CS-E4550 Advanced Combinatorics in CS
Module 3: Discrete Probability

Short assignments: Give the true/false answers to the following statements. In either case, provide a proof or a counter example.

Submit this page at the end of the tutorial session.

1. Consider a cycle of length 3, C_3. We sample each edge $e \in C_3$ independently with probability $1/2$. The event E_1 in which we sample no edge is independent from the event E_2 in which we sample all edges.

2. If events A, B are independent, then they are also mutually exclusive.

3. If $\mathbf{1}_A, \mathbf{1}_B$ are indicator variables for events $A, B \subseteq \Omega$, then $\mathbf{1}_A + \mathbf{1}_B$ is an indicator variable for $A \cup B$.

4. If $X : \Omega \to \mathbb{N}$, then $\mathrm{E}[X]$ must be an integer.

Exercise 1 (6+5 points). *Let Ω be a sample space.*

- *Suppose we have n independent non-trivial events*[1]. *Prove that $|\Omega| \geq 2^n$.*
- *For all $n \geq 3$, construct a sample space of size $|\Omega| = n+1$ with n pairwise independent non-trivial events.*

Exercise 2 (2+3+5 points). *A matrix is said to be constant if all entries are the same in the matrix. Matrix M' is said to be a submatrix of M if M' can be obtained by removing some rows and columns of M. Let $r \geq 2$ be a positive integer and $n = 2^{\lfloor r/10 \rfloor}$. We will prove that there exists an n-by-n matrix M with entries in $\{0,1,2\}$ such that M does not have any constant submatrix of size r-by-r.*

- *We construct a matrix M randomly where each $M(i,j)$ (for $i,j \in [n]$) is independently assigned a value in $\{0,1,2\}$ uniformly. Consider an r-by-r submatrix M'. Argue that the probability that M' is constant is exactly $3(\frac{1}{3})^{r^2}$.*
- *Count the number of r-by-r submatrices of M.*
- *Combine the results in the previous parts and argue (using the* union bound*) that*

$$\mathbb{P}[\textit{there is a constant } r\textit{-by-}r \textit{ submatrix of } M] < 1$$

Conclude that there exists an n-by-n matrix M with entries in $\{0,1,2\}$ such that M does not have any constant submatrix of size r-by-r

Exercise 3 (4+4+4 points). *We learned in Exercise Sheet 1 that $\chi_f(G) \leq \chi(G)$. In this exercise, we will prove that $\chi(G) = O(\chi_f(G) \ln |V(G)|)$ for all graph G.*

- *Let f be a fractional coloring scheme of G. Let $\mathcal{S} = \{X : f(X) > 0\}$ be the sets in the support of f (so each $S \in \mathcal{S}$ is an independent set). We independently sample each $X \in \mathcal{S}$ with probability $f(X)$: Denote by $\mathcal{S}' \subseteq \mathcal{S}$ the resulting collection. Argue that $\mathbb{E}[|\mathcal{S}'|] = \chi_f(G)$.*
- *Argue that the probability that $v \notin \bigcup_{X \in \mathcal{S}'} X$ is at most $1/e$.*
- *Use the previous parts to derive the result.*

Exercise 4 (3+3+3+3 points). *Parinya needs to go to a trial for teaching too much mathematical content in the CS department. In this trial, there are k members of the jury (where $k \geq 10$). Parinya would be found* innocent *if at least one jury member finds him innocent; otherwise, he would be found guilty.*

The jury's decisions are (possibly coordinated) random process, i.e. a member's decision may or may not depend on the other's decision. Let $X_1, \ldots, X_k$ be indicator variables

[1]The event E is nontrivial if $\mathbb{P}[E] \in (0,1)$

where X_i indicates the event that jury-i finds Parinya innocent. Parinya has no knowledge about the jury's random process except for the fact that $\mathbb{P}[X_i = 1] = 1/k$ (i.e. each jury's member finds him innocent with probability $1/k$).

Parinya is trying to estimate his chance and decides whether he should escape to Belize instead of going through the trial that he would lose anyway.

- *Calculate the expected number of jury's members that would find him innocent, i.e. Let $X = \sum_i X_i$, calculate $\mathbb{E}[X]$.*
- *If each jury makes decision independently, what is the probability that Parinya would be found guilty? Is this number less than half? Calculate the sum $\sum_{i,j \in [k]} \mathbb{P}[X_i = 1 \& X_j = 1]$ of the above random process.*
- *Describe a scenario where the jury's random process could find Parinya guilty with probability as close to one as possible.*[2]*. Calculate the sum $\sum_{i,j \in [k]} \mathbb{P}[X_i = 1 \& X_j = 1]$ of the random process you propose.*
- *(**) In probability theory, we know (roughly) that $\mathbb{P}[X \geq 1] = \Omega(\frac{\mathbb{E}[X]}{\Delta})$ where $\Delta = \sum_{i,j \in [k]} \mathbb{P}[X_i = 1 \& X_j = 1]$. Construct a random process of jury's decisions where $\Delta = \Theta(\log k)$, and hence the probability of Parinya's being found innocent is at least $\Omega(1/\log k)$.*

Exercise 5 (10 points). *(**) Let G be an n-vertex graph that contains a path of length $\sqrt{n}$. Describe a polynomial time randomized algorithm that finds a path of length $\Omega(\log n / \log \log n)$.*

[2] Keep in mind that your random process must be consistent with the fact that $\mathbb{P}[X_i = 1] = 1/k$