frac{\bar{F}}{F_0} - 1\right) - W$$

$$\Rightarrow h^* = \frac{0.05}{2 \cdot r \cdot 20 \cdot 0.15^2} - 500\,000$$

- If $r = 10^{-6}$ represent the risk preferences of the company, then $h^* = -444\,444$
  - $\Rightarrow$ Take a smaller short position because price is believed to increase

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