Aalto University
School of Electrical
Engineering

Lecture 9: Elementary Single-Phase Machines and Lossless Magnetic Field

ELEC-E8402 Control of Electric Drives and Power Converters

Marko Hinkkanen

Spring 2018

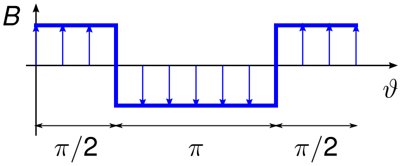
Single-Phase Machines

- Single-phase machines are not used in real applications
- Why should we study them?
- To get more thorough understanding of fundamental concepts
  - Flux linkages
  - Conservative magnetic field systems
  - Selection of state variables
  - Modeling concepts introduced are very general and powerful
- For simplicity, 2-pole machines will be considered here
- More poles can be easily taken into account

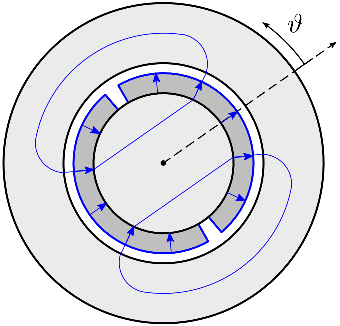
Outline

Airgap Field Produced by PMs

- Ferrite PMs are cheap and easy to manufacture
- Rare-earth magnets (e.g. NdFeB, SmCo) provide much higher energy products, but they are expensive and difficult to manufacture and handle

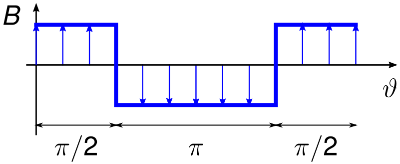


Assumed uniform radial air-gap flux density

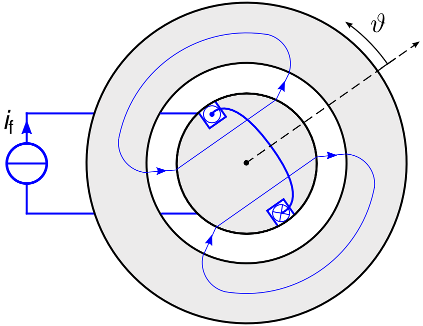


Equivalent Current Source

- PMs can be modeled with an equivalent field winding
- Permeability of the magnets is approximately the same as that of air
- Current i_f is constant

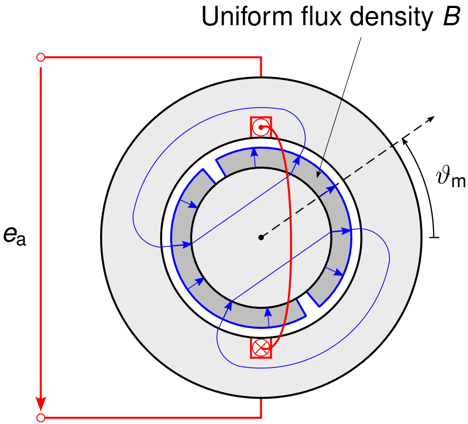


Assumed uniform radial air-gap flux density



Induced Voltage

- Rotor rotates at $\omega_m = d\vartheta_m/dt$
- Magnetic field moves at $v = r\omega_m$ with respect to the conductors
- Voltage $B\ell v$ is induced in each conductor
- N turns in the coil
- Induced voltage $e_a = 2Nr\ell B\omega_m$ is proportional to the angular speed



Phase Voltage

- Phase voltage

$$u_a = R_a i_a + \frac{d\psi_a}{dt}$$

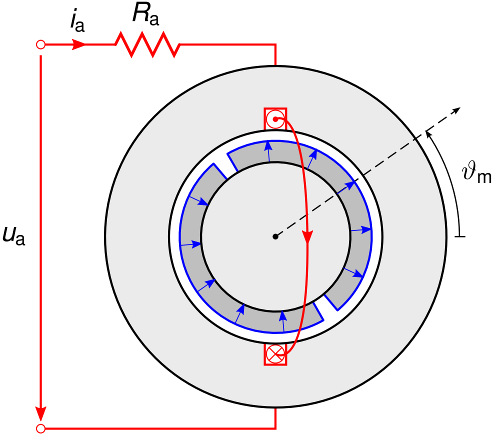
- Phase flux linkage

$$\psi_a = L_a i_a + \psi_{af}(\vartheta_m)$$

where L_a is the self-inductance and ψ_{af} is the flux linkage due to the PMs

- Based on the geometry

$$\psi_{af} = 2Nr\ell B\left(\frac{\pi}{2} - \vartheta_m\right) \quad \text{when} \quad 0 \leq \vartheta_m \leq \pi$$



Phase Voltage

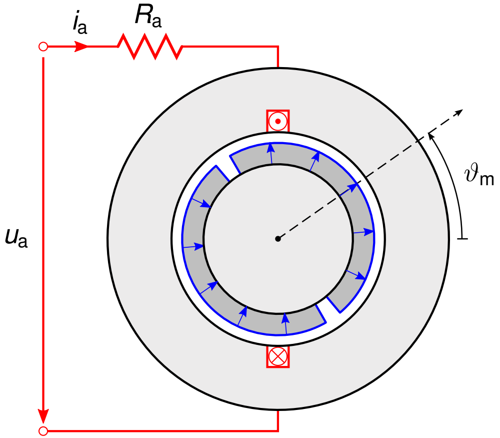
- Voltage can be expressed as

$$u_a = Ri_a + L_a \frac{\mathrm{d}i_a}{\mathrm{d}t} + e_a$$

- Voltage induced by the magnets

$$e_a = \frac{\mathrm{d}\psi_{af}}{\mathrm{d}t} = \frac{\mathrm{d}\psi_{af}}{\mathrm{d}\vartheta_m}\frac{\mathrm{d}\vartheta_m}{\mathrm{d}t} = -2Nr\ell B\omega_m$$

which is the same result as earlier



The sign in the expression for e_a is valid at $0 < \vartheta_m < \pi$, cf. the following waveforms.

Torque Production

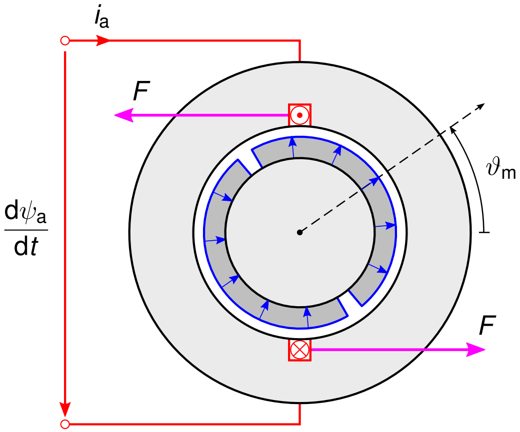
- Forces $F = NB\ell i_a$ act on both coil sides
- Opposite counterforces act on the rotor
- Electromagnetic torque

$$T_m = -2Nr\ell B i_a$$

is proportional to the current

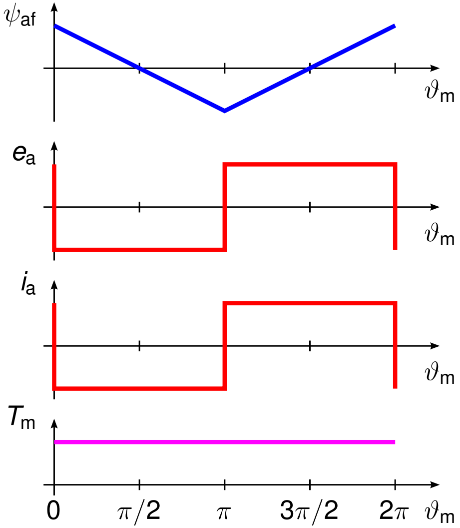
- Mechanical power

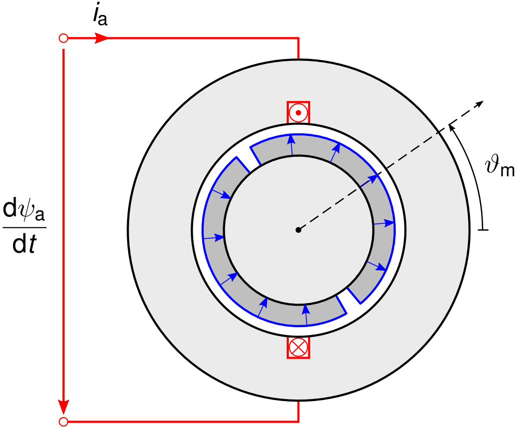
$$p_m = T_m \omega_m = e_a i_a$$



The sign in the expression for T_M is valid at $0 < \vartheta_m < \pi$, cf. the following waveforms.

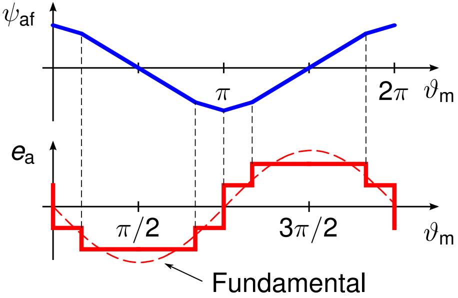
Example Waveforms

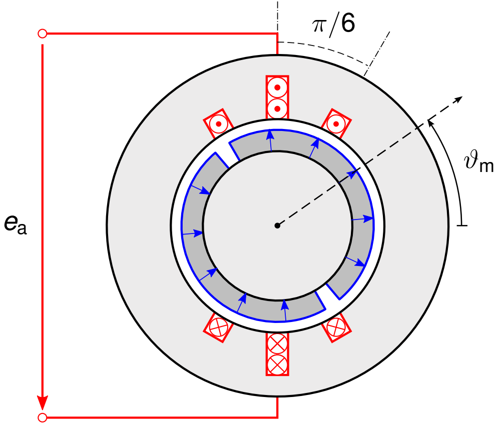




The waveform of the current i_a is chosen such that the torque T_m becomes constant.

Simple Distributed Winding



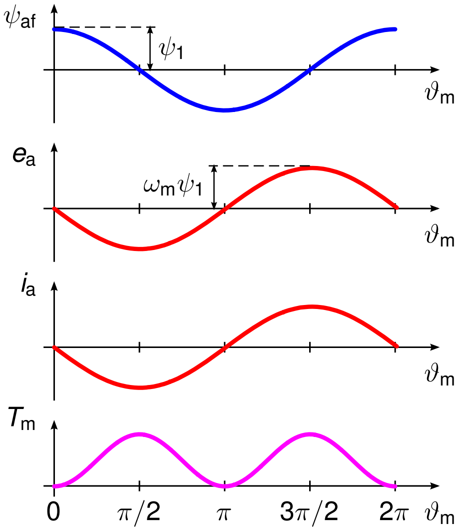


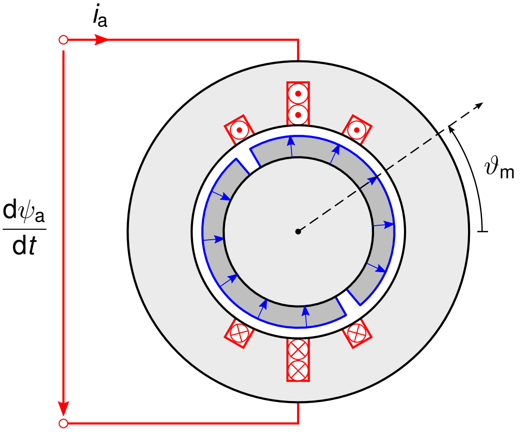
- Waveforms approximately sinusoidal

$$\psi_{af} = \psi_1 \cos(\vartheta_m)$$
$$e_a = -\omega_m \psi_1 \sin(\vartheta_m)$$

- PM-flux constant $\psi_1 \propto Nr\ell B$

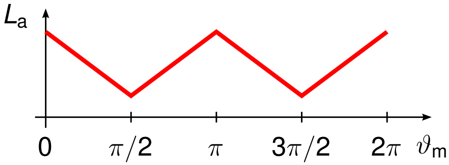
Example Waveforms: Ideal Sinusoidally Distributed Winding

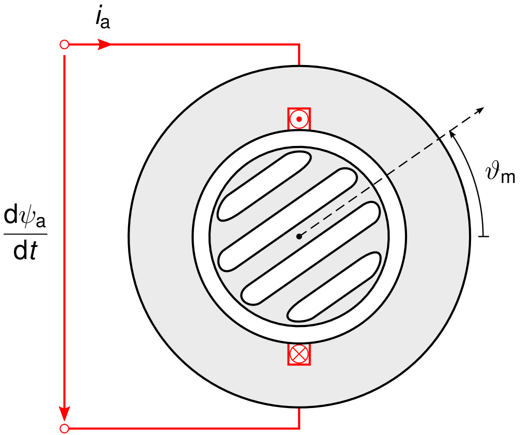




Outline

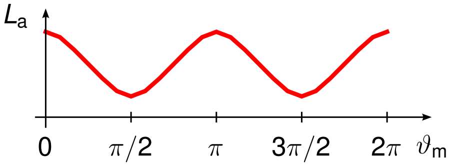
Reluctance Machine

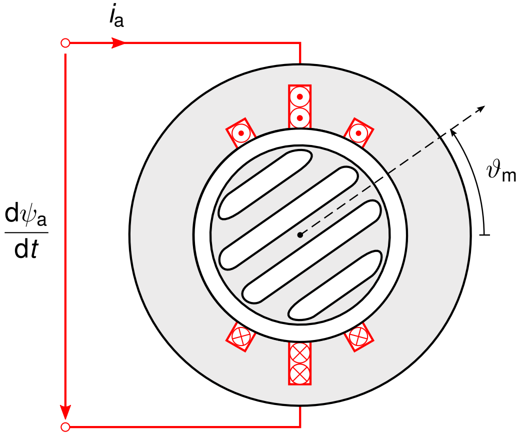




- Ideal anisotropy assumed above (with constant leakage inductance)
- Inductance L_a depends on the rotor position
- Flux linkage $\psi_a = L_a(\vartheta_m) i_a$

Simple Distributed Winding

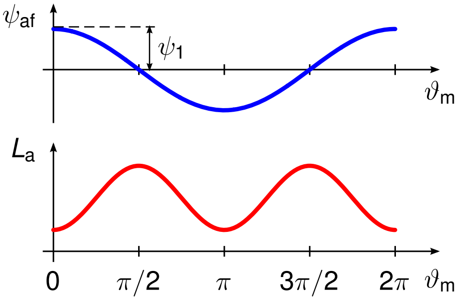


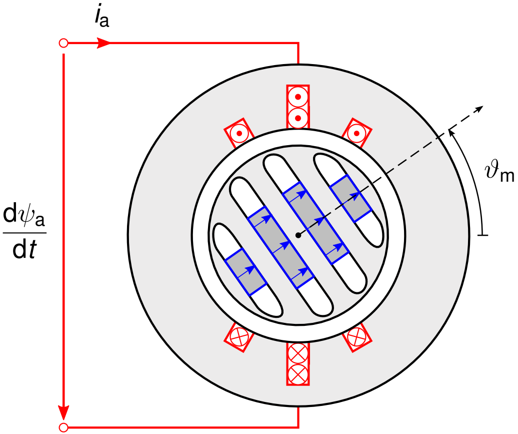


- Ideal anisotropy assumed above (with constant leakage inductance)
- If spatial harmonics are omitted

$$\psi_a = [L_0 + L_2 \cos(2\vartheta_m)]i_a$$

PM Reluctance Machine





- Approximate flux linkage

$$\psi_a = [L_0 - L_2 \cos(2\vartheta_m)] i_a + \psi_1 \cos(\vartheta_m)$$

Salient-Pole Machine With Field Winding

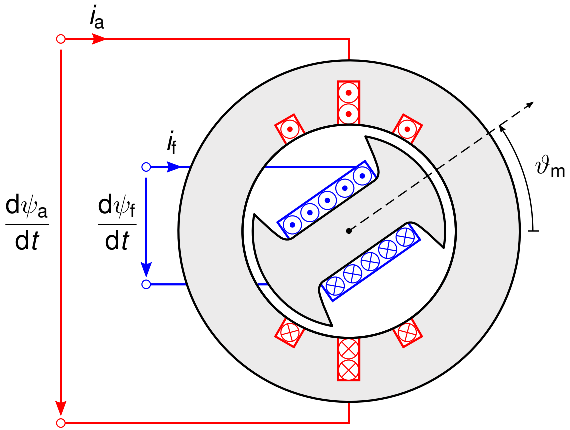
- Field winding in the rotor
- Flux linkages

$$\psi_\mathrm{a} = L_\mathrm{a}(\vartheta_\mathrm{m}) i_\mathrm{a} + L_\mathrm{af}(\vartheta_\mathrm{m}) i_\mathrm{f}$$
$$\psi_\mathrm{f} = L_\mathrm{af}(\vartheta_\mathrm{m}) i_\mathrm{a} + L_\mathrm{f} i_\mathrm{f}$$

where

$$L_\mathrm{a}(\vartheta_\mathrm{m}) = L_0 + L_2 \cos(2\vartheta_\mathrm{m})$$
$$L_\mathrm{af}(\vartheta_\mathrm{m}) = M \cos(\vartheta_\mathrm{m})$$

- How to determine torque T_m of salient-pole machines?



Outline

Lossless Magnetic Field

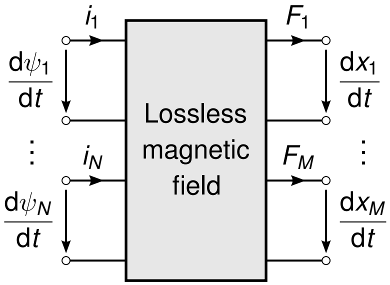
Understanding lossless magnetic field systems often helps in developing machine models for control purposes

- Very general and powerful concept
- Only assumption is that the magnetic field is lossless (conservative)
- Forces and torques in complex electromechanical systems can be determined
- Independent of machine type, number of terminals, number of poles, etc.
- Most lumped-parameter electric machine models are based on it
- Magnetic saturation and spatial harmonics can be taken into account
- Core losses can be modeled outside the lossless field system

Lossless Magnetic Field

Stored magnetic field energy W_m

- is a state function, depending only on its independent state variables
- is independent of the path used to reach the state
- can be determined completely if the electrical port relations are known
- can be evaluated by means of numerical techniques (e.g. FEM) or measurements



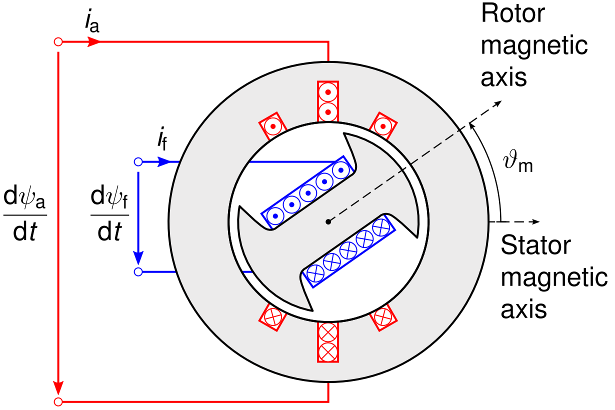
Example System: Single-Phase Machine With Field Winding

- Two electrical ports and one mechanical port
- Stored field energy

$$W_m = W_m(\psi_a, \psi_f, \vartheta_m)$$

where ψ_a, ψ_f, and ϑ_m are independent state variables

- This example system is considered in the following



- Power balance

$$\frac{\mathrm{d}W_\mathrm{m}}{\mathrm{d}t} = i_\mathrm{a}\frac{\mathrm{d}\psi_\mathrm{a}}{\mathrm{d}t} + i_\mathrm{f}\frac{\mathrm{d}\psi_\mathrm{f}}{\mathrm{d}t} - T_\mathrm{m}\frac{\mathrm{d}\vartheta_\mathrm{m}}{\mathrm{d}t}$$

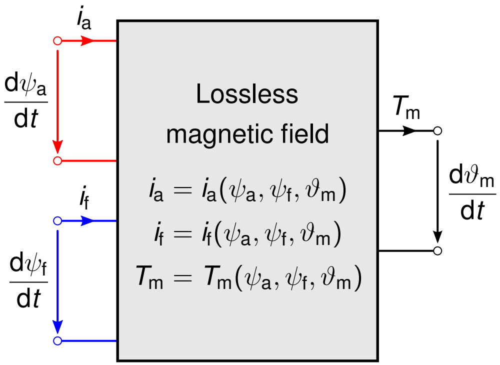
- Currents

$$i_\mathrm{a} = \frac{\partial W_\mathrm{m}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})}{\partial \psi_\mathrm{a}}$$

$$i_\mathrm{f} = \frac{\partial W_\mathrm{m}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})}{\partial \psi_\mathrm{f}}$$

- Torque

$$T_\mathrm{m} = -\frac{\partial W_\mathrm{m}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})}{\partial \vartheta_\mathrm{m}}$$



Lossless Field System Should Satisfy Reciprocity Conditions

- Incremental mutual inductances should be equal in any operating point

$$\frac{\partial i_\mathrm{a}}{\partial \psi_\mathrm{f}} = \frac{\partial i_\mathrm{f}}{\partial \psi_\mathrm{a}}$$

- If multiple mechanical ports, analogous conditions hold for them as well
- Conditions between electrical and mechanical ports

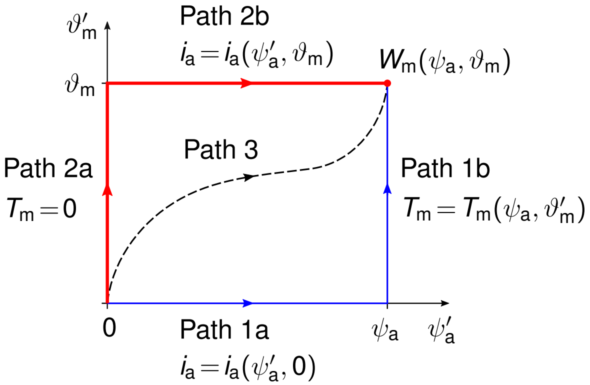
$$\frac{\partial i_\mathrm{a}}{\partial \vartheta_\mathrm{m}} = -\frac{\partial T_\mathrm{m}}{\partial \psi_\mathrm{a}} \qquad \frac{\partial i_\mathrm{f}}{\partial \vartheta_\mathrm{m}} = -\frac{\partial T_\mathrm{m}}{\partial \psi_\mathrm{f}}$$

Integration Path for the Field Energy Can Be Chosen Freely

- For illustration purposes $\psi_\mathrm{f} = 0$ assumed
- Integration along Path 1

$$W_\mathrm{m}(\psi_\mathrm{a}, \vartheta_\mathrm{m}) = \int_0^{\psi_\mathrm{a}} i_\mathrm{a}(\psi'_\mathrm{a}, 0)\mathrm{d}\psi'_\mathrm{a} - \int_0^{\vartheta_\mathrm{m}} T_\mathrm{m}(\psi_\mathrm{a}, \vartheta'_\mathrm{m})\mathrm{d}\vartheta'_\mathrm{m}$$

- We should know $T_\mathrm{m}(\psi_\mathrm{a}, \vartheta_\mathrm{m})$



- Integration along Path 2

$$W_m(\psi_a, \vartheta_m) = \int_0^{\psi_a} i_a(\psi_a', \vartheta_m) d\psi_a'$$

since $T_m(0, \vartheta_m) = 0$

- Torque is not needed in Path 2

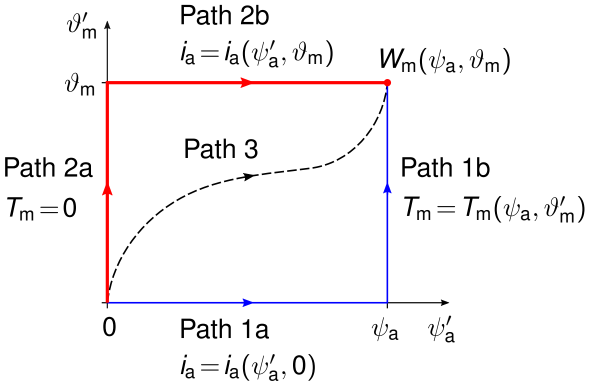
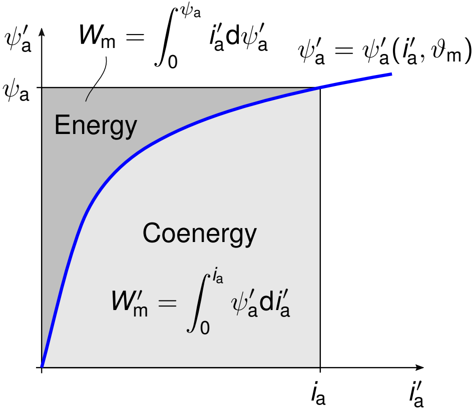


Illustration of Field Energy and Coenergy

- For illustration purposes
  $\psi_f = 0$ assumed
- Area of the rectangle $\psi_a i_a$
- Relation of coenergy to field energy

$$W_m + W'_m = \psi_a i_a$$

- Magnetically linear case: $W_m = W'_m$



Field Energy and Coenergy

- Field energy

$$W_m(\psi_a, \psi_f, \vartheta_m) = \int_0^{\psi_a} i_a(\psi_a', \psi_f, \vartheta_m) d\psi_a' + \int_0^{\psi_f} i_f(0, \psi_f', \vartheta_m) d\psi_f'$$

- Coenergy

$$W_m'(i_a, i_f, \vartheta_m) = \int_0^{i_a} \psi_a(i_a', i_f, \vartheta_m) di_a' + \int_0^{i_f} \psi_f(0, i_f', \vartheta_m) di_f'$$

- Relation of coenergy to field energy

$$W_m + W_m' = \psi_a i_a + \psi_f i_f$$

- Torque is typically easier to calculate from coenergy

Torque from Coenergy

- Power balance

$$\frac{dW'_m}{dt} = \psi_a \frac{di_a}{dt} + \psi_f \frac{di_f}{dt} + T_m \frac{d\vartheta_m}{dt}$$

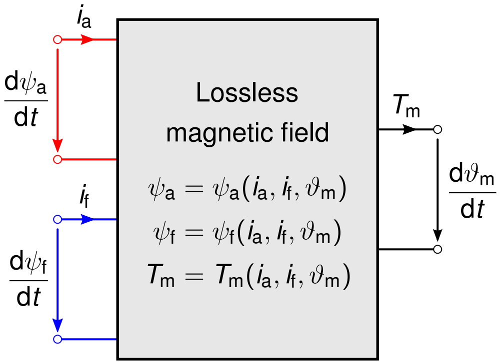
- Flux linkages

$$\psi_a = \frac{\partial W'_m(i_a, i_f, \vartheta_m)}{\partial i_a}$$

$$\psi_f = \frac{\partial W'_m(i_a, i_f, \vartheta_m)}{\partial i_f}$$

- Torque

$$T_m = \frac{\partial W'_m(i_a, i_f, \vartheta_m)}{\partial \vartheta_m}$$



Analytical Example

- Assume a magnetically linear machine with the flux linkages

$$\psi_a = L_a(\vartheta_m)i_a + L_{af}(\vartheta_m)i_f$$
$$\psi_f = L_{af}(\vartheta_m)i_a + L_f i_f$$

where the inductances are

$$L_a(\vartheta_m) = L_0 + L_2\cos(2\vartheta_m) \qquad L_{af}(\vartheta_m) = M\cos(\vartheta_m)$$

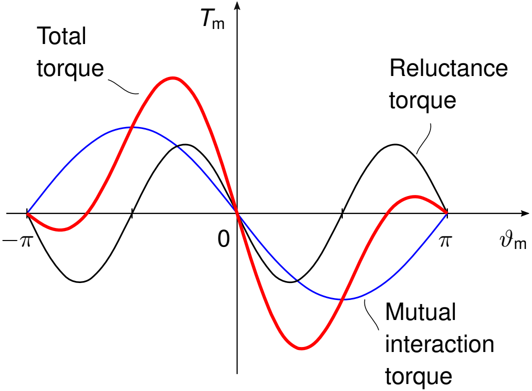
- Coenergy

$$W'_m(i_a, i_f, \vartheta_m) = \frac{1}{2}\left[L_0 + L_2\cos(2\vartheta_m)\right] i_a^2 + M\cos(\vartheta_m)i_a i_f + \frac{1}{2}L_f i_f^2$$

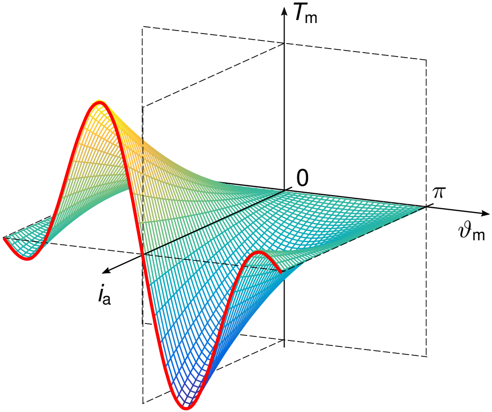
- Torque

$$T_m = -M\sin(\vartheta_m)i_a i_f - L_2\sin(2\vartheta_m)i_a^2$$

Torque



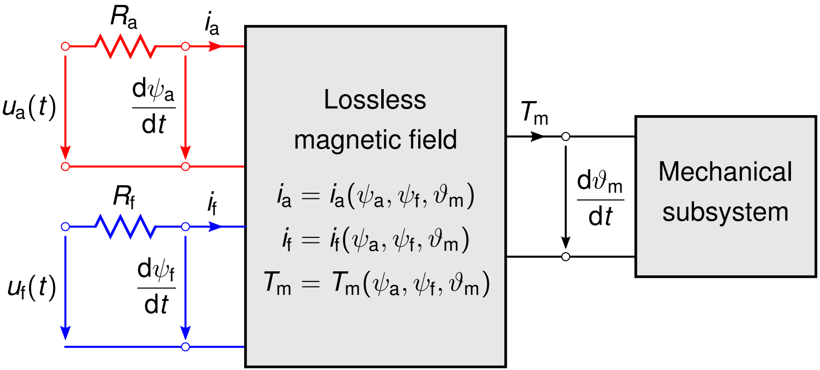
Currents i_a and i_f are constant



Field current i_f is constant

Outline

Inclusion of Voltage Equations



Voltage Equations: Flux Linkages as State Variables

- Voltage equations

$$\frac{\mathrm{d}\psi_\mathrm{a}}{\mathrm{d}t} = u_\mathrm{a} - R_\mathrm{a} i_\mathrm{a} \qquad \frac{\mathrm{d}\psi_\mathrm{f}}{\mathrm{d}t} = u_\mathrm{f} - R_\mathrm{f} i_\mathrm{f}$$

where the currents are known static functions of the state variables

$$i_\mathrm{a} = i_\mathrm{a}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m}) \qquad i_\mathrm{f} = i_\mathrm{a}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})$$

- Electromagnetic torque is the input for the mechanical subsystem

$$T_\mathrm{m} = T_\mathrm{m}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})$$

and the state variable ϑ_m is the output of the mechanical subsystem
- This set of equations is very simple to implement

The torque expression $T_\mathrm{m} = T_\mathrm{m}(i_\mathrm{a}, i_\mathrm{f}, \vartheta_\mathrm{m})$ could be used as well since the currents $i_\mathrm{a} = i_\mathrm{a}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})$ and $i_\mathrm{f} = i_\mathrm{f}(\psi_\mathrm{a}, \psi_\mathrm{f}, \vartheta_\mathrm{m})$ are known.

Voltage Equations: Currents as State Variables

- If the currents are used the state variables, the representation of the voltage equations becomes complex, for example

$$\frac{d\psi_a}{dt} = \frac{\partial \psi_a}{\partial i_a}\frac{di_a}{dt} + \frac{\partial \psi_a}{\partial i_f}\frac{di_f}{dt} + \frac{\partial \psi_a}{\partial \vartheta_m}\frac{d\vartheta_m}{dt} = u_a - R_a i_a$$

- In general case, all the partial derivatives are functions of i_a, i_f, and ϑ_m
- In the magnetically linear example case

$$L_a(\vartheta_m)\frac{di_a}{dt} + L_{af}(\vartheta_m)\frac{di_f}{dt} + \left[\frac{\partial L_a(\vartheta_m)}{\partial \vartheta_m} i_a + \frac{\partial L_{af}(\vartheta_m)}{\partial \vartheta_m} i_f\right]\frac{d\vartheta_m}{dt} = u_a - R_a i_a$$

and similarly for the rotor voltage equation

Further Reading

- H. H. Woodson and J. R. Melcher, *Electromechanical Dynamics*, John Wiley & Sons, 1968.
- A. E. Fitzgerald, C. Kingsley, Jr., and S. D. Umans, *Electric Machinery*, 6th ed., McGraw-Hill, 2003.