Supplementary Reading 31E11100: Games and Economics of Information

Juuso Välimäki

Fall 2015

1 Introduction

- Economics as a social science
- Society consists of individuals
- Individuals are the unit of analysis
- Societal consequences obtained by aggregation
- Individuals motivated by their own goals
- Are the goals within the society concordant or conflicting?
- Does individual decision making result in good collective outcomes?
- Concretely:
  - In Microeconomic Theory I:
  - $\max_{x_i} u_i(x_i)$ subject to $px_i - w_i \leq 0$
  - Here:
  - $\max_{x_i} u_i(x_i; x_j)$ subject to $g_i(x_i) \leq 0$
  - Notice that optimal x_1 depends in general on the choice of x_2.
  - Both x_i and x_j are endogenous, but i only controls x_i.
- Game theory addresses these questions:
  - what do individuals believe about other individuals' actions?
  - what action will each person take?
  - what is the outcome of these actions?
  - does it make a difference if the group interacts more than once?
  - what if each individual is uncertain about the characteristics of other players?
  - what do observed actions tell about unobservable characteristics of the player?

Examples

- Collective decision making
  - Task allocation within a firm, within a family etc.
  - Voting in municipal, national elections etc.
  - Negotiations of international treaties on trade, environmental protection etc.
- Market conduct under imperfect competition
  - Entry into a market, exit from a market
  - Prcing decisions for existing products, colluding on a price level
  - Advertising, branding decisions etc.
- Information transmission in markets
  - Can a product be any good if it is so cheap?
  - Speculation in financial markets
  - Determining prices through auctions

Analytical Framework: Non-Cooperative Game Theory

- Individuals in the society have autonomy over their decisions
- Individual decisions determine jointly the (societal) outcomes
- Individuals have preferences over the outcomes
- Individual choices are taken to yield as good outcomes (in terms of the preferences) as possible
- Strategic element: The ranking of individual choices depends in general on the decisions taken by other individuals
- Decisions and consequences are interdependent
  - Need to predict what others do
  - Current own choices may have an impact on others' future choices
  - Note the contrast to competitive markets where only prices matter

Modes of Analysis

- Positive analysis
  - Concentrates on the predictions of a model
  - How is a conflict resolved?
  - Are the outcomes Pareto efficient (first fundamental welfare theorem)?
  - What is the equilibrium price level in an oligopolistic industry?
  - Does an auction reveal all relevant information on the product?
  - Is the best candidate elected?
- Normative analysis
  - Concentrates on welfare properties of models and mechanisms
  - How to maximize sales revenue in an auction?
  - How to design elections to choose the best candidate?
  - How to ensure efficient allocation of goods?
  - How to enact/prevent collusion in a market?

2 Normal Form Games

- Full description of a social interaction
- Describe the parties involved or the players
- Define the rules of the game, i.e. what strategies are available (sometimes called actions)
- each player chooses her strategy simultaneously with other players and independently of others' choice
- Evaluate individual payoffs for outcomes in the game
- Outcomes determined by the strategy choices

A game in normal form is defined by

- Set of players $\{1, ..., N\}$
- Strategies of the players S_i for $i \in \{1, ..., N\}$
- Payoff functions $u_i(s)$ for $i \in \{1, ..., N\}$, where $s = (s_1, ..., s_N)$

2x2 Examples

- Prisoners' Dilemma
  - Players are two prisoners: $i \in \{1, 2\}$
  - Strategy: $S_i = \{C, D\}$ for $i \in \{1, 2\}$
  - Payoff $u_1(D, C) > u_1(C, C), u_1(D, D) > u_1(C, D)$
  - Assume symmetric payoffs: $u_1(s, s') = u_2(s', s)$
  - For example, we could have $u_i(C, C) = 3$ and $u_i(D, D) = 1, u_1(C, D) = 0, u_1(D, C) = 4$
  - with this specification, we have the following payoff matrix:
- One-Sided Incentive problem

  Buyer decides whether to buy B or not N, seller decides whether to supply high H or low L quality. Buyer only wants high quality, low quality is cheaper to supply.

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

Figure 1: Prisoners' Dilemma

- Players: Buyer player 1 (row), Seller player 2 (column)
- Strategies: $S_1 = \{B, N\}, S_2 = \{H, L\}$
- Payoffs $u_1(B, H) > u_1(N, H), u_1(N, L) > u_1(B, L)$
- $u_2(s, L) > u_2(s, H)$ for $s \in \{B, N\}$

For example, we could have $u_1(B, H) = 1$ and $u_1(B, L) = -1, u_1(N, s) = 0$ for $s \in H, L$

$u_2(B, H) = 1, u_2(B, L) = 2, u_2(N, H) = -1, u_2(N, L) = 0$

	H	L
B	$1,1$	$-1,2$
N	$0,-1$	$0,0$

Figure 2: One-Sided Incentive Problem

- A pure coordination Game

  Which side of the road to drive

  - Players: Two drivers $i \in \{1, 2\}$
  - Strategies: $S_i = L, R$ for $i \in \{1, 2\}$
  - Payoffs symmetric $u_i(L, L) > u_i(R, L), u_i(R, R) > u_i(L, R)$
  - For example, we could have $u_i(L, L) = u_i(R, R) = 1, u_i(L, R) = u_i(R, L) = 0$

	L	R
L	$1,1$	$0,0$
R	$0,0$	$1,1$

Figure 3: Pure Coordination

- Other two-player coordination games
  - Battle of Sexes

    Husband and wife coordinate between two events, but have opposing tastes.

	Boxing	Opera
Boxing	$3, 1$	$0, 0$
Opera	$0, 0$	$1, 3$

Figure 4: Battle of Sexes

  - Stag Hunt (compare to Bank Runs)

    A group of hunters can catch a stag if they all hunt for the stag. Each can catch a hare regardless of what the others do. Coordination yields then higher payoffs but could be more risky.

	Stag	Hare
Stag	$5, 5$	$0, 4$
Hare	$4, 0$	$4, 4$

Figure 5: Stag Hunt

  - Miscoordination Game

    Two (identical) items are for sale in two different shops. Two buyers, each deciding a unit decide where to shop. As an exercise, write the payoff matrix for this game.

	Shop 1	Shop 2
Shop 1	?	?
Shop 2	?	?

Figure 6: Stag Hunt

- Pure Conflict

  Penalty Kick in football is a pure conflict between shooter (player 1) and goalie (player 2). The shooter wants to miscoordinate with the goalie while goalie wants to coordinate with the shooter.

	L	R
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

Figure 7: Pure Conflict

- Chicken

  This game mixes miscoordination and conflict. Each player can either fight F or yield Y. Fighting while the other yields is good while fighting when the other fights is catastrophic. One payoff matrix capturing this is:

	F	Y
F	$-1, -1$	$5, 0$
Y	$0, 5$	$4, 4$

Figure 8: Chicken

Other Examples

- Two players, many strategies

  - Rock-Paper-Scissors A pure conflict with three actions

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Figure 9: Rock-Paper-Scissors

  - Location Choice Customers are located at points $\{0, 1, ..., 100\}$ on the line (two at each location). The players in the game are two firms that choose their location. If the two firms are at an equal distance from a given location of the buyers, then the two buyers are split between the firms.

	0	1	...	99	100
0	$101, 101$	$2, 200$	...	$100, 102$	$101, 101$
1	$200, 2$	$101, 101$	...	$101, 101$	$102, 100$
...	...	...	...	...	...
99	$102, 100$	$101, 101$	...	$101, 101$	$200, 2$
100	$101, 101$	$100, 102$	...	$2, 200$	$101, 101$

Figure 10: Location Choice

- Cournot Competition Two firms choose quantities $s_i \in S_i = [0, 100]$. Market price is determined as the intersection of demand and supply.

$$p = f(q)$$

$$q = s_1 + s_2$$

  Revenue of firm i is $s_i f(s_1 + s_2)$. Cost of each firm is $c_i(s_i)$.

  Payoffs are $u_i = s_i f(s_1 + s_2) - c_i(s_i)$.

- Bertrand Competition

  Two firms with identical products compete in prices for a single buyer

  Buyer with a demand for exactly one unit buys from the firm with a cheaper price

  If the firms charge same price, buyer buys from each with equal probability

  Assume no production costs

  Write down as a game between the two firms

- Many players, two strategies

  - Congestion A large (infinite) number of drivers decide whether to take the left or the right route.

    Delay depends on the fraction of drivers on the route chosen.

    Let x denote the fraction who choose L.

Then payoff from choosing L is $u_i(L, x)$ and from choosing R, it is $u_i(R, x)$.

For example, we could have $u_i(L, x) = -x^2$, $u_i(R, x) = -(1-x)^2$

- Bank Run

  A large number of depositors consider withdrawing their deposits.

  Suppose there is a fraction $\bar{x}$ such that when fraction $x \leq \bar{x}$ withdraws money from the bank, the deposit is safe and gets a positive interest. If a fraction $x > \bar{x}$ withdraws and the investor leaves money in the bank, then the deposit is lost. If an investor withdraws, then she keeps the money but gets no interest.

  Model this as a game with many players and two actions.

- Many players, many actions

  - Monopolistic competition

    A large number of firm set prices s_i for their products. Demand depends on average price $\bar{s}$ and own price: $q_i = f(s_i, \bar{s})$.

    The payoff of firm i is then $s_i f(s_i, \bar{s}) - c_i(f(s_i, \bar{s}))$.

  - Location choice with network externalities

    Firms choose the characteristics s_i of their own product.

    They have own preference for characteristics b_i, but also want to locate close to average characteristics $\bar{s}$ to ensure compatibility.

    Payoff of firm i could then be

    $u_i(s, \bar{s}) = -(s_i - b_i)^2 - \beta(s_i - \bar{s})^2$

Analysis of Normal Form Games

- Recall that a game is defined by
  - Set of Players $\{1, ..., N\}$
  - Strategies S_i for $i \in \{1, ..., N\}$
  - (Outcomes $S = \times_{i=1}^{N} S_i$)
  - Payoffs $u_i(s)$ for $s \in S$
- Strategies chosen simultaneously
- Each player maximizes own payoff

Dominant strategies

For some games, individual choices are simple. It could be that the same strategy is best for i regardless of what others do.

It is convenient to have notation $s = (s_i, s_{-i})$ where $s_{-i} \in S_{-i}$ denotes the vector of choices by other players than i.

Definition 1 (Dominant Strategy) *A strategy s_i is a dominant strategy for player i if for all $s_{-i} \in S_{-i}$ and for all $s_i' \neq s_i$,*

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

A slightly less demanding notion is given in the following:

Definition 2 (Weakly Dominant Strategy) *A strategy s_i is a weakly dominant strategy for player i if for all $s_{-i} \in S_{-i}$ and for all $s_i' \neq s_i$,*

$$\begin{aligned} & u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \\ \text{and} \qquad & u_i(s_i, s_{-i}') > u_i(s_i', s_{-i}') \end{aligned}$$

for some $s_{-i}' \in S_{-i}$

Definition 3 (Dominant Strategy Equilibrium) *If each player i has a strictly (weakly) dominant strategy s_i, then $s = (s_1, ..., s_N)$ is said to be a strictly (weakly) dominant strategy equilibrium.*

Example 1 (Second price auction)

- *Example of a game with a weakly dominant strategy equilibrium*
- *Set of players:* $\{1, ..., N\}$
- *Strategies:* $S_i = \mathbb{R}_+$ *for all* i
- *Payoffs*

$$u_i = \begin{cases} v_i - s_j^* & s_i > s_j \text{ and } s_j^* = \max_{j \neq i}\{s_j\}, \\ \frac{1}{n}(v_i - s_i) & s_i \geq s_j \text{ and } s_i = s_j \text{ for } n-1 \text{ separate } j \\ 0 & \text{otherwise} \end{cases}$$

Example 2 (Public good provision)
For example, s_i *pollution reduction,* $c_i(s_i)$ *cost of reductions.*

- *Example of a game with a strictly dominant strategy equilibrium*
- *Set of players:* $\{1, ..., N\}$
- *Strategies:* $S_i = \mathbb{R}_+$ *for all* i
- *Payoffs*

$$u_i = -c_i(s_i) + \sum_j s_j$$

Comments

- What is the difference to individual decision making?
- Recall first fundamental welfare theorem. Do we have an equivalent result here?
- In the examples of the previous lecture, decide which games have dominant strategies.

Dominated Strategies

Sometimes it is easy to see that a given strategy yields a bad payoff in comparison to another strategy regardless of how others play. In this case, we call the former strategy dominated. Formally

Definition 4 (Strictly Dominated Strategy) *A strategy $s_i \in S_i$ is a purely strictly dominated strategy for player i in Γ_N if there exists a pure strategy $s_i' \neq s_i$, such that*

$$u_i\left(s_i', s_{-i}\right) > u_i\left(s_i, s_{-i}\right) \tag{1}$$

for all $s_{-i} \in S_{-i}$

A slightly weaker notion is given in the following

Definition 5 (Weakly Dominated Strategy) *A strategy s_i is a weakly dominated strategy for player i in Γ_N if there exists a pure strategy $s_i' \neq s_i$, such that*

$$u_i(s_i', s_{-i}) \geq u_i\left(s_i, s_{-i}\right), \text{for all } s_{-i} \in S_{-i}$$
$$\text{and } u_i\left(s_i', s_{-i}'\right) > u_i\left(s_i, s_{-i}'\right), \text{for some } s_{-i}' \in S_{-i}$$

- Identifying dominated strategies is useful for the analysis of games. Since no rational opponent uses a strictly dominated strategy, each player i can safely ignore all the dominated strategies in a game.
- This results in a smaller game where new strategies can be strictly dominated. Eliminate any dominated strategies and continue with the process iteratively (i.e. repeating the previous step time after time).
- If the process of iteratively removing dominated strategies from the game results in a single strategy profile, we call the game dominance solvable.
- If a strategy gives the best payoff against some strategy profile by other players, then it is not strictly dominated.
- Such profiles are called best responses.

- A related concept called rationalizability deletes all strategies iteratively all strategies that are not best responses. For two-player games with finitely many strategies, rationalizable strategies coincide with strategies surviving iterated elimination of dominated strategies.

Iterated deletion of strictly dominated strategies

	L	M	R
T	$3,7$	$1,5$	$8,3$
M	$4,1$	$2,2$	$6,0$
B	$0,3$	$0,6$	$3,7$

Figure 11: Dominance Solution (M,M)

- 1st round: B dominated by T
- 2nd round: R dominated by L
- 3rd round: T dominated by M
- 4th round: L dominated by M
- Show that the one-sided incentive problem is dominance solvable.
- Show that the location choice game is dominance solvable.
- (More challenging) Show that Cournot competition is dominance solvable when $c_i(s_i) = 0$ and $p = 1 - s_1 - s_2$.
- (More challenging) Show that with 3 firms, Cournot competition is not dominance solvable.
- Show that none of the coordination games or conflicts is dominance solvable.

Nash Equilibrium

- Most games are not dominance solvable
- How to predict what others do?

- Look for strategy profiles where all players are satisfied
    - Take the others' actions as given
    - Ask if you could improve your payoff
    - If not, then you are happy at the current profile
- Call such profiles Nash equilibria

Definition 6 (Pure Strategy Nash Equilibrium) *A strategy profile $s = (s_1, ..., s_n)$ is a pure strategy Nash equilibrium if for every $i \in \{1, ..., N\}$,*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \text{ for all } s_i' \in S_i$$

Definition 7 (Best Response) *A strategy s_i is a best response for player i to his rivals' strategies s_{-i} if for all $s_i' \in S_i$,*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

- A strategy profile is a Nash equilibrium if each player's strategy s_i is a best response to strategy profile s_{-i} of all remaining players: $s \in BR(s)$
- Pure strategy Nash equilibrium s is a fixed-point under the best response mapping

Computing Nash equilibria

- Fix a Column and calculate best response for Row player. Fix the best response row and calculate BR for column player. Continue until you find a column and a row that are best responses to one another. If no such pair exists, that game has no pure strategy Nash equilibria.
- For games with infinitely many strategies, calculate the BR for player i for all possible s_{-i} (these are called the reaction curves of the players). This gives N equations in N unknowns (the strategies). Solve simultaneously for Nash equilibrium.

Interpretation of Nash equilibrium

- Consistency of choices across players
- Stable outcome in repetitions
- Minimal requirement

Definition 8 (Mixed strategy) *A mixed strategy for player i, $\sigma_i : S_i \to [0,1]$ assigns to each pure strategy $s_i \in S_i$, a probability $\sigma_i(s_i) \geq 0$ that it will be played, where*

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1$$

- Mixed strategy can be represented as a set of probabilities over pure strategies

$$\Delta(S_i) = \Big\{ (\sigma_i(s_1), ..., \sigma_i(s_n)) \in \mathbb{R}^n | \sigma_i(s_{ik}) \geq 0 \; \forall k, \text{ and } \sum_{k=1}^{n} \sigma_i(s_{ik}) = 1 \Big\}$$

- An alternative notation for set of mixed strategies is

$$\sigma_i \in \Sigma_i.$$

- Since players choose independently, they randomise independently
- So, probability that a pure strategy profile $s' = (s'_1, ..., s'_N)$ is chosen is given by

$$\prod_{i=1}^{N} \sigma_i(s'_i)$$

- The expected utility of any *pure strategy* s_i when some of the remaining players choose a mixed strategy profile σ_{-i} is

$$u_i(\sigma_1, ..., \sigma_{i-1}, s_i, \sigma_{i+1}, ..., \sigma_I) = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_1, ..., s_{i-1}, s_i, s_{i+1}, ..., s_I)$$

- Expected utility of any mixed strategy σ_i is

$$u_i(\sigma_1, ..., \sigma_{i-1}, \sigma_i, \sigma_{i+1}, ..., \sigma_I) = \sum_{s \in S} \left(\prod_j \sigma_j(s_j) \right) u_i(s_1, ..., s_{i-1}, s_i, s_{i+1}, ..., s_I)$$

Definition 9 (Best Response) *A strategy σ_i is a* best response *for player i to his rivals strategies σ_{-i} if*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Sigma_i$

Definition 10 (Nash Equilibrium) *A mixed strategy profile $\sigma^* = (\sigma_1^*, ..., \sigma_N^*)$ is a Nash equilibrium if for every $i \in \{1, ..., N\}$*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

for all $\sigma_i \in \Sigma_i$

We say that a game is finite if it has finitely many players, and if each player has a finite number of pure strategies.

Theorem 1 (Existence of Nash Equilibrium) *Every finite game has a mixed strategy equilibrium*

This theorem (proved by John Nash) tells us that Nash equilibrium is a well founded solution concept for finite normal form games. The theorem has been generalized for infinite strategy spaces and for infinitely many players, but these cases need additional assumptions on the payoff functions.

Computing Nash equilibria

Pure strategy equilibria in finite 2-player games

Done already

Pure strategy equilibria in N-player games with continuous actions

Cournot equilibrium

Cournot Competition N firms choose quantities $s_i \in S_i = [0, 100]$. Market price is determined as the intersection of demand and supply.

$$p = f(q)$$

$$q = \sum_i s_i$$

Revenue of firm i is $s_i f(\sum_i s_i)$. Cost of each firm is $c_i(s_i)$.
Payoffs are $u_i = s_i f(\sum_i s_i) - c_i(s_i)$.
First order condition for optimal choice of s_i for firm i

$$f(\sum_i s_i) + s_i f'(\sum_i s_i) - c_i'(s_i) = 0$$

Second order condition:

$$2f'(\sum_i s_i) + s_i f''(\sum_i s_i) - c_i''(s_i) \leq 0$$

Notice that this is satisfied if f is concave (or linear) and c_i is convex. In this case, first order condition is also sufficient so solving

$$f(\sum_i s_i) + s_i f'(\sum_i s_i) - c_i'(s_i) = 0$$

simultaneously for all i gives the solution.
If there are two firms, we get

$$\begin{aligned} f(s_1+s_2)+s_1 f'(s_1+s_2)-c_1'(s_1) &= 0, \\ f(s_1+s_2)+s_2 f'(s_1+s_2)-c_2'(s_2) &= 0. \end{aligned}$$

Assuming constant marginal costs, $c_i(s_i) = cs_i$, we get

$$2f(s_1+s_2)+(s_1+s_2)f'(s_1+s_2)-2c=0,$$

or

$$\frac{f(q)-c}{f(q)} = -\frac{qf'(q)}{2f(q)}$$

or

$$\frac{p-c}{p} = -\frac{1}{2\varepsilon_p}.$$

Location choice with network externalities

They have own preference for characteristics b_i, but also want to locate close to average characteristics $\bar{s}$ to ensure compatibility.

Payoff of firm i could then be

$u_i(s,\bar{s}) = -(s_i-b_i)^2 - \beta(s_i-\bar{s})^2$

Since the payoffs are strictly concave in own strategies, first order conditions are sufficient.

First order condition for firm i:

$$-2(s_i-b_i) = 2\beta(\frac{n-1}{n})(s_i-\bar{s})$$

Summing over i:

$$\bar{s} = \bar{b}.$$

and therefore

$$s_i(1+\beta\frac{n-1}{n}) = b_i + \beta\frac{n-1}{n}\bar{b}$$

or

$$s_i = b_i + \frac{\beta\frac{n-1}{n}}{1+\beta\frac{n-1}{n}}\sum_{j\neq i} b_j$$

		Bruce	
		Opera	Football
Sheila	Opera	$2,1$	$0,0$
	Football	$0,0$	$1,2$

Figure 12: Battle of Sexes

Computing mixed equilibria

Batlle of sexes

It is easy to see that the game has two pure strategy equilibria (O,O) and (F,F).

Recall that a mixed strategy is a probability distribution on pure strategies.

Then a strategy for Sheila has the following payoff if Bruce is playing a mixed strategy $\sigma_B(\cdot)$

$$u_S(\sigma_B, O) = 2\sigma_B(O) + 0\sigma_B(F),$$

and similarly

$$u_S(\sigma_B, F) = 0\sigma_B(O) + 1\sigma_B(F).$$

Sheila's payoffs from the two strategies:

$$u_S(\sigma_B, O) = u_S(\sigma_B, F)$$

or explicitly

$$2\sigma_B(O) + 0\sigma_B(F) = 0\sigma_B(O) + 1\sigma_B(F). \tag{2}$$

Thus when we consider whether Sheila will randomize, we examine conditions on the mixing behavior of Bruce under which Sheila is indifferent. Let

$$\sigma_B(F) = \sigma_B$$

and consequently

$$\sigma_B(O) = 1 - \sigma_B.$$

The condition (10) can then be written as

$$2(1 - \sigma_B) = \sigma_B \Leftrightarrow \sigma_B = \frac{2}{3}. \tag{3}$$

Thus if Sheila is to be prepared to randomize, Bruce must randomize according to (12).

Bruce's payoffs:

$$u_B(\sigma_S, O) = u_B(\sigma_S, F)$$

or explicitly

$$\sigma_S(O) + 0\sigma_S(F) = 0\sigma_S(O) + 2\sigma_S(F). \tag{4}$$

Thus as we consider whether Bruce is willing to randomize, we examine conditions on the mixing behavior of Sheila to make Bruce indifferent. Let

$$\sigma_S(F) = \sigma_S$$

and consequently

$$\sigma_S(O) = 1 - \sigma_S.$$

The condition (4) can then be written as

$$(1 - \sigma_S) = 2\sigma_S \Leftrightarrow \sigma_S = \frac{1}{3}. \tag{5}$$

Thus we reached the conclusion that the following is a mixed strategy Nash equilibrium

$$\sigma_B^* = \left(\sigma_B(O) = \frac{1}{3}, \sigma_B(F) = \frac{2}{3}\right)$$
$$\sigma_S^* = \left(\sigma_S(O) = \frac{2}{3}, \sigma_S(F) = \frac{1}{3}\right).$$

Crime and punishment

The payoff from crime depends on whether or not criminal gets caught.

Assume that the benefit of crime when not caught is $b > 0$.

If caught, punishment yields disutility of $-d < 0$.

Not committing the crime yields a payoff of 0.

The police authority decides whether to patrol the street of the criminal.

Patrolling involves a cost of c, not patrolling involves no cost.

If a crime is committed, the police gets utility u from catching the criminal.

If the criminal is not caught, police suffers a loss of l.

		Criminal	
		Commit	Don't
Police	Patrol	$u - c, -d$	$-c, 0$
	Don't	$-l, b$	$0, 0$

Figure 13: Monitoring game

Assume that $u > c$.

This yields the following game:

Does this game have pure strategy equilibria?

What happens to equilibrium level of crime if punishments are made harsher or if benefits from crime increase?

What if the cost of patrolling goes up?

Other interpretations for this game: Moral hazard with costly monitoring. Worker may exert effort at a positive cost. Supervisor may monitor at a positive cost. In equilibrium, workers work with positive probability and monitoring takes place with positive probability.

Continuous mixed strategies

- Two firms compete for a single buyer.
- Each has a single indivisible unit of the good and the buyer has unit demand, i.e. demands at most one unit of the good.
- The good is worth v to the buyer.
- Firms compete in offering prices p_i.
- Buyer buys from the firm with the lower price (with ties, she randomizes)
- If both firms always make offers, this is Bertrand competition and equilibrium prices are at marginal costs (zero here).
- Suppose next that it costs c (small) for a firm to make an offer (regardless of the price quote)
- Firms decide simultaneously the whether to make an offer and the price they quote

- Buyer buys from the cheapest firm as long as price offer is below v
- No pure strategy equilibria.
- If both make offers, then prices driven to 0 by Bertrand competition. This results in a loss of c for each firm.
- If i makes no offer, then j offers $p_j = v$. But then i should make an offer $p_i = v - \epsilon$.
- It is not possible that both randomize over prices but mix over prices
    - Suppose they do, and consider the highest price.
    - Highest price cannot be offered with a strictly positive probability.
    - Since highest price wins only when the opponent also prices at the highest price, there would be an incentive to undercut the highest price slightly.
    - Hence the highest price always loses.
    - Therefore the highest price cannot exist in this case with $c > 0$.
- The only possibility is then that offers are made with probability x where $0 < x < 1$.
- Since a firm makes zero profit by not offering, equilibrium profit from all offers must be zero.
- Prices are offered on $[c, v]$.
- Each price must result in the same profit.
- Price $p_i = c$ wins with probability 1. $p_i = v$ wins with probability x.
- Therefore $x = \frac{c}{v}$.
- Also for all p_i, $1 - F(p_i) = \frac{c}{p_i}$, where $F(p)$ is the distribution of offered prices (whose total mass is $1 - x$ by the previous argument)

Interpretation of mixed strategy equilibria

Finally, consider the following interpretations of a mixed strategy Nash equilibrium:

1. mixed strategies as objects of choice
2. steady state, probability as a frequency of certain acts
3. mixed strategies as pure strategies in a game where the randomness is introduced through some random private information (Harsanyi's purification argument)

Correlated randomizations

- In the battle of sexes, the mixed strategy equilibrium used private randomization by each player.
- Suppose instead they would have a common (or public) randomization device, say a coin, then they could realize a payoff of $(\frac{3}{2}, \frac{3}{2})$ by e.g. both going to opera if heads and both going to football if tails.
- Observe that this payoff is far superior to the mixed strategy equilibrium payoff
- The notion of public randomization is used in the *correlated equilibrium*. In macroeconomic settings, correlated equilibria are often referred to as sunspot equilibria.
- more complicated correlation structures yield even better outcomes in games such as chicken

3 Extensive form Games

Games of Perfect Information

- Games where timing plays a role
- Defining properties of games of perfect information?
  - At each point in time, a single player moves
  - All past moves are observed
  - Players receive payoffs at the end of the game when all play has ended
- Examples
  1. oligopolistic competition: first- or second-mover advantage?
     - commitment versus flexibility to react
     - how do we decide which firm moves first?
  2. bargaining: how should two individuals share a surplus?
     - players take turns in making proposals
     - each proposal is either accepted or rejected
     - acceptance ends the game, rejection leads to a proposal by the rejecting player
     - how to think about first proposer?
  3. parlour games
     - chess
     - tick-tack-toe
- How to analyze games of perfect information?
  - graphical representation as game trees
  - nodes correspond to decision moments
  - edges correspond to actions (sometimes, edges are drawn as arrows)

- the game tree has a single node with no preceding edges called the initial node
- terminal nodes nodes with no further edges
- no cycles: each node is reached from the initial node by a different sequence of edges
- payoffs are assigned to each terminal node

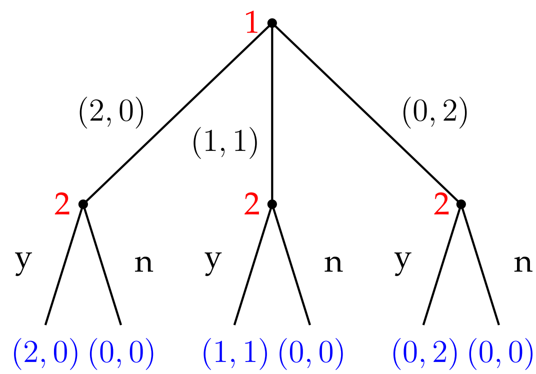
- Interpreting the perfect information game tree
  - each decision node corresponds to a unique sequence of actions called the history of the game at the decision node
  - the history summarizes all relevant information in the game up to the decision point
  - outcomes in the game are associated with complete paths of play, i.e. with terminal nodes
  - payoffs evaluate the consequences

- Strategy: a full contingent plan of action for entire game
  - includes selection of action for all possible decision nodes
  - for mixed strategies, player could choose a mixed action at each decision node item is this really all that a mixed strategy could do? (Kuhn's theorem)
  - player should be able to pass plan to his lawyer (who then conducts the play) so that lawyer does not need to contact player once game starts
  - if all players do this, the strategy profile results in a single outcome (if players use pure strategies)
  - how large are the sets of pure strategies for the players in simple games?

- Example game: bargaining
  - 2 players are trying to share 2 indivisible units of a good
  - player 1 makes a take-it-or-leave-it offer to player 2

- player 2 after having observed offer decides whether to accept or reject offer
- payoffs described in figure 14

Figure 14: An extensive form game



- Set of players: $N = \{1, 2\}$
- Set of histories of game

$$\mathcal{H} = \{\emptyset, (2,0), (1,1), (0,2), [(2,0), y], [(2,0), n], [(1,1), y], [(1,1), n], [(0,2), y], [(0,2), n]\}$$

- Player function describing which player chooses an action after which (non-terminal) history

$$P(\emptyset) = 1,\ P((2,0)) = P((1,1)) = P((0,2)) = 2$$

- Players' payoffs associated with each terminal history

$$\begin{aligned} u_1([(2,0), y]) &= 2, & u_2([(2,0), y]) &= 0, \\ u_1([(2,0), n]) &= 0, & u_2([(2,0), n]) &= 0, \\ u_1([(1,1), y]) &= 1, & u_2([(1,1), y]) &= 1, \\ u_1([(1,1), n]) &= 0, & u_2([(1,1), n]) &= 0, \\ u_1([(0,2), y]) &= 0, & u_2([(0,2), y]) &= 2, \\ u_1([(0,2), n]) &= 0, & u_2([(0,2), n]) &= 0 \end{aligned}$$

- Strategies specifying actions chosen by each player for every history
  - player 1: $s_1(\emptyset) \in \{(2,0),(1,1),(0,2)\}$
  - hence player 1 has 3 strategies
  - player 2: $s_2(2,0), s_2(1,1), s_2(0,2) \in \{y,n\}$
  - hence player 2 has 8 strategies: $(y,y,y),(y,y,n),(y,n,n),(y,n,y),(n,y,y),(n,y,n),$ $(n,n,y),(n,n,n)$
- What are the Nash equilibria (in pure strategies) of this game?
  - there are many
  - e.g.: $((2,0),(y,y,y))$; $((2,0),(y,y,n))$; $((2,0),(y,n,n))$ are all Nash equilibria (why?)
  - find all the other pure strategy Nash equilibria
- To answer this question, may be easier to look at normal form associated with extensive form game

	(y,y,y)	(y,y,n)	(y,n,n)	(y,n,y)	$\ldots$
$(2,0)$	$2,0$				
$(1,1)$					
$(0,2)$					

- Problem with Nash equilibria: not all of them are 'reasonable'
  - e.g., what do we think about the Nash equilibrium $(0,2),(n,n,y)$?
  - do we believe that player 2 would choose n after player 1 chooses $(1,1)$?
- Alternative method of solution: backward induction
  - start at end of game
  - at last decision nodes before terminal nodes, decision problems
  - replace these node with the outcome resulting from the decision problem

- this results in a shorter game tree
- repeat the step with the new shorter tree
- after finite steps, the every game of finite length is solved

- In bargaining game

  - if at $(2, 0)$ branch: player 2 is indifferent between y and n
  - if at $(1, 1)$ branch: player 2 strictly prefers y to n
  - if at $(0, 2)$ branch: player 2 strictly prefers y to n

  $\Rightarrow$ only relevant strategies are (y, y, y) and (n, y, y)

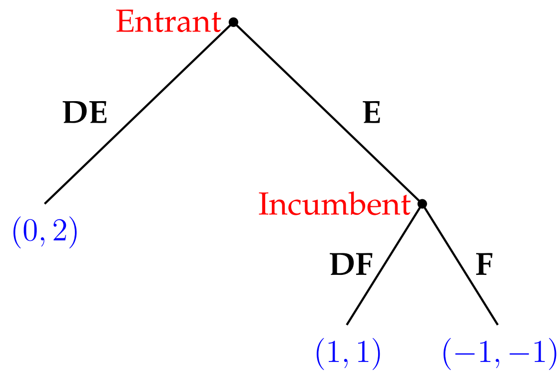
Zermelo's theorem:

if at most one player moves at any time and the game ends after finitely many moves, a solution by backwards induction exists

Remark: E.g. chess has therefore a backwards induction solution. Of course, we do not know whether white wins, black wins or if the game has a draw as its solution.

If there are no ties in payoffs for any player, then the backwards induction outcome is unique.

Figure 15: An entry game



- 2 Nash equilibria: $(DE, (\cdot, F)), (E, (\cdot, DF))$
- 1 solution through backward induction: $(E, (\cdot, DF))$
- Backward induction can capture idea of credibility which is missing in Nash eqm
- Strategic reasoning also seems less demanding than for Nash eqm

Stackelberg competition

- Quantity competition is a symmetric market. Three output levels: $\{L, M, H\}$.
- Firm 1 chooses the quantity (row) first
- Firm 2 observes firm 1′s choice and chooses her quantity (column)
- The payoffs from all outcomes are given in the table below
- What are the strategies of the two firms?

- Who gets higher payoff, who gains relative to normal form game, what does first mover advantage mean?

	L	M	H
L	$18, 18$	$15, 20$	$9, 18$
M	$20, 15$	$16, 16$	$8, 12$
H	$18, 9$	$12, 8$	$0, 0$

Figure 16: Finite Stackelberg model

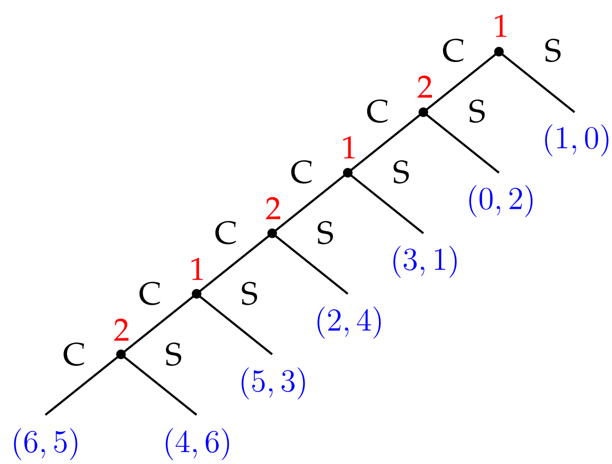
For games of perfect information, each decision node starts a shorter game tree within the original called a subgame.

The following notion is slightly more general but close to the idea of backwards induction:

Definition 11 (Subgame perfection) *A strategy profile in the original game is a subgame perfect equilibrium if play in every continuation (or sub-) game forms a Nash equilibrium, i.e., the strategies of each player are optimal given the strategies of all other players*

- Extreme example: centipede game
  - unique SPE: each player chooses S at every history
  - is this prediction intuitively appealing?
  - subgame perfection should capture implications of rational play along the game tree
  - what to conclude about rationality if moves by the opponent contradict subgame perfection?

Figure 17: The centipede game



Bargaining games

- Which allocations in Edgeworth box are reasonable agreement points?
  - Edgeworth: depends on 'bargaining power' which is best left for psychologists
  - Rubinstein: assume that delay is costly and solve for a SPE in a game with alternating offers
- Simple example
  - 2 players have to divide 1 unit between them
  - player A discounts future at α, player B at β
  - 3 periods
  - player A makes offer in 1st period; player B in 2nd period; player A in 3rd period
  - player not offering can accept or reject
  - if accept, game ends; if reject, go to next period

- if reject in 3rd period, players receive 0
- what is the SPE?

- Start in last period: player A will offer player B zero
- In 2nd period
  - player A is guaranteed a payoff of α by rejecting any offer
  - hence player B has to offer player A at least α
  - so player B offers $(\alpha, 1-\alpha)$
- In 1st period
  - player B is guaranteed a payoff of $\beta(1-\alpha)$ by rejecting any offer
  - hence player A has to offer player B at least $\beta(1-\alpha)$
  - so player A offers $(1-\beta(1-\alpha), \beta(1-\alpha))$
- Notice that if player A is perfectly patient ($\alpha = 1$), gets whole surplus
- What about infinite horizon, with alternating offers?
  - player A makes offers in odd periods, player B in even periods
  - game is stationary: continuation game starting at every odd (even) period is same
  - let x_i^H = highest equilibrium share player A can get in a subgame starting in a period in which player $i \in \{A, B\}$ makes an offer
  - let x_i^L = lowest equilibrium share player A can get in a subgame starting in a period in which player $i \in \{A, B\}$ makes an offer
- Equilibrium conditions
  - it must be the case that

$$x_B^H \leq \alpha x_A^H, \qquad 1 - x_A^L \leq \beta(1 - x_B^L)$$

    i.e., opponent will offer less than when it's your turn to offer

- also

$$x_B^L \geq \alpha x_A^L, \qquad 1 - x_A^H \geq \beta(1 - x_B^H)$$

  otherwise reject and wait until next period
- substituting

$$x_A^L \geq \frac{1-\beta}{1-\alpha\beta} \geq x_A^H, \qquad x_B^L \geq \frac{\alpha(1-\beta)}{1-\alpha\beta} \geq x_B^H$$

- an equilibrium solution is therefore

$$x_A = \frac{1-\beta}{1-\alpha\beta}, \qquad x_B = \frac{\alpha(1-\beta)}{1-\alpha\beta}$$

  where x_A is offered in odd periods, x_B is offered in even periods
- and in fact, this is unique SPE

• Features of SPE

  - player B accepts 1st offer made
  - eqm payoffs:

$$\frac{1-\beta}{1-\alpha\beta}, \frac{\beta(1-\alpha)}{1-\alpha\beta}$$

  - $\alpha > \beta \Rightarrow$ player A gets most surplus

Multistage games with observed actions

- Many players may move at each point in time
- All past moves are observed
- How to draw game trees: information sets
- How to analyze: Subgame perfect equilibrium

Information sets

- represent what is know to a player at the moment when called to move

- if more than one node in information set, then player cannot tell apart nodes
    - therefore must have same actions at all these nodes
    - must have taken same own actions to reach these nodes
    - must have had same prior information at all nodes on the path to current node
- normal form game is represented by a tree where each player has a single information set
- no actions of other players can be told apart
- in multistage games with observable actions, the information of each player is determined by past play in the previous stages
- information set then coincides with the history of the game
- each history induces a new subgame
- hence subgame perfect equilibrium is the appropriate concept to analyze these games

Repeated games

- same normal form game (called the stage game) played in peach stage or period over a finite or infinite horizon
- payoffs obtained by summing over the stage game payoffs (discounted in the infinite horizon case)
- how does the repetition of the interaction change equilibrium outcomes?
- will the players be able to achieve better outcomes in prisoners' dilemma or one-sided incentive problem?
- properties of repeated game strategies

Analysis

- how to use subgame perfection?
- with finite horizon, start with the last period, find Nash equilibria
- play in next to last stage must be followed by play of one of the Nash equilibria in the last
- if the stage game has a unique Nash equilibrium, then finitely repeated versions of the same game have a unique subgame perfect Nash equilibrium
- do we conclude that repetition makes no difference?
- Suppose stage game has multiple equilibria
- recall from normal form games:

	F	Y
F	$-1,-1$	$5,0$
Y	$0,5$	$4,4$

Figure 18: Chicken

- notice that the game has two pure strategy Nash equilibria
- suppose the game is played twice, can Y, Y be supported as part of equilibrium in first period?
- suppose the game is played three times
- the key to achieving good outcomes is the possibility to adjust future play to reward good behavior and to punish bad behavior

Infinite horizon

- there is always a future (that looks the same as the entire game)
- to have well defined payoffs, assume constant discount factor between periods, δ.

- players then maximize the discounted sum of payoffs
- with infinite horizon, no natural place to start backwards induction
- does this change the analysis of games with a single stage game Nash equilibrium
- strategies (even pure strategies) in infinitely repeated games are complicated
- to proceed with analysis, consider simple strategies
- for example, divide the set of all histories into good and bad histories
- if history is good, play cooperatively using strategies that are not stage game Nash equilibrium strategies
- if history is bad, play stage game Nash equilibrium

Illustration with prisoners' dilemma

	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

Figure 19: Prisoners' Dilemma

- a history is good if no player has ever played D
- a history is bad if some player has played D at some past point
- claim: a pair of strategies where both players play C as long as history is good and D if history is bad forms a subgame perfect Nash equilibrium
- consider first behavior for bad histories
    - regardless of own choices, all future histories will be classified as bad
    - therefore opponent always plays D

- therefore each player should always choose D

- consider next good histories
  - opponent plays C
  - therefore history in next period is good if player plays C in current period and bad in player plays D
  - the payoff from the first is (by one shot deviation principle)

$$3 + \frac{3\delta}{1-\delta}$$

  - payoff from D is

$$4 + \frac{\delta}{1-\delta}$$

  - therefor it is optimal to play C at good histories if

$$3 + \frac{3\delta}{1-\delta} > 4 + \frac{\delta}{1-\delta}$$

  - this is the same as

$$2\delta \geq 1 - \delta \Leftrightarrow \delta \geq \frac{1}{3}$$

- in equilibrium, the short run incentive to gain is tempered with the threat of losing future benefits from cooperation
- the general point is that it is possible to enforce good behavior as long as the discount factors are sufficiently high
- this result is known in the literature under the name 'Folk Theorem'
- reasoning along these lines has been applied in:
  - models of collusion in oligopolies (firms price cooperatively highly near monopoly prices because of a threat to revert to price wars).
  - models of public goods provision over time such as participation in trade agreements, climate agreements etc
  - informal compensation in employment agreements: bonuses etc.
  - formation of reputation for good quality in one-sided incentive problems

War of Attrition

- An example of games of timing
- Two players
- Each player has a single decision, when to quit
- The game ends at first quit
- Payoffs depend on the identity of the player to quit and the period where quitting takes place
- As long as both stay in the game, both players lose c per period
- If i quits, then j makes a profit of v in each future period (i makes neither losses nor profits)
- The game has many equilibria: i quits in all periods and j never quits is a subgame perfect equilibrium (why?)
- Find a symmetric equilibrium
    - Both players must randomize (why?)
    - Therefore both players must make an expected profit of 0 (why?)
    - Let p denote the probability of exit
    - Then $-c + p\frac{v}{1-\delta} = 0$ or
    - $p = \frac{(1-\delta)c}{v}$
- This model is the starting point for analyzing exit in declining industries, lobbying for political favor, competition for an indivisible resource such as a mate in biology

Export Subsidies

- An example of a two-stage structure
- The players at each of the stages may be the same or different

- Analysis cares mostly bout the preferences of those that move in the first stage
- First stage actions determine the game for the second stage players
- More examples of the structure at the end
- Two governments in a two country model decide export subsidies in the first stage
- In the second stage two firms (one in each country) choose production levels given tariffs
- Positive analysis: How will the governments set tariffs?
- Normative analysis: How could one overcome resulting inefficiencies?
- Find subgame perfect equilibria and start at end.
- Suppose subsidy in country i is set at t_i for $i \in \{1, 2\}$.
- Denote the production levels by q_i^j for the quantity that firm located in i produces for market in j.
- Assume linear demand:

$$p = 1 - q_1 - q_2$$

- Assume also constant marginal costs, and normalize the cost to zero.
- The company located in country i competes both at home and abroad.
- Notice that in this simple example, we can analyze the two countries independently.
- For concreteness, consider competition in country 2.
- If country 1 pays an export subsidy for its own firm, then profits are:

$$u_1(q_1^2, q_2^2, s_1) = (1 - q_1^2 - q_2^2)q_1^2 + s_1 q_1^2$$

- Since this is a concave function, first order conditions are sufficient.
- F.O.C.:

$$q_1^2 = \frac{1 + s_1 - q_2^2}{2}$$

$$u_2(q_1^2, q_2^2, s_1) = (1 - q_1^2 - q_2^2)q_2^2$$

- Again F.O.C. is sufficient:

$$q_2^2 = \frac{1 - q_1^2}{2}$$

- Solving simultaneously:

$$q_1^2 = \frac{1 + 2s_1}{3}$$
$$q_2^2 = \frac{1 - s_1}{3}$$

- Consider the first stage choice of optimal s_i for government in country i.
- Objective function: to maximize the profit of own country company net of subsidy.
- Profit of firm 1 in country 2 is

$$q_1^2((1 - q_1^2 - q_2^2) + s_1)q_1^2$$

or

$$\frac{1 + 2s_1(1 - s_1)}{9}$$

- Maximizing over s_1 yields $s = \frac{1}{4}$.

Discussion

- Is the outcome Pareto efficient ?
- What are the right welfare criteria?

- How are the consumers taken into account?
- How can inefficiencies be avoided?
- Repeated games provide an easy way out of these problems.
- How about international agreements?
- Then need to worry about participation in the agreements

Other examples

- Could do similar analysis for tariffs etc.
- Two firms choose product quality in the first period, then compete. Some examples of this in Microeconomic Theory III
- Owners of competing firms first sign contracts with managers (how to reward). In the second stage, managers choose firm actions to maximize their payoff stemming from the contract.

4 Games of incomplete information (Bayesian Games)

- An example
  - N firms are bidding for a contract to build Musiikkitalo
  - each firm has superior knowledge about its cost of completing project
  - rivals have imperfect information about true costs
  - exact building conditions are to some extent uncertain to all parties
  - each party commissions a survey about general site condition
- To model this situation, need to address 2 sources of uncertainty
  - privately known costs of bidding firms: private values information
  - general condition of site: common values information
- How will firms bid for the project?
  - if a firm has low costs, it might bid quite aggressively (a low price) in order to have a good chance of winning project
  - but it does not pay to be too aggressive since lower bids eat away profit when winning project
  - costs of each firm affect equilibrium payoffs: firm j is more likely to win when firm i has a high cost (as long as bids are monotone functions of cost)
- Rules of bidding
  - most commonly used bidding game is an auction
  - most commonly used auction form for procurement is a sealed bid first price auction
  - all bidders simultaneously submit bids in sealed envelopes to auctioneer who then proceeds to open them in public

- bidder with lowest quoted price for completing the project is awarded project
- variations: use a reservation price; entry fees

- Ascending price auction or English auction
  - often used for selling an object
  - all bidders are present and call prices to auctioneer
  - each called price has to exceed previous bid by a prespecified amount
  - once nobody wants to raise price already called, object is awarded to last bidder at price called

- Descending price or Dutch auction
  - auctioneer starts at a very high price
  - price is lowered as long as at least one of bidders accepts the price
  - first bidder to accept a price gets object at that price

- Questions arising from example
  - theoretical: how to capture essential features of such a complicated interaction in a model that is amenable to game theoretic analysis?
  - empirical: is solution to game a good prediction of bidding in actual auctions?
  - normative: is solution good from a normative point of view?

- Two main objectives: Pareto efficiency, revenue maximization

- Issue belongs to broader class: mechanism design
  find (design) a game (mechanism) that performs well according to some criterion (revenue max or Pareto efficiency)

Modelling Games of Incomplete Information

- Harsanyi: a game of incomplete information is given by

  1. set of players: $i \in \{1, 2, ..., N\}$
  2. actions available to player i: A_i for $i \in \{1, 2, ..., N\}$. Let $a_i \in A_i$ denote a typical action for player i
  3. sets of possible types for all players: Θ_i for $i \in \{1, 2, ..., N\}$. Let $\theta_i \in \Theta_i$ denote a typical type of player i
  4. let $a = (a_1, ..., a_N)$, $\theta = (\theta_1, ..., \theta_N)$, $a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_N)$, $\theta_{-i} = (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_N)$ etc
  5. natures move: θ is selected according to a joint probability distribution $p(\theta)$ on $\Theta = \Theta_1 \times \cdots \times \Theta_N$
  6. strategies: $s_i : \Theta_i \to A_i$, for $i \in \{1, 2, ..., N\}$. $s_i(\theta_i) \in A_i$ is then the action that type θ_i of player i takes
  7. payoffs: $u_i(a_1, ..., a_N; \theta_1, ..., \theta_N)$

-

- Game proceeds as follows

  - nature chooses θ according to $p(\theta)$
  - each player i observes realized type $\widehat{\theta}_i$ and updates her beliefs
    - each player comes up with conditional probability on remaining types conditional on $\theta_i = \widehat{\theta}_i$
    - denote distribution on θ_{-i} conditional on $\widehat{\theta}_i$ by $p_i(\theta_{-i}|\widehat{\theta}_i)$
  - players take actions simultaneously

- Solution concept: Bayesian Nash equilibrium

Definition 12 *A strategy profile $(s_1(\theta_1), ..., s_N(\theta_N))$ is a (pure strategy) Bayesian Nash Equilibrium if $s_i(\theta_i)$ is a best response to $\boldsymbol{s}_{-i}(\boldsymbol{\theta}_{-i})$ for all $\theta_i \in \Theta_i$ and for all i*

- Action specified by strategy of any given player has to be optimal given strategies of all other players and beliefs of player

- Expected payoff
  - given strategy $s_i(\cdot)$, type θ_i of player i plays action $s_i(\theta_i)$
  - with vector of types $\theta = (\theta_1, ..., \theta_N)$ and strategies $(s_1, ..., s_N)$, realized action profile is $(s_1(\theta_1), ..., s_N(\theta_N))$
  - player i of type $\widehat{\theta}_i$ has beliefs about types of other players given by conditional probability distribution $p_i(\theta_{-i}|\widehat{\theta}_i)$
  - expected payoff from action s_i is

$$\sum_{\boldsymbol{\theta}_{-i}} u_i(s_i, s_{-i}(\theta_{-i}), \theta) p_i(\theta_{-i} \mid \widehat{\theta}_i)$$

- Best response: action $s_i(\widehat{t}_i)$ is a best response to $s_{-i}(t_{-i})$ if and only if for all $s_i' \in A_i$

$$\sum_{\boldsymbol{\theta}:_{-i}} u_i(s_i(\widehat{\theta}_i), s_{-i}(\theta_{-i}), \theta) p_i(\theta_{-i} \mid \widehat{\theta}_i) \geq \sum_{\boldsymbol{\theta}:_{-i}} u_i(s_i', s_{-i}(\theta_{-i}), \theta) p_i(\theta_{-i} \mid \widehat{\theta}_i)$$

A First Look at Auctions

- Seller puts a single object up for auction
- n potential buyers; each buyer i has a valuation of θ_i
- Each buyer known her own θ_i but other θ_{-i} are not known
- All buyers have common prior distributions for valuations of other buyers
  - assume all buyers are ex ante identical
  - values are distributed according to the uniform distribution on [0,1]
- Action of each buyer: announcement of a bid, $b_i \in \mathbb{R}_+$
- Strategy of a buyer: function from valuations to bids, i.e., $s_i(t_i) \in \mathbb{R}_+$

- First price auction:
    - all buyers announce their bids simultaneously
    - object is awarded to buyer submitting highest bid for a price equal to bid
    - in case of a tie, a symmetric randomization is used
- Payoff to bidder i as a function of her own type and all of bids is (ignoring ties)

$$u^i(\theta_i, b_1, ..., b_n) = \begin{cases} \theta_i - b_i & \text{if } b_i > b_j \text{ for all } j \neq i. \\ 0 & \text{otherwise} \end{cases}$$

item We start with a simple brute force computation

- Concentrate on simplest case of finding a symmetric equilibrium in linear bidding strategies

$$b_i = \gamma\theta_i$$

- Easy to see that $0 < \gamma < 1$ in equilibrium
- Assume now that such an equilibrium exists and compute γ
- A bid of b_i wins auction if for all $j \neq i, b_j < b_i$
- If other players use equilibrium strategies, then for all $b_i \leq \gamma$, the probability of winning is

$$\left(\frac{b_i}{\gamma}\right)^{N-1}$$

- Payoff conditional on winning is $\theta_i - b_i$
- Player i of type θ_i maximizes by choosing b_i:

$$\max_{b_i}(\theta_i - b_i)\left(\frac{b_i}{\gamma}\right)^{N-1}$$

- Concave payoff: first-order condition characterizes the maximum

$$\frac{(N-1)}{\gamma}\left(\frac{b_i}{\gamma}\right)^{N-2}(\theta_i - b_i) - \left(\frac{b_i}{\gamma}\right)^{N-1} = 0$$
$$b_i = \frac{N-1}{N}\theta_i$$

- Hence a linear equilibrium exists with

$$\gamma = \frac{N-1}{N}$$

- Bidding is more aggressive when there is more competition
- Which is better for seller: first price or second price auction?

Solving auctions with envelope theorem

- Recall from optimization the definition of a maximum value function

$$V(\theta) = \max_x u(x;\theta)$$

- Envelope theorem states that

$$V'(\theta) = \frac{\partial}{\partial\theta}u(x;\theta)$$

- Consider now an auction where the bid b is chosen optimally.
- There are N bidders.
- We can let the (symmetric) distribution of valuations to be an arbitrary distribution $F(\theta_i)$.
- Bidder i has an expected value

$$v(b_i, \theta_i) = (\theta_i - b_i)\text{Pr}(b_j < b_i \text{ for all} j \neq i)$$

  from bidding b_i when her type is θ_i.
- Denote the value function of bidder i by $v(\theta_i)$.

- Envelope theorem tells us that

$$v'(\theta_i) = \Pr(b_j < b_i \text{ for all} j \neq i)$$

- Therefore

$$v(\theta_i) = \int_0^{t_i} \Pr(b_j < b_i \text{ for all} j \neq i)$$

- If all players use symmetric strictly increasing strategies $b(\theta)$, then

$$\Pr(b_j < b_i \text{ for all} j \neq i) = F(\theta_i)^{N-1}$$

- Therefore

$$(\theta_i - b_i)F(\theta_i)^{N-1} = \int_0^{\theta_i} F(\theta)^{N-1} d\theta$$

  and

$$b_i = \theta_i - \frac{\int_0^{\theta_i} F(\theta)^{N-1} d\theta}{F(\theta_i)^{N-1}}$$

- Check that you get the same result as before for the uniform distribution case.

Continuing further

- Observe that nothing in the above derivation until the last step of solving $b(t)$ depended on the nature of the auction.
- Therefore we have much more generally

$$v(\theta_i) = \int_0^{\theta_i} F(\theta)^{N-1} d\theta.$$

- This holds for second price auction, for all pay auction and many more.
- The allocation determines entirely the payoff for the bidders.
- This is a special case of a quite general theorem derived in mechanism design and we'll say a bit more in the section on adverse selection.

- In our case, it leads immediately to the celebrated revenue equivalence theorem for private value auctions.
- if the expected payoffs to bidders are the same across auctions and if the allocations are the same (efficient), then the expected revenues to the seller must be the same.

5 Economics of Information

- basic model: principal-agent
  - principal has bargaining power
  - principal makes a take-it-or leave it offer to the agent
  - different preferences for the two parties
  - agent has superior information
  - how should the principal motivate the agent?
- 2 dimensions to information asymmetry: about
  - what an agent does: hidden action
  - who an agent is: hidden information
- 2 dimensions to who has the information
  - agent taking the initiative
  - agent not taking the initiative
- Leads to 3 families of models
  - adverse selection: uninformed party is imperfectly informed about characteristics of informed party; uninformed party moves first
  - signalling: same as above, but informed party moves first
  - moral hazard: uninformed party moves first and is imperfectly informed about actions of informed party
  - In these notes, we only cover Adverse Selection.

Market for lemons

- This section is based on Akerlof 1970, QJE, 'The Market for Lemons: Quality Uncertainty and the Market Mechanism'.
- Suppose an object with value $\theta \sim \mathcal{U}[0,1]$ is offered by the seller.
- The object is worth

  $$u_s = v_s \theta$$

  in monetary units to the seller and worth

  $$u_b = v_b \theta$$

  to the buyer with $v_b > v_s$.
- Hence trading is always pareto-optimal.
- The twist in this model is that only the seller knows θ.
- The motivating example for markets of this type used by Akerelof was the market for used cars. This explains why the model is sometimes referred to as the 'lemons' problem.
- In markets trade has to be voluntary, i.e. both parties must agree to the sale.We then ask if there is a price at which voluntary trade occurs.
- Suppose that there is trade at price p. What properties should the price have in order to induce trade. The seller sells if

  $$\theta v_s \leq p$$

  and thus by selling the object he *signals* that

  $$\theta \leq \frac{p}{v_s} \tag{6}$$

- The buyer buys the object if

  $$v_b \mathbb{E}[\theta] \geq p \tag{7}$$

  and as he knows that (6) has to hold, he can form a conditional expectation, that

  $$v_b \mathbb{E}[\theta | p] \geq p \Leftrightarrow v_b \frac{p}{2v_s} \geq p \tag{8}$$

- Thus for the sale to occur,

$$v_b \geq 2v_s. \tag{9}$$

- Thus unless, the tastes differ substantially, the market breaks down completely:
- Market mechanism in which a lower prices increases sales fails to work as lowering the price decreases the average quality, lower price is "bad news".
- Market may not disappear but display lower volume of transaction than socially optimal trading.

Extensions

- Generalize the distribution of the seller's types to a uniform density with *varying support* and constant mean:

$$\theta \sim \mathcal{U}\left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$$

with $\varepsilon \in \left[0, \frac{1}{2}\right]$.

- We can formalize the *amount* of private information by ε and how it affects the efficiency of the trade.
- Redo the analysis above where the seller signals by selling the object at a fixed price p that

$$\theta \leq \frac{p}{v_s}.$$

- The buyer buys the object if

$$v_b \mathbb{E}\left[\theta \left| p\right.\right] \geq p,$$

- The expected value is now given by

$$\mathbb{E}\left[\theta \left| p\right.\right] = \frac{\frac{1}{2} - \varepsilon + \frac{p}{v_s}}{2}$$

and hence for sale to occur

$$v_b \frac{\frac{1}{2} - \varepsilon + \frac{p}{v_s}}{2} \geq p. \tag{10}$$

- This inequality prevails if, provided that $\varepsilon \in [0, \frac{1}{2})$ and $2v_s > v_b$,

$$p \leq v_b v_s \frac{\frac{1}{2} - \varepsilon}{2v_s - v_b}$$

- Consider next the efficiency issue. All types of sellers are willing to sell if

$$p = v_s \left(\frac{1}{2} + \varepsilon \right) \tag{11}$$

in which case the expected conditional value for the buyer is

$$\frac{1}{2} v_b \geq v_s \left(\frac{1}{2} + \varepsilon \right)$$

or equivalently full trading requires

$$v_b \geq v_s (1 + 2\varepsilon) . \tag{12}$$

- Alternatively, the effect of asymmetric information can be reduced if only sellers with low valuations trade.

- Sellers below type $\left(\frac{1}{2} + x\right)$ trade if $p > v_s \left(\frac{1}{2} + x\right)$ and buyeraccept these trades if

$$v_b \frac{\frac{1}{2} - \varepsilon + \left(\frac{1}{2} + x\right)}{2} \geq p$$

- Hence we have some trading if

$$v_b \frac{1 - \varepsilon + x}{2} \geq v_s \left(\frac{1}{2} + x \right)$$

.

- Maximal trade:

$$x = \frac{v_b (1 - \varepsilon) - v_s}{2v_s - v_b}.$$

- Thus x is increasing in v_b and decreasing in ε,confirming now in terms of the volume of trade our intuition about private information and efficiency.

Screening and Adverse Selection

- Consider next principal-agent setting.
- A decision $x \in X$ must be taken.
- Agent has private information $\theta \in \Theta$. Principal believes that $\theta \sim F(\theta)$.
- The agent makes a payment to principal $t \in \mathbb{R}$. Negative payments are then transfers from principal to agent.
- Agent's utility function:
$$u(x, \theta) - t.$$
- Assume that u is twice differentiable and concave.
- Principal's utility function
$$v(x, \theta) + t.$$
- Assume that v is twice differentiable and concave.
- In a screening model, the principal proposes a menu of different contracts $\{x(\theta), t(\theta)\}$ to the agent.
- Agent chooses the one that is best for her.
- *Incentive compatibility* (IC^{θ}) requires that agent of type θ likes best contract $(x(\theta), t(\theta))$ or
$$u(x(\theta), \theta) - t(\theta) \geq u(x(\theta'), \theta) - t(\theta') \text{ for all } \theta'.$$
- In words, agent of type θ gets at least as high utility from the contract meant to her $(x(\theta), t(\theta))$ thank from any contract meant for another type θ', i.e. $(x(\theta'), t(\theta'))$.
- *Participation Constraint* or *Individual Rationality* (IR^{θ}) requires that the agent be better off accepting than refusing the contract:
$$u(x(\theta), \theta) - t(\theta) \geq U,$$
where U denotes the utility from not accepting the contract.

Examples

- Insurance: Monopolist insurance company is the principal, insured is the agent. x is the amount of insurance, θ is the risk type (probability of accident) and t is the insurance premium.
- Regulation: Regulator as principal, regulated firm as agent. θ the privately known marginal cost, x production target, t the transfer from regulator to the firm to cover costs.
- Finance: Monopolist banker principal, entrepreneur agent. θ probability of success for entrepreneur's project, x amount of lending, t the repayment scheme.
- Nonlinear pricing: Monopolist seller as principal, buyer as agent. θ a taste parameter, x quantity of sales (or quality), t total price of the package. This is of course the most importan case for 31E11100.
- Employment: Employer as principal, employee as agent. θ demand conditions, x level of sales, t wage payment from principal to agent.
- ...Many others

Basic Approach

$$\max \mathbb{E}v\left(x,\theta\right)+t$$
$$\text{subject to IC,IR.}$$

First best

- We cover the case $\theta \in \{\theta^H, \theta^L\}$ with $\theta^H > \theta^L$ in this lecture. Let $p = \Pr\{\theta = \theta^H\}$.
- Consider first-best problem for the social planner, i.e. maximization of surplus for a known θ.
- Then we have
$$\max_x v\left(x,\theta\right)+u\left(x,\theta\right).$$

- First order condition (necessary and sufficient by concavity):

$$v_x(x,\theta) + u_x(x,\theta) = 0.$$

- Exercise: Think about surplus maximization and first order condition for all the examples above.
- Let $\widehat{x}^s$ be the solution to the first best problem when $\theta = \theta^s$.
- If principal gets to propose a contract and observes θ, then $x =$ and transfer is solved from:

$$u(\widehat{x}^s, \theta^s) - \widehat{t}^s = U \text{ for all } s.$$

Second best

- Is it optimal to propose these contracts when θ is not observable?
- Suppose first that u is increasing in θ (i.e. for all x, $u(x,\theta^H) > u(x,\theta^L)$).
- If monopolist offers $\{(\widehat{x}^H, \widehat{t}^H), (\widehat{x}^L, \widehat{t}^L)\}$ as above, then

$$u(\widehat{x}^L, \theta^H) - \widehat{t}^L > u(\widehat{x}^L, \theta^L) - \widehat{t}^L = U = u(\widehat{x}^H, \theta^H) - \widehat{t}^H.$$

- Therefore type θ^H will not choose $(\widehat{x}^H, \widehat{t}^H)$.
- What about setting $x^s = \widehat{x}^s$ and $t^L = \widehat{t}^L$ and

$$t^H = \widehat{t}^L + u(\widehat{x}^H, \theta^H) - u(\widehat{x}^L, \theta^H).$$

- With this specification, incentive compatibility holds for type θ^H as an equality (we say that IC^H binds).
- Expected profit is then:

$$p[v(\widehat{x}^H, \theta^H) + u(\widehat{x}^L, \theta^L) - U + u(\widehat{x}^H, \theta^H) - u(\widehat{x}^L, \theta^H)] + (1-p)[v(\widehat{x}^L, \theta^L) + u(\widehat{x}^L, \theta^L) - U].$$

- Since $\widehat{x}^L$ maximizes $v(x^L, \theta^L) + u(x^L, \theta^L)$, we know that a small change in $\widehat{x}^L$ has no effect on the second term in the profit expression.

- It has an effect on the first:

$$\frac{\partial}{\partial \widehat{x}^L}\left(u\left(\widehat{x}^L,\theta^L\right)-u\left(\widehat{x}^L,\theta^H\right)\right).$$

- This is in general nonzero and as a result, it is optimal to choose $x^L \neq \widehat{x}^L$ in general.

- Notice that the general expression for the profit is

$$p[v\left(x^H,\theta^H\right)+u\left(x^L,\theta^L\right)-U+u\left(x^H,\theta^H\right)-u\left(x^L,\theta^H\right)]+(1-p)\left[v\left(x^L,\theta^L\right)+u\left(x^L,\theta^L\right)-U\right]$$

  for any pair $\left(x^H,t^H\right),\left(x^L,t^L\right)$ such that IC^H is binding and IR^L is binding.

- It is immediately clear that for any such case, it is optimal to choose $x^H=\widehat{x}^H$.

- Hence the only remaining task is to argue that IC^H and IR^L are both binding.

- Write out the constraints in full:

$$\begin{aligned}
u\left(x^H,\theta^H\right)-t^H &\geq u\left(x^L,\theta^H\right)-t^L, \quad \left(IC^H\right)\\
u\left(x^L,\theta^L\right)-t^L &\geq u\left(x^H,\theta^L\right)-t^H, \quad \left(IC^L\right)\\
u\left(x^H,\theta^H\right)-t^H &\geq 0, \quad \left(IR^H\right)\\
u\left(x^L,\theta^L\right)-t^L &\geq 0 \quad \left(IR^L\right)
\end{aligned}$$

IR^L

- If $u\left(x^L,\theta^L\right)-t^L>0$, then $u\left(x^H,\theta^H\right)-t^H>0$ since $u\left(x^H,\theta^H\right)-t^H\geq u\left(x^L,\theta^H\right)-t^L>u\left(x^L,\theta^L\right)-t^L>0$, where the next to last inequality follows since u is increasing in θ.

- Therefore the prinipac could charge $t^H+\varepsilon$ and $t^L+\varepsilon$ for the same choices of x and increase her profit.

- As a result, IR^L must bind at optimum.

IC^H

- If $u\left(x^H, \theta^H\right) - t^H > u\left(x^L, \theta^H\right) - t^L$, then $u\left(x^H, \theta^H\right) - t^H > 0$ by the previous argument.
- Consider then an increase to $t^H + \varepsilon$ leaving everything else as before.
- IR^L is not affected, IC^L must still hold and IC^H and IR^H hold for small enough ε since those constraints were not binding.
- Hence purchases are as before but profit is increased.
- we conclude that IC^H must bind at optimum.

General properties of solution

- Since $\left(x^H, x^L\right)$ solves

$$\max_{x^H, x^L} p[u\left(x^H, \theta^H\right) + u\left(x^L, \theta^L\right) - U + u\left(x^H, \theta^H\right) - u\left(x^L, \theta^H\right)]$$
$$+ (1-p)\left[u\left(x^L, \theta^L\right) + u\left(x^L, \theta^L\right) - U\right],$$

  we have:

- $x^H = \widehat{x}^H$.
- x^L is solved from:

$$(1-p)\frac{\partial}{\partial x^L}\left(v\left(x^L, \theta^L\right) + u\left(x^L, \theta^L\right)\right)$$
$$-p\frac{\partial}{\partial x^L}\left(u\left(x^L, \theta^H\right) - u\left(x^L, \theta^L\right)\right) = 0$$

- $x^L < \widehat{x}^L$ if

$$\frac{\partial^2}{\partial x \partial \theta} u(x, \theta) > 0 \text{ and } \frac{\partial}{\partial x} u(x, \theta) > 0.\text{and } \frac{\partial}{\partial \theta} u(x, \theta) > 0.$$

- $\left(u\left(x^L, \theta^H\right) - u\left(x^L, \theta^L\right)\right)$ measures the surplus going to type θ^H agent.

- It is called the informaiton rent to agent θ^H and it measures the net utility she would get from agent θ^L deal.
- Hence this form of the maximization problem shows clearly the main trade-off.
- The principal trades off economic efficiency in form $\left(v\left(x^L,\theta^L\right)+u\left(x^L,\theta^L\right)\right)$ and information rent.
- Sum together the two IC constraints to get:

$$u\left(x^H,\theta^H\right)-u\left(x^L,\theta^H\right)\geq u\left(x^L,\theta^H\right)-u\left(x^L,\theta^L\right).$$

- We say that a function $f:X\times Y\rightarrow R$ is supermodular is for all $x^H>x^L$ and $y^H>y^L$, we have

$$u\left(x^H,\theta^H\right)+u\left(x^L,\theta^L\right)\geq u\left(x^L,\theta^H\right)+u\left(x^H,\theta^L\right).$$

- Hence if u is supermodular, then we have $x^H>x^L$. A sufficient condition for supermodularity is $\frac{\partial^2}{\partial x\partial\theta}u\left(x,\theta\right)>0.$
- Exercise: think about how these results would be changed if we had different signs for these derivatives.
- Exercise: Consider the applications above and think about what might be good assumptions on the payoff functions u and v in each of the applications.

Nonlinear Pricing

Two types of agents

- A simple model of wine merchant and wine buyer
  - buyer could have either coarse or sophisticated tastes,
  - buyer's tastes are unobservable to merchant
  - what qualities should the merchant offer and at what price?

- Utility function of buyer

$$v(\theta_i, q_i, t_i) = u(\theta_i, q_i) - t_i = \theta_i q_i - t_i, \quad i \in \{l, h\}$$

  - θ_i represent marginal willingness to pay for quality q_i
  - t_i is transfer (price) buyer i has to pay for quality q_i
  - assume θ_i satisfies $0 < \theta_l < \theta_h < \infty$

- Cost of producing quality $q \geq 0$ is given by

$$c(q) \geq 0, \quad c'(q) > 0, \quad c''(q) > 0$$

- Ex ante (prior) probability that buyer has a high willingness to pay is $p = \Pr(\theta_i = \theta_h)$

- Important: difference in utility for high and low valuation buyer for any given quality q is increasing in q

$$\frac{\partial}{\partial q}(u(\theta_h, q) - u(\theta_l, q)) > 0$$

  Spence-Mirrlees sorting condition

  - if θ_i were a continuous variable, sorting condition could be written in terms of second cross derivative:

$$\frac{\partial^2 u(\theta, q)}{\partial \theta \partial q} > 0$$

  - i.e., taste θ and quality q are complements

- Profit for seller from a bundle (q, t) is

$$\pi(t, q) = t - c(q)$$

- Consider first socially optimal solution
- As different types have different preferences, they should consume different qualities
- Social surplus for each type can be maximized separately by solving

$$\max_{q_i} \theta_i q_i - c(q_i)$$

- First order conditions

$$c'(q_i^*) = \theta_i \Rightarrow q_l^* < q_h^*$$

- Efficient solution is equilibrium outcome if
  - monopolist can perfectly discriminate between types: first degree price discrimination
    - monopolist sets $t_i = \theta_i q_i$
    - solves for each type separately:

$$\max_{\{t_i, q_i\}} \theta_i q_i - c(q_i)$$

  - there is perfect competition
    - sellers break even, get zero profit and set prices at $t_i = c(q_i^*)$
    - buyer gets entire surplus
- Consider next situation under asymmetric information
- Perfect discrimination is now impossible, since

$$\theta_h q_l^* - t_l^* = (\theta_h - \theta_l) q_l^* > 0 = \theta_h^* q_h^* - t_h$$

- Sorting is possible: problem for monopolist is now

$$\max_{\{t_l,q_l,t_h,q_h\}}(1-\pi)(t_l-c(q_l))+\pi(t_h-c(q_h))$$

subject to individual rationality constraints

$$\theta_i q_i - t_i \geqq 0 \qquad (IR_i)$$

and incentive compatibility constraints

$$\theta_i q_i - t_i \geqq \theta_i q_j - t_j \quad (IC_i)$$

- Question is: how to separate?
- Main steps
  - show that binding constraints are IR_l and IC_h This was already done above.
  - solve for t_l and t_h
  - solve for q_h and q_l
- Solve for $\hat{q}_l$

$$\max_{q_l}\{(1-p)(\theta_l q_l(c(q_l))+p(\theta_h(q_h^*-q_l)+\theta_l q_l-c(q_h^*)))\}$$

- But q_h^* is a constant, can simplify to

$$\max_{q_l}\theta_l q_l - c(q_l)-\frac{p}{1-p}(\theta_h-\theta_l)q_l$$

- First order condition

$$\theta_l - c'(q_l)-\frac{p}{1-p}(\theta_h-\theta_l)=0$$

- So $c'(\hat{q}_l)<\theta_l$, and hence $\hat{q}_l<q_l^*$
- Information rent for high valuation buyer

$$I(q_l)=(\theta_h-\theta_l)q_l$$

- Summary
  - quality supplied to high valuation buyer is efficient
  - quality supplied to low valuation buyer is inefficiently low (with possibility of complete exclusion)
  - low quality distortion reduces information rent for high valuation buyer

Continuous type model (strictly at your own risk)

Utilities

- Consider a more general contracting problem between an agent and a principal.
- Type of the agent is a single-dimensional real variable. $\theta \in [\underline{\theta}, \overline{\theta}]$
- We take $\underline{\theta} = 0$.
- Preferences of the agent:

$$U(x, \theta, t) = u(x, \theta) - t$$

- and the principal:

$$V(x, \theta, t) = v(x, \theta) + t$$

- Interpretation: x represents the action that the agent agrees to take, and t is the monetary transfer from the agent to the principal. The transfer could be positive or negative depending on application.
  - Regulation: x is the output, θ privately known cost parameter, t transfer from regulator to regulated.
  - Insurance: x amount of coverage, θ risk type, t insurance premium.

- Optimal sales: x quantity or quality provided, θ preference parameter, t price.

- Social surplus is given by

$$S(x,\theta) = u(x,\theta) + v(x,\theta).$$

- uncertainty about the type of the agent is given by:

$$f(\theta), F(\theta).$$

- Assume the (strict) Spence-Mirrlees conditions

$$\frac{\partial u}{\partial \theta} > 0, \ \frac{\partial^2 u}{\partial \theta \partial x} > 0.$$

Incentive Compatibility

Definition 13 *An allocation is a mapping*

$$\theta \to y(\theta) = (x(\theta), t(\theta)).$$

Definition 14 *An allocation is implementable if $y = (x,t)$ satisfies truthtelling:*

$$u(x(\theta), \theta) - t(\theta) \geq u\left(x\left(\widehat{\theta}\right), \theta\right) - t\left(\widehat{\theta}\right), \ \ \forall \theta, \widehat{\theta} \in \Theta.$$

Definition 15 *Net utility for agent of type θ from reporting type $\hat{\theta}$ is denoted by*

$$U\left(\hat{\theta}\,|\theta\right) = u\left(x\left(\widehat{\theta}\right), \theta\right) - t\left(\widehat{\theta}\right).$$

Truthful net utility is denoted by

$$U(\theta) \triangleq U(\theta\,|\theta) = u(x(\theta), \theta) - t(\theta).$$

- We are interested in (i) describing which contracts can be implemented and (ii) in describing optimal contracts.

Theorem 2 *The direct mechanism* $y(\theta) = (x(\theta), t(\theta))$ *is incentive compatible if and only if:*

1. *the truthtelling utility is described by:*

$$U(\theta) - U(0) = \int_0^\theta u_\theta(x(s), s)\, ds,$$

2. $x(s)$ *is nondecreasing.*

- The first condition can be restated in terms of first order conditions:

$$\frac{dU}{d\theta} = \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial \theta} = 0.$$

- Recall our result from the section on auctions.