Separable Models, Introduction to Dynamic Systems

Roland Hostettler

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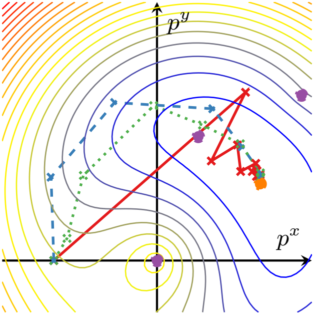
Recap

- Gauss–Newton algorithm:

$$\hat{\boldsymbol{\theta}}^{(i+1)} = \hat{\boldsymbol{\theta}}^{(i)} + \gamma(\boldsymbol{G}_{\boldsymbol{\theta}}^{\mathsf{T}}\boldsymbol{R}^{-1}\boldsymbol{G}_{\boldsymbol{\theta}})^{-1}\boldsymbol{G}_{\boldsymbol{\theta}}^{\mathsf{T}}\boldsymbol{R}^{-1}(\boldsymbol{y} - g(\hat{\boldsymbol{\theta}}^{(i)}))$$

- Levenberg–Marquardt algorithm:

$$\hat{\boldsymbol{\theta}}^{(i+1)} = \hat{\boldsymbol{\theta}}^{(i)} + (\boldsymbol{G}_{\boldsymbol{\theta}}^{\mathsf{T}}\boldsymbol{R}^{-1}\boldsymbol{G}_{\boldsymbol{\theta}} + \lambda\boldsymbol{I})^{-1}\boldsymbol{G}_{\boldsymbol{\theta}}^{\mathsf{T}}\boldsymbol{R}^{-1}(\boldsymbol{y} - g(\hat{\boldsymbol{\theta}}^{(i)}))$$



Intended Learning Outcomes

After this lecture, you will be able to:

- identify and describe separable models,
- recognize the problems with dynamic systems,
- describe the idea of state-space modeling.

Example: Localization using Signal Strength

- Measurement model:

$$
\begin{aligned}
y_n &= P_{0,\text{dBm}} - 10\beta \log_{10}(|\Delta p_n|) + r_n \\
&= P_{0,\text{dBm}} - \frac{10\beta}{\log(10)} \log(|\Delta p_n|) + r_n
\end{aligned}
$$

  with $\Delta p_n = p_t - p_n$

- Parameters are $\boldsymbol{\theta} = \begin{bmatrix} P_{0,\text{dBm}} & \beta & p_t^x & p_t^y \end{bmatrix}^\mathsf{T}$
- The Jacobian is given by

$$
\boldsymbol{G}_{\boldsymbol{\theta}} = \begin{bmatrix}
1 & -10\log_{10}(|\Delta p_1|) & \frac{-10\beta}{\log(10)|\Delta p_1|^2}\Delta p_n^x & \frac{-10\beta}{\log(10)|\Delta p_1|^2}\Delta p_n^y \\
1 & -10\log_{10}(|\Delta p_2|) & \frac{-10\beta}{\log(10)|\Delta p_2|^2}\Delta p_n^x & \frac{-10\beta}{\log(10)|\Delta p_2|^2}\Delta p_n^y \\
\vdots & \vdots & \vdots & \vdots \\
1 & -10\log_{10}(|\Delta p_N|) & \frac{-10\beta}{\log(10)|\Delta p_N|^2}\Delta p_n^x & \frac{-10\beta}{\log(10)|\Delta p_N|^2}\Delta p_n^y
\end{bmatrix}
$$

Separable Models: Motivation & Definition

- Numerical optimization methods:
  - Can handle arbitrary nonlinear least squares problem
  - Are computationally expensive
  - Can be challenging to implement
- In many problems, only a few parameters enter the problem nonlinearly, the remaining ones enter linearly
- The separable model is:

$$\boldsymbol{y} = \boldsymbol{G}(\boldsymbol{\theta}_1)\boldsymbol{\theta}_2 + \boldsymbol{r},$$

  where
  - $\boldsymbol{\theta}_1$ are the nonlinear parameters
  - $\boldsymbol{\theta}_2$ the linear parameters
- Can we exploit this property?

Separable Models: Derivation (1/2)

- The separable model is:

$$\boldsymbol{y} = \boldsymbol{G}(\boldsymbol{\theta}_1)\boldsymbol{\theta}_2 + \boldsymbol{r}$$

- Weighted least squares cost function:

$$J(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = (\boldsymbol{y} - \boldsymbol{G}(\boldsymbol{\theta}_1)\boldsymbol{\theta}_2)^\mathsf{T}\boldsymbol{R}^{-1}(\boldsymbol{y} - \boldsymbol{G}(\boldsymbol{\theta}_1)\boldsymbol{\theta}_2)$$

- Weighted least squares estimator for $\boldsymbol{\theta}_2$ given $\boldsymbol{\theta}_1$:

$$\hat{\boldsymbol{\theta}}_2 = (\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1))^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)\boldsymbol{R}^{-1}\boldsymbol{y}$$

- Substitution of $\boldsymbol{\theta}_2$ in $J(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$:

$$\begin{aligned} J(\boldsymbol{\theta}_1) = (&\boldsymbol{y} - \boldsymbol{G}(\boldsymbol{\theta}_1)(\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1))^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)\boldsymbol{R}^{-1}\boldsymbol{y})^\mathsf{T}\boldsymbol{R}^{-1} \\ &\times (\boldsymbol{y} - \boldsymbol{G}(\boldsymbol{\theta}_1)(\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1))^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)\boldsymbol{R}^{-1}\boldsymbol{y}) \end{aligned}$$

Separable Models: Derivation (2/2)

- Modified cost function:

$$J(\boldsymbol{\theta}_1) = \Big(\boldsymbol{y} - \boldsymbol{G}(\boldsymbol{\theta}_1)(\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1))^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)\boldsymbol{R}^{-1}\boldsymbol{y}\Big)^\mathsf{T}\boldsymbol{R}^{-1} \\ \times \Big(\boldsymbol{y} - \boldsymbol{G}(\boldsymbol{\theta}_1)(\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1))^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)\boldsymbol{R}^{-1}\boldsymbol{y}\Big)$$

- Expansion and simplification yields:

$$J(\boldsymbol{\theta}_1) = \boldsymbol{y}^\mathsf{T}\boldsymbol{y} - \boldsymbol{y}^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)(\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1))^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{y}$$

- Estimating $\boldsymbol{\theta}_1$:

$$\begin{aligned}\hat{\boldsymbol{\theta}}_1 &= \underset{\boldsymbol{\theta}_1}{\operatorname{argmin}} -\boldsymbol{y}^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)\left(\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)\right)^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{y} \\ &= \underset{\boldsymbol{\theta}_1}{\operatorname{argmax}}\, \boldsymbol{y}^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)\left(\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)\right)^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{y}\end{aligned}$$

Separable Models: Practical Considerations

- Estimating $\boldsymbol{\theta}_1$:

$$\hat{\boldsymbol{\theta}}_1 = \underset{\boldsymbol{\theta}_1}{\operatorname{argmax}}\, \boldsymbol{y}^\mathsf{T} \boldsymbol{R}^{-1} \boldsymbol{G}(\boldsymbol{\theta}_1) \left(\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T} \boldsymbol{R}^{-1} \boldsymbol{G}(\boldsymbol{\theta}_1) \right)^{-1} \boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T} \boldsymbol{R}^{-1} \boldsymbol{y}$$

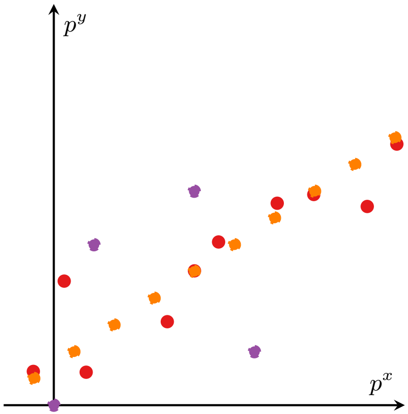
- Cost function is not in least squares form anymore (but we still solve the original least squares problem)
- Gradient descent can readily be used (but needs recalculating the gradient)
- Gauss–Newton and Levenberg–Marquardt need more work

Separable Models: Example

Localizing a Moving Target (1/4)

Assume that:

- Target moves rather than being stationary
- Sensors measure periodically, e.g. every second



Localizing a Moving Target (2/4)

- Straight line model:

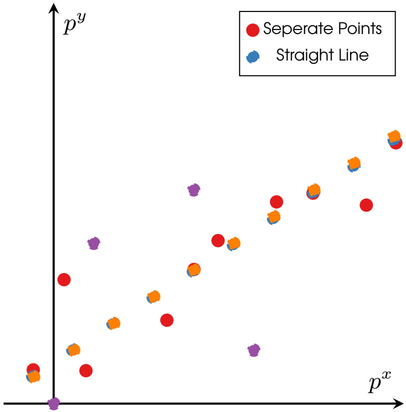
$$p_t(t) = p_t(0) + vt$$

- Measurement model

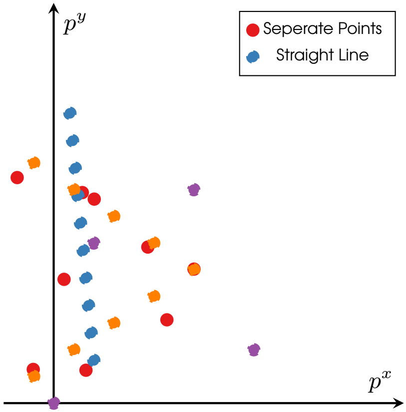
$$\begin{aligned} \boldsymbol{y}_n(t) &= |\Delta p_n(t)| + r_n(t) \\ &= |p_t(0) + vt - p_n| + r_n(t) \end{aligned}$$

- Only need to estimate 4 parameters:
  $\boldsymbol{\theta} = \begin{bmatrix} p_t^x(0) & p_t^y(0) & v^x & v^y \end{bmatrix}^\mathsf{T}$

Localizing a Moving Target (3/4)



Localizing a Moving Target (4/4)

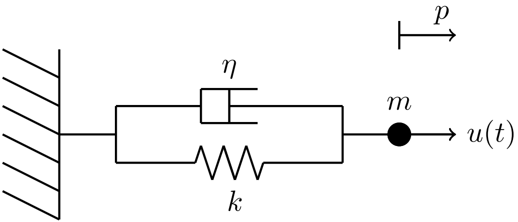


Localizing a Moving Target: Conclusions

- The static approach is not too well suited for time-varying processes
- A systematic method that relates (time-wise) related measurements is needed
- Solution: Use differential (and difference) equations to model the time-varying, i.e. dynamic, system

ODE Modeling of Dynamic Systems

- ODEs can be used to describe many dynamic systems
- Example: Spring-damper system:



- Second order ordinary differential equation:

$$ma(t) = -kp(t) - \eta v(t) + u(t)$$

- Other examples: Newtonion/Hamiltonian dynamics, kinematic models, heat and mass transfer, wave equations, ...

Example: State-Space Representation of ODEs

- Second order ordinary differential equation:

$$ma(t) = -kp(t) - \eta v(t) + u(t)$$

- Equation system representation:

$$\begin{bmatrix} v(t) \\ a(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

- First order ODE equation system:

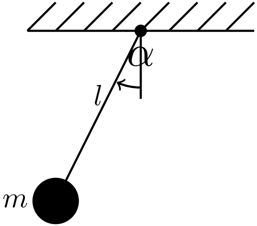
$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

- $\boldsymbol{x}(t) = \begin{bmatrix} p(t) & v(t) \end{bmatrix}^{\mathsf{T}}$ is the state of the system

Exercise: ODE to State-Space Representation

Tasks:

1. Find the (nonlinear) ODE for the pendulum below
2. Linearize the ODE using Taylor expansion around $\alpha = 0$
3. Write the linearized ODE as an ODE equation system



Hint: The moment of inertia for a pendulum is $I = ml^2$

Summary

- Separable models:

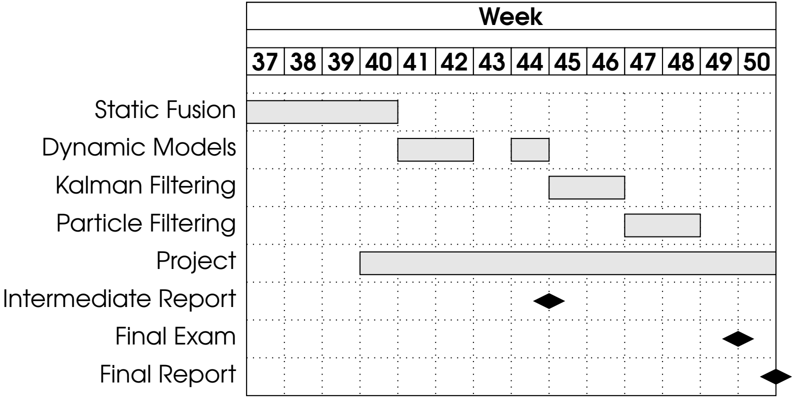
$$\boldsymbol{y} = \boldsymbol{G}(\boldsymbol{\theta}_1)\boldsymbol{\theta}_2 + \boldsymbol{r}$$

- Cost function for the nonlinear parameters $\boldsymbol{\theta}_1$:

$$J(\boldsymbol{\theta}_1) = -\boldsymbol{y}^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)(\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1))^{-1}\boldsymbol{G}(\boldsymbol{\theta}_1)^\mathsf{T}\boldsymbol{R}^{-1}\boldsymbol{y}$$

- ODEs can be used to describe dynamic behavior
- Higher order ODEs can be reduced to first-order vector ODEs

Schedule


Week
37
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47
48
49
50
Static Fusion
Dynamic Models
Kalman Filtering
Particle Filtering
Project
Intermediate Report
Final Exam
Final Report