Exercise 1. Let $f : \mathbb{R} \to \mathbb{R}$ be any nondecreasing function and

$$f^{\leftarrow}(y) = \inf\{x \mid f(x) \geq y\} \qquad f^{\rightarrow}(y) = \inf\{x \mid f(x) > y\}$$

($f^{\leftarrow}$ is the left-continuous inverse of f and $f^{\rightarrow}$ is the right-continuous inverse of f.) Check that

1. $(f^{\leftarrow})^{\leftarrow} = f^{-}$ with f^{-} the left-continuous version of f (i.e. $f^{-}(x) = \lim_{t\uparrow x} f(t)$)

2. $(f^{\rightarrow})^{\rightarrow} = f^{+}$ with f^{+} the right-continuous version of f (i.e. $f^{+}(x) = \lim_{t\downarrow x} f(t)$)

3. $f^{-}(f^{\leftarrow}(t)) \leq t \leq f^{+}(f^{\leftarrow}(t))$

Solution: (1): By definition

$$(f^{\leftarrow})^{\leftarrow}(t) = \inf\{y \mid f^{\leftarrow}(y) \geq t\} = \inf\{y \mid \inf\{x \mid f(x) \geq y\} \geq t\} := \inf S$$

Notice that the left-continuous inverse $f^{\leftarrow}$ is increasing. Thus if $x \in S$ then $x + M \in S$ for all $M > 0$ and S is an interval (half-open or open) from some $s \in \mathbb{R}$ to ∞. Consequentially, $\inf S = \sup(\mathbb{R} \setminus S) := \sup T$. Hence, we want to find

$$\sup\{y \mid \inf\{x \mid f(x) \geq y\} < t\}.$$

Notice that for any $u \in \mathbb{R}$

$$\inf\{x \mid f(x) \geq f(u)\} \leq u$$

thus $f(t - \varepsilon) \in T$ for all $\varepsilon > 0$ and

$$\sup_{\varepsilon>0} f(t-\varepsilon) = \lim_{x\uparrow t} f(x) \leq \sup T.$$

On the other hand,

$$\inf\left\{x \;\middle|\; f(x) \geq \lim_{x\uparrow t} f(x) + \varepsilon\right\} \geq t$$

for all $\varepsilon > 0$. Hence $\sup T \leq \lim_{x\uparrow t} f(x) + \varepsilon$ for all $\varepsilon > 0$ and

$$\lim_{x\uparrow t} f(x) \leq \sup T \leq \lim_{x\uparrow t} f(x),$$

hence the claim.

(2): By definition

$$\left(f^{\rightarrow}\right)^{\rightarrow}(t) = \inf\left\{y \mid f^{\rightarrow}(y) > t\right\} = \inf\left\{y \mid \inf\left\{x \mid f(x) > y\right\} > t\right\} := \inf S$$

Notice that for any $u \in \mathbb{R}$

$$\inf\left\{x \mid f(x) > f(u)\right\} \geq u$$

hence $f(t+\varepsilon) \in S$ for all $\varepsilon > 0$ and

$$\inf_{\varepsilon>0} f(t+\varepsilon) = \lim_{x\downarrow t} f(x) \geq \inf S.$$

On the other hand

$$\inf\left\{x \,\middle|\, f(x) > \lim_{x\downarrow t} f(x) - \varepsilon\right\} \leq t$$

Hence $\inf S \geq \lim_{x\downarrow t} f(x) - \varepsilon$ for all $\varepsilon > 0$. Hence

$$\lim_{x\downarrow t} f(x) \leq \inf S \leq \lim_{x\downarrow t} f(x)$$

and the claim follows.

(3): Let $t \in \mathbb{R}$. Let $x_t = f^{\leftarrow}(t) = \inf\left\{x \mid f(x) \geq t\right\} = \sup\left\{x \mid f(x) < t\right\}$. Now since $f(x) < t$ for all $x < x_t$ and $f(x) \geq t$ for all $x > x_t$

$$f^{-}(x_t) = \lim_{x\uparrow x_t} f(x) \leq t \leq \lim_{x\downarrow x_t} f(x) = f^{+}(x)$$

Definition 1. A measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ that is positive after $M > 0$ ($f(x) > 0$ for $x > M$), is **regularly varying** with index α if

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^{\alpha}.$$

If $\alpha = 0$, f is said to be **slowly varying**.

Exercise 2. Find an integer-valued random variable X whose distribution F has a regularly varying tail with $\alpha > 0$ (i.e. $1 - F$ is regularly varying).

Solution: The function

$$f(x) = \lfloor x \rfloor^{-\alpha}$$

is regularly varying:

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = \lim_{t\to\infty} \frac{(tx + O(1))^{-\alpha}}{(t + O(1))^{-\alpha}} = \lim_{t\to\infty} \frac{(x + o(1))^{-\alpha}}{(1 + o(1))^{-\alpha}} = x^{-\alpha}.$$

Hence a random variable X whose distribution is $F(x) = 1 - \lfloor x \rfloor^{-\alpha}$, $x > 1$, has a regularly varying tail. For an integer $n \geq 2$

$$\mathbb{P}\left[X = n\right] = n^{-\alpha} - (n-1)^{-\alpha}$$

otherwise the probability is zero.

Exercise 3. Find examples of the following:

1. A regularly varying function f with $\alpha > 0$ that's discontinuous for every $x > M$ for some $M \in \mathbb{R}$.
2. A slowly varying function satisfying $f \to \infty$ as $x \to \infty$.
3. A bounded function whose derivative converges to 0 as $x \to \infty$ that is not regularly varying for any α.

Solution:

1. By the solution of the previous exercise, $f(x) = x^{-\alpha} + O(1)$ is regularly varying. Choose $f(x) = x^{-\alpha} + \chi_{\mathbb{Q}}$.
2. Choose $f(x) = \exp\left(\sqrt{\log x}\right)$:

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = \exp\left(\sqrt{\log tx} - \sqrt{\log t}\right) = \exp\left(\sqrt{\frac{\log tx}{\log t}} - 1\right)$$

now

$$\frac{\log tx}{\log t} = \frac{\log t + \log x}{\log t} = \frac{1 + o(1)}{1}$$

which yields the claim.

3. Let $f(x) = 2 + \sin \log x$. Note that $f'(x) = \frac{\cos \log x}{x}$ which clearly converges to 0 as $n \to \infty$. However, for example

$$\frac{2 + \sin \log e^{\pi} x}{2 + \sin \log x} = \frac{2 + \sin(\log x + \pi)}{2 + \sin \log x}$$

Take the sequences $y_n = e^{2\pi n}$ and $z_n = e^{\frac{\pi}{2} + 2\pi n}$. Notice that

$$\frac{2 + \sin(\log y_n + \pi)}{2 + \sin \log y_n} = \frac{2 + \sin((2+1)\pi n)}{2 + \sin 2\pi n} = 1$$

however,

$$\frac{2 + \sin\left(\frac{3}{2}\pi + 2\pi n\right)}{2 + \sin\left(\frac{\pi}{2} + 2\pi n\right)} = \frac{1}{3}$$

Hence,

$$\lim_{t \to \infty} \frac{f(tx)}{f(t)}$$

doesn't exist for $x = e^{\pi}$.