MS-E2114 Investment Science
Lecture 9: Basic options theory

A. Salo, T. Seeve

Systems Analysis Laboratory
Department of System Analysis and Mathematics
Aalto University, School of Science

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Overview

Price processes

Options

Options pricing theory

This lecture

- The previous lecture covered contracts for forwards and futures
  - Simple derivative securities, theoretical price is straightforward to calculate using no arbitrage assumption
- This lecture covers price processes and options
  - The timing of the payoff of an option may vary

    $\Rightarrow$ Options pricing theory calls for the modelling of asset prices

Overview

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Price processes

- Multiperiod investment decisions can be analyzed by modelling asset prices as stochastic processes
- Discrete processes $\Rightarrow$ binomial lattices
  - These are simple and suitable for the analysis of many types of investments
  - Additive model

$$S(k+1) = aS(k) + u(k)$$

  - Multiplicative model

$$S(k+1) = u(k)S(k)$$

- Continuous processes $\Rightarrow$ Itô-processes
  - Price can change by any amount in a given time interval
  - Some Itô-processes have analytical solutions
  - Itô-process

$$\mathrm{d}x(t) = a(x,t)\,\mathrm{d}t + b(x,t)\,\mathrm{d}z$$

Additive model

- Consider the price process

$$S(k+1) = aS(k) + u(k), \quad k = 0, 1, \ldots, N-1,$$

  where $u(k)$ is random and a is constant (usually $a > 0$)
- $S(k)$ is therefore ($k = 1, 2, \ldots, N$)

$$\begin{aligned}
S(1) &= aS(0) + u(0) \\
S(2) &= aS(1) + u(1) = a^2 S(0) + au(0) + u(1) \\
S(3) &= aS(2) + u(2) = a^3 S(0) + a^2 u(0) + au(1) + u(2) \\
&\vdots \\
S(k) &= a^k S(0) + \sum_{i=0}^{k-1} a^{k-1-i} u(i)
\end{aligned}$$

Additive model

- Additive price process

$$S(k) = a^k S(0) + \sum_{i=0}^{k-1} a^{k-1-i} u(i)$$

- If $u(k)$ is normally distributed, the price process is a sum of normal random variables and hence normally distributed
- If $u(k)$ has zero expectation, the expected value of additive price process is

$$\mathbb{E}[S(k)] = a^k S(0)$$

- The additive model is partly unrealistic
  - $u(i)$'s can be negative $\Rightarrow$ $S(k)$ can become negative, too
  - The volatility of $S(k+1)$ given $S(k)$ is not proportional to $S(k)$, contrary to what is suggested by actual asset prices

Multiplicative model

- In the multiplicative model

$$S(k+1) = u(k)S(k), \quad k = 0, 1, \ldots, N-1$$

- Independent random variables $u(k)$ model the relative change of the price in one period
- The multiplicative model is additive in terms of the logarithms of price, because

$$\begin{aligned} S(k+1) &= u(k)S(k) && \mid \text{take } \ln \\ \Rightarrow \ln S(k+1) &= \ln S(k) + \ln u(k) \end{aligned}$$

- If $w(k) = \ln u(k)$ are normally distributed, then $u(k)$ are lognormally distributed and

$$\ln S(k) = \ln S(0) + \sum_{i=0}^{k-1} w(i)$$

Multiplicative model

- Multiplicative price process

$$\ln S(k) = \ln S(0) + \sum_{i=0}^{k-1} w(i)$$

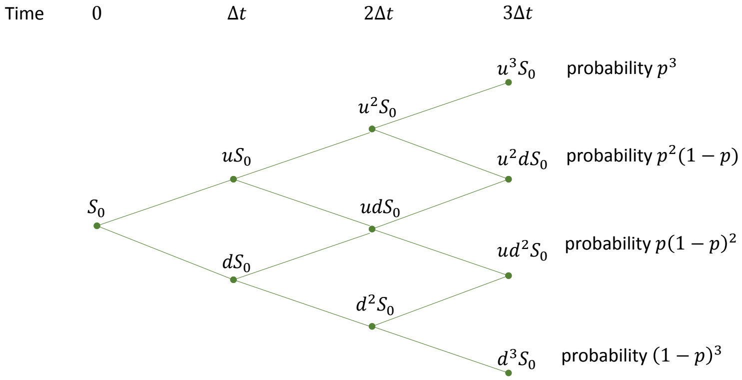
- If $\mathbb{E}[w(k)] = \nu$ and $\mathrm{Var}[w(k)] = \sigma^2$, then

$$\mathbb{E}[\ln S(k)] = \ln S(0) + k\nu$$
$$\mathrm{Var}[\ln S(k)] = k\sigma^2$$

- Real stock prices are approximately lognormal
- However, empirical distributions tend to have thicker tails than those of the lognormal distribution
  - $\Rightarrow$ Extreme price changes are more frequent than predicted by the lognormal distribution
- Otherwise lognormal distribution fits usually quite well

Binomial lattice

- Model parameters S_0, Δt, d, u and p
  1. Initial price S_0
  2. Period length Δt (e.g., one week)
  3. Relative price changes down d and up u
  4. Probability of price going up p



Binomial lattice

- The lattice can be re-parameterised in terms of

$$\nu = \mathbb{E}[\ln(S_T/S_0)]$$
$$\sigma^2 = \mathrm{Var}[\ln(S_T/S_0)]$$

- The parameters can be estimated from data

$$\hat{\nu} = \frac{1}{N}\sum_{k=0}^{N-1} \ln u(k) = \frac{1}{N}\sum_{k=0}^{N-1} \ln \frac{S(k+1)}{S(k)}$$
$$= \frac{1}{N}\sum_{k=0}^{N-1} [\ln S(k+1) - \ln S(k)] = \frac{1}{N} \ln \frac{S(N)}{S(0)}$$
$$\hat{\sigma}^2 = \widehat{\mathrm{Var}}[w(k)] = \widehat{\mathrm{Var}}[\ln u(k)]$$
$$= \frac{1}{N-1}\sum_{k=0}^{N-1} \left[\ln \frac{S(k+1)}{S(k)} - \hat{\nu}\right]^2$$

Fitting the lattice parameters

- Let $S(0) = 1$ so that

$$\mathbb{E}[\ln S(1)] = \mathbb{E}[\ln S(0) + w(0)] = p \ln u + (1-p) \ln d$$

- Then the variance of the logarithmic price is

$$\begin{aligned}\mathrm{Var}[\ln S(1)] =& p\left(\ln u - p \ln u - (1-p)\ln d\right)^2 \\ &+ (1-p)\left(\ln d - p\ln u - (1-p)\ln d\right)^2 \\ =& p(1-p)\left(\ln u - \ln d\right)^2\end{aligned}$$

- Denote $U = \ln u$ and $D = \ln d$ to obtain

$$pU + (1-p)D = \nu \Delta t$$
$$p(1-p)(U-D)^2 = \sigma^2 \Delta t$$

- The quantities ν and σ^2 can be estimated from data: there are three unknown parameters and only two equations

Fitting the lattice parameters

- This extra degree of freedom can be exploited to assign $d = 1/u$ (i.e., $D = -U$) to get

$$(2p - 1)U = \nu \Delta t$$
$$4p(1-p)U^2 = \sigma^2 \Delta t$$

- These equations give

$$p = \frac{1}{2} + \frac{1/2}{\sqrt{\sigma^2/\left(\nu^2 \Delta t\right) + 1}}, \quad U = \ln u = \sqrt{\sigma^2 \Delta t + (\nu \Delta t)^2}$$

- For small Δt, these are approximately equal to

$$p = \frac{1}{2}\left(1 + \frac{\nu}{\sigma}\sqrt{\Delta t}\right), \quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}$$

- We have now fitted the parameters of the binomial lattice to match the observed expectation and variance of the price process

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Options pricing theory

Options

- Option = A contract which gives its owner the right, but not obligation, to sell or buy a commodity at prespecified terms
  - Right to buy 1000 shares of company A for 20 € per share on 30 May 2018
  - Right to sell 10 tons of oil for 100 € / barrel in March 2019
- Terminology
  - Underlying asset = the commodity which the option gives the right to buy or sell
  - Call option = option to buy the commodity
  - Put option = right to sell the commodity
  - Expiration date = date by/upon which the option must be exercised (and after which the option expires)
  - Exercise/strike price = price paid for the asset when the option is exercised
  - Premium = price of the option

Options

- American option can be exercised at any time before expiration
- European option can be exercised only on expiration date
  - Classification refers to contract type, not location!
- Determining the premium
  - If the option is established between two parties, then the premium can be negotiated between them
  - The premium of options traded in exchanges are determined in the market
    - Commodity quantities, expiration dates and strike prices are all standardized
- Upon expiry, the value of an option depends on the price of the commodity and the strike price
  - If the price of company A stock is 20 € /share, the value of an expiring put option for selling 1000 shares at 25 € /share is $1000 \times (25 - 20)$ € $=$ 5000 €

Risks of options

- The risk associated with an option is asymmetric for the seller and the buyer
  - The buyer has the right - but no the obligation - to exercise the option
    - $\Rightarrow$ Possible loss is limited to the size of the premium
  - The seller of the options must fulfil her obligation if the buyer chooses to exercise the option
    - $\Rightarrow$ The seller may incur losses (e.g., when selling call options and the price of the asset increases considerably above the strike price)
    - $\Rightarrow$ Sellers are required to have margin accounts, which often must have a deposit worth as much as 50% of the value of the underlying asset
- Options are often purchased for purpose of hedging one's position from risks

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Value of an option

- The value of an option depends on:
  1. Price of underlying asset
     - Difference between the price of the underlying asset and the strike price
  2. Time value
     - Depends on the time to expiry; volatility of the price of the underlying asset; interest rates; and possible dividends of the asset
  3. Other factors
     - Interest rates
     - Asset price volatility
     - Etc.

Option value at expiration

- Consider a call with strike price K
  - If at the time of expiry T, the price of underlying asset S is higher than K, then the value of the call is $S - K$
  - If S is less than K, then the option is worthless

  $\Rightarrow$ Upon expiry, the value of the call is

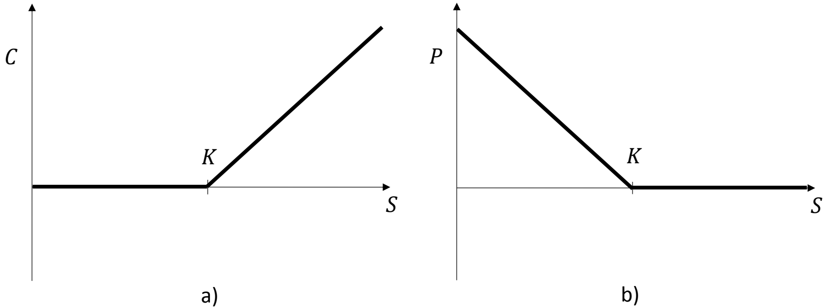
$$C = \max\{0, S - K\}$$

- Consider a put with strike price K
  - If at the time of expiry T, the price of underlying asset S is lower than K, then the value of the put is $K - S$
  - If S is greater than K, then the option is worthless

  $\Rightarrow$ Upon expiry, the value of the put is

$$P = \max\{0, K - S\}$$

Option value at expiration

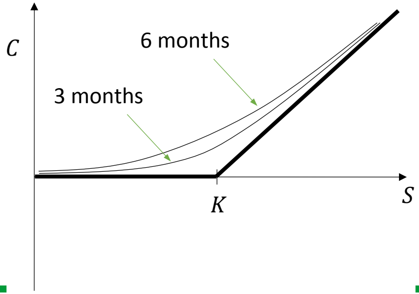


a) The value $C = \max\{0, S - K\}$ of a call

b) The values $P = \max\{0, K - S\}$ of a put

Time value of an option

- Let S_t be the price of the underlying asset at time $t < T$
- A call option is
  - **In the money** if $S_t > K$
  - **At the money** if $S_t = K$
  - **Out of the money** if $S_t < K$
- Even if the call option is out of the money, the option still has value, because the price of the underlying asset may become higher before expiry



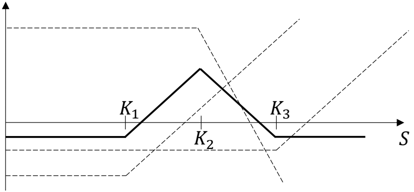
Other factors affecting the value of an option

- Consider a call option which is out of the money
  - The more volatile the asset price, the greater the chance that the price will exceed the strike price
- Higher interest rates make call options more valuable
  - A call is a method of purchasing the asset at time T by paying the strike price K at T and the premium immediately
    - $\Rightarrow$ K can be invested for time period T

Factor	Impact when factor increases: Call	Put
Price of underlying asset	+	-
Strike price	-	+
Time to expiry	+	+
Price volatility of underlying asset	+	+
Prevailing interest rate	+	-
Dividends	-	+

Combining options

- Options are often combined to construct a given desired financial position
- Example: Butterfly spread
  - Buy two calls with strike prices K_1 and K_3 such that $K_3 > K_1$
  - Sell two calls with strike price K_2 such that $K_1 < K_2 < K_3$
    - Usually K_2 chosen so that it is close to the price of the underlying asset
  - This portfolio has the following properties:
    A: It yields a profit if the price of the underlying asset does not change much
    B: It has a low risk even if the price of the underlying asset would change significantly



Put-call parity

Theorem

*(**Put-call parity**) Let C and P be the prices of a European call and a European put, both with a strike price of K and defined on the same stock with price S. The put-call parity states that*

$$C - P + dK = S,$$

where d is the risk-free discount factor to the expiration date.

Put-call parity

Proof: Consider the following position at time $t < T$:

1. Buy a call
2. Sell a put option
3. Deposit dK at the risk-free rate

A: If $S \geq K$ at T, then the call yields a profit $S - K$, the put is worthless, and the deposit yields the cash flow K

 $\Rightarrow$ Total cash flow is $(S - K) + K = S$

B: If $S < K$ at T, then the call is worthless, the short position on put yields a loss $K - S$ and the deposit yields the cash flow K

 $\Rightarrow$ Total cash flow is $K - (K - S) = S$

Thus, the positions have the same cash flow as the underlying asset at time T

$\Rightarrow$ The position and the asset must have the same value at the preceding time t, too

Hence $C - P + dK = S$. □

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