CS-E4550 Advanced Combinatorics in CS

Module 2: Forbidden Patterns

Short assignments: Give the true/false answers to the following statements. In either case, provide a proof or a counter example.

Submit this page at the end of the tutorial session.

1. $\log^* n = o(\log \log n)$.

2. Let $\alpha(\cdot)$ be an inverse Ackermann function. Then, $\log \alpha(n) = \Theta(\alpha(n))$

3. The sequence $< 2, 1, 3, 4, 1, 2, 1, 3, 1, 2, 1 >$ is a DS$(3, 5)$ sequence.

4. The sequence $< 5, 2, 1, 3, 4, 1, 2, 1, 3, 4, 1, 2, 4, 5 >$ is a DS$(5, 4)$ sequence but not a DS$(5, 3)$ sequence.

Exercise 1 (10 points). *Prove that* $\lambda_2(n) = 2n - 1$.

Exercise 2 (7 + 5 points). *A divide-and-conquer program, on input of size* n*, partitions the array into* $\frac{n}{\log n}$ *pieces, each of size* $\log n$[1]. *Then, the algorithm recursively solves the sub-problem on each of the smaller pieces. Assume that the algorithm pays the cost of* $O(n)$ *to do such partitioning.*

- *What is the running time of this algorithm?*
- *(*) Instead, if the algorithm partitions the array into* $\frac{n}{\log^2 n}$ *pieces, each of size* $\log^2 n$*, what would be the running time?*

Exercise 3 (7+5 points). *Let* M *be a* $\{0,1\}$*-matrix of dimension* n*-by-*n*. Let* P *be a* k*-by-*k $\{0,1\}$*-matrix where* $k < n$*. We say that matrix* M *contains pattern* P *if there are* k *rows* $r_1, \ldots, r_k \in [n]$ *and* k *columns* $c_1, \ldots, c_k \in [n]$ *such that, for all* $i, j \in [k]$ *we have* $M(r_i, c_j) \geq P(i, j)$.

- *Prove that, if matrix* M *does not contain a pattern*

  $$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  then the total number of ones in M *is at most* $O(n)$.

- *(*) Consider an* n*-by-*n *matrix* M *that does not contain a pattern:*

  $$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

  Prove that the number of ones in M *cannot be more than* $O(n)$.

Exercise 4 (10 points). *Prove that* $\lambda_s(n)$ *is finite for all* n *and* s.

Exercise 5 (7+7+<u>1000</u> points). *Let* $X \subseteq \mathbb{R}^2$ *be a set of points on the plane. How many pairs of points in* X *can be unit distance from each other? That is, how many pairs* p, q *for which* $||p - q|| = 1$*? We will upper bound this in two steps.*

- *Construct graph* G *as follows:* $V(G) = X$ *and* $E(G) = \{(p,q) \in X \times X : ||p - q|| = 1\}$*, i.e. there is an edge between any two points having unit Euclidean distance between each other. Prove that* G *does not contain* $K_{2,3}$ *as a subgraph.*
- *(**) Upper bound* $|E(G)|$ *where* G *does not contain* $K_{2,3}$ *as a subgraph.*
- *(open) Prove or disprove: the number of such pairs is at most* $|X|^{1+o(1)}$.

[1] To get rid of stupid technicality, let us assume that all these logarithms and divisions are integers.