Exercise 1. Show that the Cauchy distribution $F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x$ is heavy-tailed with $\gamma = 1$.

Solution: By l'Hospital's rule:

$$\lim_{t\to\infty} \frac{\frac{1}{2} - \frac{1}{\pi}\arctan(tx)}{\frac{1}{2} - \frac{1}{\pi}\arctan(t)} = \lim_{t\to\infty} \frac{x(1+t^2)}{1+(tx)^2} = x^{-1}.$$

Exercise 2. Let $E_1, E_2, \ldots$ be i.i.d. and standard exponential. Show that

$$\frac{1}{\log n} E_{(n,n)} \to_P 1.$$

Solution: By definition

$$\mathbb{P}\left[\frac{E_{(n,n)}}{\log n} \le a\right] = \mathbb{P}\left[E_{(n,n)} \le a \log n\right]^n = \left(1 - n^{-a}\right)^n$$

Write $a = 1 + b$, then the above expression is

$$\left(1 + \frac{-n^{-b}}{n}\right)^n.$$

If $b > 0$, the expression is increasing in n^{-b}. Hence for any $x > 0$ there is large n_0 such that for all $n > n_0$

$$\left(1 + \frac{-n^{-b}}{n}\right)^n \ge \left(1 + \frac{-x^{-b}}{n}\right)^n \to \exp\left(-x^{-b}\right).$$

Hence the limit of the left-hand side is at least $\exp(-x^{-b})$ for any $x > 0$ i.e. one. Similarly for $b < 0$ the limit is zero. Hence the ratio converges in probability to one.

Exercise 3. Let $X_1, X_2, \ldots$ be i.i.d. and assume that the corresponding distribution $F \in \mathcal{D}(G_\gamma)$, with $\gamma > 0$. Show that

$$\frac{\log X_{(n,n)} - \log X_{(n-k_n,n)}}{\log k_n} \to_P \gamma,$$

where k_n is such that $k_n/n \to 0$, $k_n \to \infty$ as $n \to \infty$. *Hint: Refer to the proof of Hill estimator's consistency. Apply similar techniques combined with exercise 2.*

Solution: Recall that $U(Y_i) =_D X_i$ with $Y_i \sim 1 - 1/x$. Thus

$$\frac{\log X_{(n,n)} - \log X_{(n-k_n,n)}}{\log k_n} =_D \frac{\log\left(\frac{U(Y_{(n,n)}))}{U(Y_{(n-k_n,n)}))}\right)}{\log k_n} = \frac{1}{\log k_n} \log\left(\frac{U(tx)}{U(t)}\right)$$

with $t = X_{(n-k_n,n)}$ and $x = X_{(n,n)}/X_{(n-k_n,n)}$. For $\varepsilon, \delta > 0$, there is large t_0 such that for $t > t_0$, $x > 1$

$$\frac{1}{\log k_n} \log\left(\frac{U(tx)}{U(t)}\right) \leq \frac{(1+\varepsilon)(\gamma+\delta)}{\log k_n} \log x = \frac{(1+\varepsilon)(\gamma+\delta)}{\log k_n} \log\left(\frac{Y_{(n,n)}}{Y_{(n-k_n,n)}}\right)$$

$\log Y_i$ follows a standard exponential distribution. By Rényi representation theorem

$$\begin{aligned}\frac{1}{\log k_n}\left(E_{(n,n)} - E_{(n-k_n,n)}\right) &=_D \frac{1}{\log k_n}\left(\sum_{i=1}^{n} \frac{E_i^*}{n-i+1} - \sum_{i=1}^{n-k_n} \frac{E_i^*}{n-i+1}\right) \\ &= \frac{1}{\log k_n} \sum_{i=n-k_n+1}^{n} \frac{E_i^*}{n-i+1} =_D \frac{1}{\log k} E_{(k_n,k_n)}\end{aligned}$$

Since $\varepsilon, \delta > 0$ are arbitrary, the claim follows.