Aalto University
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MS-E2114 Investment Science
Lecture 11: Options pricing in continuous time

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Overview

Stochastic processes

Black-Scholes equation

Application of Black-Scholes equation

Synthetic and exotic options

Computational methods

Previous lecture

- In the last lectures, we have priced options in the binomial lattice
  - The asset price dynamics were calculated using the binomial lattice formulas derived in Lecture 9
  - Lecture 10 presented recursive formulas for calculating the arbitrage-free price of a derivative
- In this lecture, we will price options using a continuous-time model for the price of the underlying asset
  - Dynamics modelled with stochastic processes
  - Arbitrage-free prices of derivatives expressed with stochastic differential equations

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Wiener-process

- Consider the stochastic process

$$\begin{array}{rl} z(t_{k+1}) & = z(t_k) + \varepsilon(t_k)\sqrt{\Delta t} \\ t_{k+1} & = t_k + \Delta t \end{array}, \quad k = 0, 1, \ldots, N,$$

  where $\varepsilon(t_i)$ and $\varepsilon(t_j)$ ($i \neq j$) are independent and $\varepsilon(t_i) \sim \mathcal{N}(0, 1)$
  - $\mathrm{Cov}[\varepsilon(t_i), \varepsilon(t_j)] = 0$ and $\varepsilon(t_i)$ are normally distributed with mean 0 and variance 1
  - This process is a random walk
- For any $j < k$, it holds that

$$z(t_k) - z(t_j) = \sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t}$$

Wiener-process

- As a result, $z(t_k) - z(t_j)$ is normally distributed and

$$\mathbb{E}\left[z(t_k) - z(t_j)\right] = \sum_{i=j}^{k-1} \mathbb{E}[\varepsilon(t_i)]\sqrt{\Delta t} = 0$$

$$\text{Var}\left[z(t_k) - z(t_j)\right] = \mathbb{E}\left[\sum_{i=j}^{k-1} \varepsilon(t_i)\sqrt{\Delta t}\right]^2 = \mathbb{E}\left[\sum_{i=j}^{k-1} \varepsilon(t_i)^2 \Delta t\right]$$
$$= (k-j)\Delta t = t_k - t_j$$

- In the limit $\Delta t \to 0$, the random walk $z(t)$ becomes the Wiener process (or alternatively, the standard Brownian motion) defined by the equation

$$\mathrm{d}z = \varepsilon(t)\sqrt{\mathrm{d}t},$$

where $\varepsilon(t) \sim \mathcal{N}(0, 1)$

Wiener-process

- Wiener process $z(t)$ is characterized by the following properties
  1. For any $s < t$, $z(t) - z(s)$ is normally distributed such that $\mathbb{E}[z(t) - z(s)] = 0$ and $\text{Var}[z(t) - z(s)] = t - s$
  2. For any $0 \leq t_1 < t_2 \leq t_3 < t_4$, differences $z(t_2) - z(t_1)$ and $z(t_4) - z(t_3)$ are uncorrelated
  3. $z(t_0) = 0$ with probability 1
- The Wiener process in not differentiable anywhere, because

$$\mathbb{E}\left[\left(\frac{z(s) - z(t)}{s - t}\right)^2\right] = \frac{s - t}{(s - t)^2} = \frac{1}{s - t} \to \infty$$

when $s \to t$

  - The term term $\mathrm{d}z/\,\mathrm{d}t$ is called white noise

Generalized Wiener-process and Itô process

- The generalized Wiener process (or alternatively, Brownian motion with drift) is an extension of the Wiener process which has a term for deterministic shift

$$\mathrm{d}x(t) = a\,\mathrm{d}t + b\,\mathrm{d}z,$$

where $x(t)$ is a stochastic process, a and b are constants, dt is differential in time, and z is a Wiener process
- Integration yields

$$x(t) = x(0) + at + bz(t)$$

- Itô-process is an extension of the generalized Wiener process such that the deterministic and stochastic shifts are functions on x and t, defined through

$$\mathrm{d}x(t) = a(x,t)\,\mathrm{d}t + b(x,t)\,\mathrm{d}z,$$

where $a(x,t)$ and $b(x,t)$ are integrable functions

Geometric Brownian motion and standard Itô form

- A special case of the Itô-process is the geometric Brownian motion, defined through

$$\mathrm{d}x(t) = ax\,\mathrm{d}t + bx\,\mathrm{d}z$$

- The multiplicative model

$$\ln S(k+1) - \ln S(k) = w(k), \quad w(k) \sim \mathcal{N}(\nu, \sigma^2)$$

has the continuous time counterpart

$$\mathrm{d}\ln S(t) = \nu\,\mathrm{d}t + \sigma\,\mathrm{d}z$$

- This continuous time model is a geometric Brownian motion, which can be written[1] in the standard Itô form as

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}z, \quad \text{where } \mu = \nu + \frac{1}{2}\sigma^2$$

[1] This form is obtained by expanding $\mathrm{d}\ln S(t)$ using differential calculus

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Itô's lemma

- There is the extra term $\sigma^2/2$ in the Itô equation of $S(t)$, because the random variables have order $\sqrt{\mathrm{d}t}$, and hence their squares produce first-order (rather than second order) effects
- The systematic method for making such transformations generally is the Itô's lemma
- The proof of Itô's lemma is omitted here; however, a sketch of the proof can be found in Luenberger's book

Itô's lemma

Theorem

*(**Itô's lemma**) Suppose that the random process x is defined by the Itô process*

$$\mathrm{d}x(t) = a(x,t)\,\mathrm{d}t + b(x,t)\,\mathrm{d}z, \tag{1}$$

where $z(t)$ is a standard Wiener process. If the process $y(t)$ is defined by $y(t) = F(x,t)$, then $y(t)$ satisfies the Itô equation

$$\mathrm{d}y(t) = \left(\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}b^2\right)\mathrm{d}t + \frac{\partial F}{\partial x}b\,\mathrm{d}z,$$

where z is the same Wiener process as in Equation (1).

Itô's lemma example

- For example, define $x(t) = \ln S(t)$ by the Itô process

$$\mathrm{d}x = \mathrm{d}\ln S(t) = \nu\,\mathrm{d}t + \sigma\,\mathrm{d}z$$

- By Itô's lemma, the random process $y(t) = F(x,t) = S(t) = e^{\ln S(t)} = e^x$ satisfies

$$\begin{aligned}
\mathrm{d}y &= \left(\frac{\partial e^x}{\partial x}\nu + \frac{\partial e^x}{\partial t} + \frac{1}{2}\frac{\partial^2 e^x}{\partial x^2}\sigma^2\right)\mathrm{d}t + \frac{\partial e^x}{\partial x}\sigma\,\mathrm{d}z \\
&= \left(\nu + \frac{1}{2}\sigma^2\right)e^x\,\mathrm{d}t + e^x\sigma\,\mathrm{d}z \\
\Rightarrow \frac{\mathrm{d}S(t)}{S(t)} &= \left(\nu + \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\,\mathrm{d}z
\end{aligned}$$

Black-Scholes equation

- We have priced options in binomial lattices
- Analogous results can be derived by using stochastic differential equations
- Assume that
  - Price of underlying asset S follows the geometric Brownian motion

    $$\mathrm{d}S = \mu S\,\mathrm{d}t + \sigma S\,\mathrm{d}z, \tag{2}$$

    where z is a Wiener process
  - Value of the risk-free asset B satisfies the differential equation

    $$\mathrm{d}B = rB\,\mathrm{d}t$$

  - $f(S, t)$ is the value of a derivative security of the underlying asset S at time t

Black-Scholes equation

Theorem

*(**Black-Scholes equation**) Suppose that the price of a security is governed by the geometric Brownian motion* (2) *and the interest rate is r. A derivative of this security has a price $f(S, t)$, which satisfies the partial differential equation*

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = rf.$$

Black-Scholes equation

Proof: Apply Itô's lemma to $f(S, t)$:

$$\mathrm{d}f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \mathrm{d}t + \frac{\partial f}{\partial S} \sigma S \, \mathrm{d}z$$

Form a replicating portfolio from the underlying asset and the risk-free asset, i.e., invest x_t in the underlying asset and y_t in the risk-free asset. The value differential of this portfolio is

$$\mathrm{d}G = x_t \, \mathrm{d}S + y_t \, \mathrm{d}B = x_t(\mu S \, \mathrm{d}t + \sigma S \, \mathrm{d}z) + y_t r B \, \mathrm{d}t$$

$$\Rightarrow \mathrm{d}G = (x_t \mu S + y_t r B) \, \mathrm{d}t + x_t \sigma S \, \mathrm{d}z$$

The amounts x_t, y_t are selected such that the value G of the portfolio is identical to the value of the derivative $f(S, t)$. As a result, the coefficients of differentials dt and dz are identical. From the coefficients of dz, we obtain

$$x_t \sigma S = \frac{\partial f}{\partial S} \sigma S \Rightarrow x_t = \frac{\partial f}{\partial S}.$$

Black-Scholes equation

Furthermore, the portfolio must have the same value as the derivative asset:

$$G = x_t S + y_t B = \frac{\partial f}{\partial S} S + y_t B = f(S,t)$$
$$\Rightarrow y_t = \frac{1}{B}\left[f(S,t) - S\frac{\partial f}{\partial S}\right]$$

Substitute x_t and y_t and choose the coefficients of the dt terms so that they are identical to obtain

$$\frac{\partial f}{\partial S}\mu S + \frac{1}{B}\left[f(S,t) - S\frac{\partial f}{\partial S}\right] rB = \frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2$$
$$\Rightarrow \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf. \qquad \square$$

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Application of Black-Scholes equation

- In general, the Black-Scholes equation does not have a closed form solution
- Certain special cases satisfy the equation
  - A derivative whose value is the same as that of the underlying asset (i.e., $f(S,t) = S$)

$$\frac{\partial f}{\partial t} = 0 \wedge \frac{\partial f}{\partial S} = 1 \wedge \frac{\partial^2 f}{\partial S^2} = 0$$
$$\Rightarrow 0 + 1rS + 0 = rS$$

  - Risk-free asset as a derivative instrument (i.e., $f(S,t) = e^{rt}$)

$$\frac{\partial f}{\partial t} = re^{rt} \wedge \frac{\partial f}{\partial S} = 0 \wedge \frac{\partial^2 f}{\partial S^2} = 0$$
$$\Rightarrow re^{rt} + 0 + 0 = re^{rt}$$

Application of Black-Scholes equation

- How to use Black-Scholes equation?
  - Pick or guess $f(S,t)$: If it does not satisfy the BS-equation, there are arbitrage opportunities

    $\Rightarrow$ Mispriced asset!
  - Give boundary conditions (e.g., value of option on expiry) and solve the equation
    - For example, the boundary conditions of a European call option are

$$C(0,t) = 0, \quad C(S,T) = \max(S-K,0)$$

    - An American call can be exercised before expiry

      $\Rightarrow$ The value of the option satisfies

$$C(S,t) \geq \max(S-K,0)$$

Application of Black-Scholes equation

- Example: Consider an American call with unlimited time to expiry (perpetual call)
  - Boundary conditions

$$C(S,t) \geq \max\{S - K, 0\}$$
$$C(S,t) \leq S$$

  - Solution $C(S,t) = S$ satisfies these conditions
  - Interpretation: The price of the underlying asset will in the long run increase so much that the strike price of the option becomes irrelevant
    - $\Rightarrow$ The option and the call have the same value
- The BS equation has an analytical solution for the European call

Call option formula

Theorem

(***Black-Scholes call option formula***) *Consider a European call option with strike price K and expiration time T. If the underlying stock pays no dividends during the time $[0, T]$ and if interest is compounded continuously at a constant rate r, the Black-Scholes solution is $f(S, t) = C(S, t)$, defined by*

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad \textit{where}$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

and where $N(x)$ denotes the standard cumulative normal probability distribution $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,\mathrm{d}y$

Call option formula example

- Let us revisit the example in Lecture 10 in which the stock price is 80 € and volatility 0.40
- Consider a European call which expires in four months with the strike price 85 €
- What is the price of the option, when the risk-free rate is 8% and no dividends are paid?

$$S = 80, \quad K = 85 \quad r = 0.08, \quad \text{and } \sigma = 0.40, \text{ so}$$

$$d_1 = \frac{\ln(80/85) + (0.08 + 0.40^2/2)(4/12)}{0.40\sqrt{4/12}} = 0.0316$$

$$d_2 = d_1 - 0.40\sqrt{4/12} = -0.2625$$

and $N(d_1) = 0.4874, N(d_2) = 0.3965,$ and hence

$$C(S,t) = 80 \cdot 0.4874 - 85 \cdot e^{-0.08(4/12)} \cdot 0.3965 = 6.18$$

- The value of the call is 6.18 €, which is slightly less than the price we obtained from the binomial lattice (6.40 €)

Delta Δ

- <u>Delta Δ</u> measures how sensitive the value of the derivative (e.g., an option) is with respect to changes in the price of the underlying asset

$$\Delta = \frac{\partial f(S,t)}{\partial S} \approx \frac{\Delta f(S,t)}{\Delta S}$$

  - By Black-Scholes, the delta of a European call is $N(d_1)$
- E.g., an investor thinks that a call is over priced (so selling them can lead to arbitrage)
  - Sell n call options
  - Buy $\Delta \cdot n$ shares of the underlying asset
- This portfolio is delta-neutral (i.e. immune to changes in the value of the underlying asset)

$$\frac{\partial}{\partial S}(-nC(S,t) + \Delta \cdot nS) = n(-\Delta + \Delta) = 0$$

- Delta depends on S and t

$\Rightarrow$ Portfolio must be rebalanced (continuously)

Gamma Γ and theta Θ

- The amount of required rebalancing is related to the curvature of the value (second order derivative) ; this is the gamma

$$\Gamma = \frac{\partial^2 f(S,t)}{\partial S^2}$$

- Theta Θ is the change in the value of a derivative with respect to time

$$\Theta = \frac{\partial f(S,t)}{\partial t}$$

- Over time, the value of the option approaches the value that is has at exercise
  - Time value diminishes $\Rightarrow$ Θ is negative for options
- The Taylor approximation for option value

$$\delta f \approx \Delta \cdot \delta S + \frac{1}{2}\Gamma \cdot (\delta S)^2 + \Theta \cdot \delta t$$

Option value Taylor approximation

- Let $S = 43$ €, volatility $\sigma = 0.20$, and risk-free rate $r = 0.10$
- Consider a European call which expires in $T - t = 6$ months with strike price $K = 40$ €
- The option price is calculated with Black-Scholes call option formula

$$d_1 = 0.936, \quad d_2 = 0.794 \Rightarrow C = 5.56$$

- Delta, gamma and theta are

$$\Delta = N(d_1) = 0.825, \quad \Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T-t}} = 0.143$$

$$\Theta = -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-rT}N(d_2) = -6.127$$

- If the price of stock rises by one euro in a week, the value of the option becomes

$$C' \approx C + \delta C = 5.56 + \Delta \cdot 1 + \frac{1}{2}\Gamma \cdot 1^2 + \Theta \cdot \frac{1}{52} = 6.22$$

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Synthetic options

- A return identical to an option can be obtained from a portfolio of the underlying asset and the risk-free asset
  - Portfolio value tracks the value of the option, but the portfolio must be continuously rebalanced
- A synthetic option can be constructed as follows:
  1. Define the value C of an option (e.g., using binomial lattice or Black-Scholes)
  2. Invest ΔS in the underlying asset and the rest $C - \Delta S$ at the risk-free rate
  3. Rebalance the portfolio frequently so that the portfolio has the required Δ

Synthetic option example

Weeks remaining	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
XON price	35.50	34.63	33.75	34.75	33.75	33.00	33.88	34.50	33.75	34.75	34.38	35.13	36.00	37.00	36.88	38.75	37.88	38.00	38.63	38.50	37.50
Call price	2.62	1.96	1.40	1.89	1.25	0.85	1.17	1.42	0.96	1.40	1.10	1.44	1.94	2.65	2.44	4.10	3.17	3.21	3.76	3.57	2.50
Delta	0.701	0.615	0.515	0.618	0.498	0.397	0.494	0.565	0.456	0.583	0.522	0.624	0.743	0.860	0.858	0.979	0.961	0.980	0.998	1.000	
Portfolio value	2.62	1.96	1.39	1.87	1.22	0.81	1.14	1.41	0.96	1.38	1.13	1.49	2.00	2.69	2.53	4.08	3.16	3.22	3.76	3.57	2.50
Stock portfolio	24.89	21.28	17.37	21.47	16.79	13.09	16.74	19.48	15.39	20.27	17.94	21.92	26.74	31.80	31.65	37.92	36.39	37.25	38.56	38.50	
Bond portfolio	-22.27	-19.32	-15.98	-19.59	-15.58	-12.28	-15.60	-18.07	-14.43	-18.89	-16.81	-20.43	-24.75	-29.11	-29.12	-33.84	-33.23	-34.03	-34.79	-34.93	

- A synthetic call option on Exxon stock with a strike price of 35 € and with 20 weeks to expiration is constructed by buying the stock and selling the risk-free asset at 10%
- The portfolio is adjusted each week based on the value of delta Δ at that time

Exotic options

- Some options are more complicated than the American and European options we have treated
  1. Bermudan option: Early exercise possible on specific dates before expiry
  2. Compound option: An option on another option
  3. Chooser option: The holder specifies after a given time whether the option is a call or a put
  4. CAP option: Automatically exercised if the price of underlying asset exceeds the specified given limit
     - E.g., if a 20 € CAP-call option has strike 60 €, it will be automatically exercised when the stock price exceeds 80 €
  5. Knockout option:Expires if the price of underlying asset reaches the specified level
     - Call expires if price of underlying asset below knockout level ("down and out")
     - Put expires if price of underlying asset above knockout level ("up and out")

Exotic options

6. Discontinuous option: Profit depends discontinuously on the price of the underlying asset
   - E.g., return 100 € if the price of the underlying asset is above the strike price at expiry; otherwise 0
7. Digital option: Has a payoff 1 € if the corresponding European option is in the money and 0 € otherwise
8. Lookback option: Exercise price is determined by the minimum and maximum values obtained by the underlying asset during the period of the option
   - Put option exercise price = highest value of the underlying asset during the option period
   - Call exercise price = lowest value of the underlying asset during the option period

   $\Rightarrow$ Lookback options have always a positive value at expiry

   $\Rightarrow$ They are expensive
9. Asian option: Profit depends on the average underlying asset price S_{avg} during the period of the option

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Monte-Carlo simulation

- Let the underlying asset of a derivative follow geometric Brownian motion
$$dS = rS\,dt + \sigma S\,dz,$$
where r is the risk-free rate and z is a Wiener process
  - Note that when pricing using risk neutral probabilities, the "drift"-term is r (and not μ)
- Value of the underlying asset can be simulated as
$$S(t_k + \Delta t) = S(t_k) + rS(t_k)\Delta t + \sigma S(t_k)\varepsilon(t_k),$$
where $\varepsilon(t_k)$ are normally distributed with expected value 0 and variance Δt

  $\Rightarrow$ Obtain $S(t)$ and $f(S, t)$ $\Rightarrow$value of derivative at time T
- With linear pricing, the value of the derivative is the discounted risk neutral expected value of $f(S, T)$
  - The value can be estimated as
$$\hat{P} = e^{-rT}\text{Avg}\left[f(S, T)\right]$$

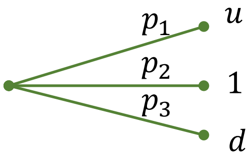
Finite difference methods

- Black-Scholes can discretized by considering finite price and time differences ΔS and Δt

$$\frac{\partial f}{\partial S} \approx \frac{f_{i+1,j} - f_{i,j}}{\Delta S}$$
$$\frac{\partial^2 f}{\partial S^2} \approx \frac{(f_{i+1,j} - f_{i,j}) - (f_{i,j} - f_{i-1,j})}{(\Delta S)^2}$$
$$\frac{\partial f}{\partial t} \approx \frac{f_{i,j+1} - f_{i,j}}{\Delta t}$$

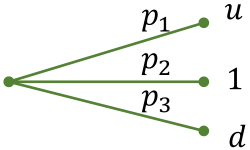
- Solve by
  1. Setting boundary conditions
  2. Solving interior points iteratively by starting from the boundary

Binomial and trinomial lattices



- Binomial lattice treated in Lecture 10
  - Good for explaining theory, not state-of-the-art
- Trinomial lattice more accurate than binomial lattice, but then the value of an option cannot be replicated by only using the underlying asset and the risk-free asset (three possible outcomes, only two free parameters to fit)
  - Pricing follows from no arbitrage condition: On risk-neutral expectation, the return of every asset is the risk-free rate

Binomial and trinomial lattices



- If the one-period (length Δt) mean value of the underlying asset is $1 + \mu\Delta t$ and the variance is $\sigma^2\Delta t$, then the probabilities are selected to satisfy

$$p_1 + p_2 + p_3 = 1$$
$$up_1 + p_2 + dp_3 = 1 + \mu\Delta t$$
$$u^2p_1 + p_2 + d^2p_3 = \sigma^2\Delta t + (1 + \mu\Delta t)^2,$$

  where the last line represents $\mathbb{E}[x^2] = \text{Var}[x] + \mathbb{E}[x]^2$
- To use this lattice for pricing, the risk-neutral probabilities q_1, q_2, q_3 have to be used instead
  - These are found by solving the same set of equations such that the mean value is changed to $1 + r\Delta t$

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