Exercise 1. Let $\eta : \mathcal{B}(\mathbb{X}) \to \mathbb{N}$ be a Poisson point process with intensity λ, i.e. $\eta(A)$ is the number of points in $A \in \mathcal{B}(\mathbb{X})$. Let $0 < p \leq 1$. Remove each point produced by the process independently with probability $1-p$. Show that the resulting point process η' is a Poisson point process with intensity $p\lambda$.

Solution: The independence of the random variables $\eta'(A)$ and $\eta'(B)$ in $A \cap B = \varnothing$ is clear from the assumptions. Hence we only need to show that the said random variables are Poisson distributed.

Write $\lambda_A = \lambda\mu(A)$. Then for all non-negative integers n

$$\mathbb{P}[\eta(A) = n] = \frac{\lambda_A^n}{n!} e^{-\lambda_A}.$$

Hence

$$\mathbb{P}[\eta'(A) = k] \;=\; \sum_{n=k}^{\infty} \mathbb{P}[\eta(A) = n] \mathbb{P}\left[B_n = k\right].$$

where B_n are independent random variables with $B_n \sim \mathrm{Bin}\binom{n}{p}$. Hence

$$\begin{aligned}
\mathbb{P}[\eta'(A) = k] &= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda_A^n}{n!} e^{-\lambda_A} \\
&= \frac{p^k}{k!} e^{-\lambda_A} \lambda_A^k \sum_{n=k}^{\infty} \frac{(\lambda_A - p\lambda_A)^{n-k}}{(n-k)!} \\
&= \frac{p^k}{k!} e^{-\lambda_A} \lambda_A^k e^{\lambda_A - p\lambda_A} \\
&= \frac{(p\lambda_A)^k}{k!} e^{-p\lambda_A}
\end{aligned}$$

Exercise 2. Consider a finite measure space X and a sequence of measurable sets $A_1, A_2 \cdots \subset X$. Show that if

$$\limsup \mu\left(A_n\right) \geq \lambda.$$

Then $\mu\left(\limsup A_i\right) \geq \lambda$. *Hint: Fatou's lemma.*

Solution: By Fatou's lemma

$$\lambda \leq \limsup \mu(A_n) = \mu(X) - \liminf \int_X \chi_{A_n} \,\mathrm{d}\mu \leq \mu(X) - \int_X \liminf \chi_{A_n} \,\mathrm{d}\mu$$
$$= \mu(X) - \mu\left(\liminf A_n\right) = \mu\left(\limsup A_n\right)$$

Exercise 3. Use Karamata's representation theorem to verify the following facts for $f \in RV_\alpha$, $g \in RV_\beta$:

- If $\alpha > 0$ then $f \to \infty$, if $\alpha < 0$, then $f \to 0$ as $x \to \infty$.
- If $\alpha > \beta$ then $f/g \to \infty$.
- $f + g \in RV_{\max\{\alpha,\beta\}}$

Proof. Assume $\alpha > 0$. By Karamata's representation theorem for $t > t_0$

$$f(t) = c(t) \exp\left(\int_{t_0}^{t} \frac{a(s)}{s} \,\mathrm{d}s\right).$$

Then for $t > t_\delta$ we have $a(s) \geq \alpha/2$ and

$$f(t) = c(t) \exp\left(\int_{t_\delta}^{t} \frac{a(s)}{s} \,\mathrm{d}s + \int_{t_0}^{t_\delta} \frac{a(s)}{s} \,\mathrm{d}s\right) \geq c(t) \exp\left(\int_{t_\delta}^{t} \frac{a(s)}{s} \,\mathrm{d}s + O(1)\right)$$
$$= \exp\left(\frac{\alpha}{2} \log t + O(1)\right) \to \infty.$$

The case $\alpha < 0$ is similar.

If f/g in $RV_{\alpha-\beta}$. If $\alpha > \beta$ the result follows from the previous part.

Assume that $\alpha > \beta$. Note that

$$\frac{f(tx) + g(tx)}{f(t) + g(t)} = \frac{\frac{f(tx)}{f(t)} + \frac{g(tx)}{f(t)}}{1 + \frac{f(t)}{g(t)}} = \frac{f(tx)}{f(t)} + o(1) \to x^\alpha.$$

□