CS-E4550 Advanced Combinatorics in CS
Module 1: Graph Theory

Short assignments: Give the true/false answers to the following statements. When your answer is true, provide a proof. When it is false, provide a counter example.

<u>Submit this page at the end of the tutorial session.</u>

1. Graph G is a clique if and only if $\bar{G}$ is an independent set.

2. A bipartite graph may contain C_5 as a subgraph.

3. For all $n \geq 2$, graph $K_{n,n}$ has more edges than K_n.

4. The diameter of the graph is the same as the lengths of its longest path.

Exercise 1 (Chromatic number and clique). *Let G be a graph.*

- *Prove that $\chi(G) \geq \omega(G)$.*
- *Draw the smallest possible graph G (with smallest number of vertices) where $\chi(G) \neq \omega(G)$.*
- *Prove that $\chi(G)\alpha(G) \geq n$.*

Exercise 2 (Fractional chromatic number). *Let G be a graph. We define a* fractional chromatic number *of G (denoted by $\chi_f(G)$). A* valid fractional coloring scheme *of G is a collection of independent sets $S_1, S_2, \ldots, S_\ell \subseteq V(G)$ together with real values $f(S_1), f(S_2), \ldots, f(S_\ell) \in (0,1]$ such that, for each node $v \in V(G)$, $\sum_{i:v \in S_i} f(S_i) \geq 1$. The value of this fractional coloring is $\sum_i f(S_i)$.*

The value $\chi_f(G)$ is defined as the minimum k for which there exists a fractional coloring scheme of value k.

- *Show that $\omega(G) \leq \chi_f(G) \leq \chi(G)$.*
- *Give the smallest possible graph G (with smallest number of vertices) for which $\omega(G) < \chi_f(G) < \chi(G)$.*

Exercise 3. *Prove that in any graph $G = (V, E)$, there are two vertices $u, v \in V$ where $d_G(v) = d_G(u)$.*

Exercise 4. *Let G and H be undirected graphs. Consider the OR product $G \times H$ defined as:*

$$\begin{aligned} V(G \times H) &= V(G) \times V(H) \\ E(G \times H) &= \{(u,a)(v,b) : uv \in E(G) \textbf{ OR } ab \in E(H)\} \end{aligned}$$

- *Draw the graph $C_4 \times C_4$. What is $\alpha(C_4 \times C_4)$?*
- *(*) Prove that $\alpha(G \times H) = \alpha(G) \times \alpha(H)$.*
- *(**) Prove that $\chi_f(G)\chi(H) \leq \chi(G \times H) \leq \chi(G)\chi(H)$*

Exercise 5. *Let G be a connected n-vertex graph. Prove that the vertices of graph G can be labeled by $\{1, 2, \ldots, n\}$ such that, for all i, $G[\{1, \ldots, i\}]$ is connected.*

Exercise 6 (*). *Let G be a connected graph and $\delta(G)$ be a minimum degree of G. Show that there is a path of length $\min\{2\delta(G), |V(G)| - 1\}$ in G.*

Exercise 7 (**). *Let G be a a bipartite graph with n vertices on each side of the partition. If G does not contain any cycle of length 4 as a subgraph, then $|E(G)| = O(n^{3/2})$.*