Chapter 4

State-Space Models

Most sensor fusion problems involve dynamically changing variables. For example, in autonomous driving, other road users continuously appear, disappear, and move in the traffic scene. In this case, we are interested in estimating dynamically varying parameters. We refer to these parameters as *states* or *the state*, because they describe the state a system is in (e.g., the location and velocity of a car, or the position, attitude, and velocity of a unmanned aerial vehicle, etc.).

In such dynamic scenarios, we have to employ sensors that provide repeated measurements in time, that is, they sample periodically. In between those samples, the state of the system evolves according to some process and thus, the states at different times are different from each other. However, since the evolution of the state follows some dynamic process, they are related and thus, we can relate the states at two times to each other. To do that, we need a principled way of describing how the state evolves between samples. We find a suitable approach in differential equations (for continuous-time processes) and difference equations (for discrete-time processes), which can be used to describe dynamic processes. Indeed, we will make use of differential equations to derive *state-space models* that turn an Lth order differential (or difference) equation into a first order vector-valued differential (or difference) equation.

4.1 Continuous-Time State-Space Models

4.1.1 Deterministic Linear State-Space Models

We start the derivation of the state-space formulation based on an example. Figure 4.1 shows a spring-damper system, a typical mechanical system. This system is governed by Newton's second law of motion, from which we can find the following inhomogeneous ordinary differential equation (ODE):

$$ma(t) = -kp(t) - \eta v(t) + u(t), \tag{4.1}$$

where $p(t)$, $v(t)$, and $a(t)$ denote the position, velocity, and acceleration of the mass m, respectively, k is the spring constant, and η is the damping constant. Furthermore, the forcing function $u(t)$ is a deterministic input to the system.

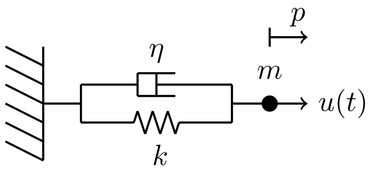

Figure 4.1: Example of a mechanical dynamic system: A spring-damper system with forcing function $u(t)$. The positive direction of the motion p is defined to the right.

By introducing a second equation, $v(t) = v(t)$, which always holds, and dividing (4.1) by m, we obtain the equation system

$$v(t) = v(t), \tag{4.2a}$$

$$a(t) = -\frac{k}{m}p(t) - \frac{\eta}{m}v(t) + \frac{1}{m}u(t). \tag{4.2b}$$

This can be rewritten in matrix form, such that we obtain

$$\begin{bmatrix} v(t) \\ a(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\eta}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t). \tag{4.3}$$

Observe that in (4.3), the vector on the left hand side of the equation is the derivative of the vector on the right hand side. Hence, we can define the vector

$$\boldsymbol{x}(t) = \begin{bmatrix} p(t) \\ v(t) \end{bmatrix},$$

and rewrite (4.3) as

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{\eta}{m} & -\frac{k}{m} \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t), \tag{4.4}$$

where

$$\dot{\boldsymbol{x}}(t) = \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}$$

denotes the time-derivative.

The form (4.4) is the state-space representation of the dynamic system in Figure 4.1 and the vector $\boldsymbol{x}(t)$ is called the state vector (or simply state). The state vector now encodes all the information about the state the system is in (position, speed, etc.). Therefore, solving this first-order vector valued ODE representation rather than the original differential equation in 4.1 provides richer information about the system.

We can now generalize this approach. Consider the Lth order ODE of the form

$$\frac{\mathrm{d}^L x(t)}{\mathrm{d}t^L} = a_0 x(t) + a_1 \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \cdots + a_{L-1} \frac{\mathrm{d}^{L-1} x(t)}{\mathrm{d}t^{L-1}} + b_1 u(t). \tag{4.5}$$

Then we can introduce $L-1$ equations of the form $\mathrm{d}^l x(t)/\mathrm{d}t^l = \mathrm{d}^l x(t)/\mathrm{d}t^l$ to obtain the equation system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}, \tag{4.6a}$$

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}, \tag{4.6b}$$

$$\vdots \tag{4.6c}$$

$$\frac{\mathrm{d}^Lx(t)}{\mathrm{d}t^L} = a_0x(t) + a_1\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \cdots + a_{L-1}\frac{\mathrm{d}^{L-1}x(t)}{\mathrm{d}t^{L-1}} + b_1u(t). \tag{4.6d}$$

Rewriting (4.6) in vector form yields

$$\begin{bmatrix} \frac{\mathrm{d}x(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} \\ \vdots \\ \frac{\mathrm{d}^Lx(t)}{\mathrm{d}t^L} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ a_0 & a_1 & & \dots & a_{L-1} \end{bmatrix} \begin{bmatrix} x(t) \\ \frac{\mathrm{d}x(t)}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}^{L-1}x(t)}{\mathrm{d}t^{L-1}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_1 \end{bmatrix} u(t).$$

This in turn, can be written compactly as the dynamic model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u\boldsymbol{u}(t), \tag{4.7}$$

where

$$\boldsymbol{x}(t) = \begin{bmatrix} x(t) \\ \frac{\mathrm{d}x(t)}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}^{L-1}x(t)}{\mathrm{d}t^{L-1}} \end{bmatrix}, \ \boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ a_0 & a_1 & & \dots & a_{L-1} \end{bmatrix}, \ \boldsymbol{B}_u = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_1 \end{bmatrix}, \ \boldsymbol{u}(t) = u(t) \tag{4.8}$$

While quite general, eq. (4.7) is a state-space representation for models of the type in (4.5). However, we can also derive the same type of representation for other dynamic models. For example, consider a freshly brewed hot cup of coffee. Newton's law of cooling then states that the change in temperature of the coffee is proportional to the temperature difference between the coffee and its surrounding (assuming that the surroundings are much larger than the coffee cup). This can be formulated as a differential equation of the form

$$\frac{\mathrm{d}T_\mathrm{c}(t)}{\mathrm{d}t} = -k_1(T_\mathrm{c}(t) - T_\mathrm{r}(t)), \tag{4.9}$$

where $T_\mathrm{c}(t)$ denotes the coffee's temperature, $T_\mathrm{r}(t)$ the room temperature, and k_1 is a constant. The change in room temperature on the other hand depends on the heat that is produced inside the room (e.g., due to heating) as well as the losses to the outside. Again assuming that Newton's law of cooling applies, we can write the differential equation as

$$\frac{\mathrm{d}T_\mathrm{r}(t)}{\mathrm{d}t} = -k_2(T_\mathrm{r}(t) - T_\mathrm{a}(t)) + h(t), \tag{4.10}$$

where k_2 is a constant, $T_\mathrm{a}(t)$ denotes the outside temperature and $h(t)$ is the heating input. Observe that (4.9)–(4.10) is a coupled system that can be written as the equation system

$$\frac{\mathrm{d}T_\mathrm{r}(t)}{\mathrm{d}t} = -k_2(T_\mathrm{r}(t) - T_\mathrm{a}(t)) + h(t) \tag{4.11a}$$
$$\frac{\mathrm{d}T_\mathrm{c}(t)}{\mathrm{d}t} = -k_1(T_\mathrm{c}(t) - T_\mathrm{r}(t)), \tag{4.11b}$$

or equivalently on matrix form as

$$\begin{bmatrix} \frac{\mathrm{d}T_\mathrm{r}(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}T_\mathrm{c}(t)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -k_2 & 0 \\ k_1 & -k_1 \end{bmatrix} \begin{bmatrix} T_\mathrm{r}(t) \\ T_\mathrm{c}(t) \end{bmatrix} + \begin{bmatrix} k_2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_\mathrm{a}(t) \\ h(t) \end{bmatrix}. \tag{4.12}$$

This has again the form of (4.7), where

$$\boldsymbol{x}(t) = \begin{bmatrix} T_\mathrm{r}(t) \\ T_\mathrm{c}(t) \end{bmatrix}, \ \boldsymbol{A} = \begin{bmatrix} -k_2 & 0 \\ k_1 & -k_1 \end{bmatrix}, \ \boldsymbol{B}_u = \begin{bmatrix} k_2 & 1 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{u}(t) = \begin{bmatrix} T_\mathrm{a}(t) \\ h(t) \end{bmatrix}. \tag{4.13}$$

Hence, the model (4.7) is a quite general *dynamic model* of the time-varying behavior for many different processes. Recall that our original aim was to estimate the dynamically changing state $\boldsymbol{x}(t)$. This requires that we measure $\boldsymbol{x}(t)$ in some way, that is, we need to combine the dynamic model with a *measurement model.* The simplest case is that of a linear measurement model of the form (2.7) where we replace the static parameters $\boldsymbol{\theta}$ with the state $\boldsymbol{x}(t)$. Since the measurements are obtained at discrete time instances t_n (i.e., at $t_1, t_2, \dots$), the measurement model becomes

$$\boldsymbol{y}_n = \boldsymbol{G}\boldsymbol{x}(t_n) + \boldsymbol{r}_n, \tag{4.14}$$

where $\boldsymbol{r}_n$ denotes the measurement noise with covariance matrix $\mathrm{Cov}\{\boldsymbol{r}_n\} = \boldsymbol{R}_n$.

Defining $\boldsymbol{x}_n \triangleq \boldsymbol{x}(t_n)$ and combining the dynamic model (4.7) and the measurement model (4.14) then yields the deterministic *linear state space model*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u\boldsymbol{u}(t), \tag{4.15a}$$
$$\boldsymbol{y}_n = \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n. \tag{4.15b}$$

4.1.2 Stochastic Linear State-Space Models

Often, the behavior of dynamic systems is not completely deterministic, or the input $u(t)$ is not known. For example, when tracking an airplane using radar, we do not know the inputs made by the pilots. Additionally, variables such as wind or pressure fields are not known either and it is difficult to model them accurately. Instead, such random effects can be modeled by including a stochastic process as an input to the differential equation (and hence to its state-space representation), which turns an ODE into a *stochastic differential equation* (SDE; Øksendal (2010); Särkkä and Solin (2018)).

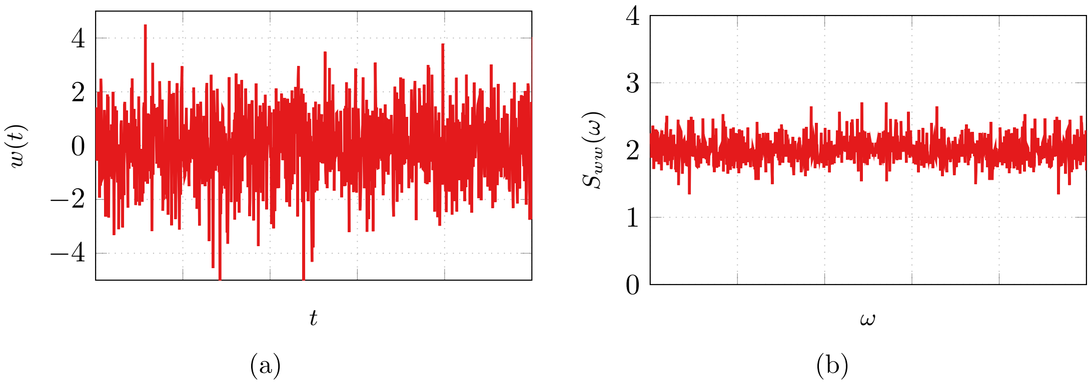

Figure 4.2: Example of a white noise process $w(t)$ with $\sigma_w^2 = 2$. (a) One realization of the process, and (b) its power spectral density $S_w(\omega)$ averaged over 100 realizations.

A stationary stochastic process $w(t)$ can be characterized by its autocorrelation function

$$R_{ww}(\tau) = \mathrm{E}\{w(t+\tau)w(t)\}$$

or, equivalently, the power spectral density (which is the Fourier transform of the autocorrelation function; Papoulis (1984)). A suitable assumption is that the stochastic process is a white noise process. In this case, the autocorrelation function is

$$R_{ww}(\tau) = \sigma_w^2 \delta(\tau), \tag{4.16}$$

where $\delta(\tau)$ denotes the Dirac delta function. Hence, the power spectral density is a constant, that is,

$$S_{ww} = \sigma_w^2. \tag{4.17}$$

In other words, the white noise process contains equal contributions from each frequency. One realization of such a process, together with its power spectral density are shown in Figure 4.2.

The derivation of the state-space representation of the resulting SDE follows the same steps as for the deterministic case, where the random process takes the role of the input. Consider the Lth order SDE given by

$$\frac{\mathrm{d}^L x(t)}{\mathrm{d}t^L} = a_0 x(t) + a_1 \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \cdots + a_{L-1} \frac{\mathrm{d}^{L-1} x(t)}{\mathrm{d}t^{L-1}} + b_1 w(t), \tag{4.18}$$

which is of the same form as (4.5) but with the stochastic process $w(t)$ as the input. Rewriting (4.18) into matrix form yields

$$\begin{bmatrix} \frac{\mathrm{d}x(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} \\ \vdots \\ \frac{\mathrm{d}^L x(t)}{\mathrm{d}t^L} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ a_0 & a_1 & & \dots & a_{L-1} \end{bmatrix} \begin{bmatrix} x(t) \\ \frac{\mathrm{d}x(t)}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}^{L-1} x(t)}{\mathrm{d}t^{L-1}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_1 \end{bmatrix} w(t).$$

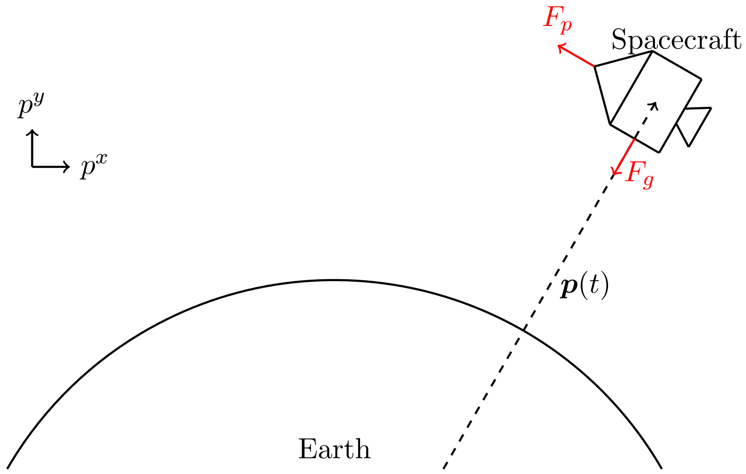

Figure 4.3: Illustration of a spacecraft orbiting the earth.

Hence, this can also be written compactly in the form of a first order vector valued SDE system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_w \boldsymbol{w}(t).$$

Naturally, also coupled models (like the coffee-cup example) can be written in this form. Together with a linear measurement equation, the stochastic linear state-space model becomes

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_w \boldsymbol{w}(t), \tag{4.19a}$$

$$\boldsymbol{y}_n = \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n. \tag{4.19b}$$

Note that some dynamic models may contain both deterministic inputs $\boldsymbol{u}(t)$ and stochastic inputs $\boldsymbol{w}(t)$. Naturally, both of these can be incorporated in the model as well.

4.1.3 Nonlinear State-Space Models

So far, we have only discussed the state-space form of linear ODEs and SDEs. However, many dynamic systems actually behave nonlinearly and fortunately, the approach can be extended to nonlinear systems as well.

We start our discussion again based on an example. Figure 4.3 shows an illustration of a spacecraft orbiting the earth. The forces acting upon the craft are the gravitational pull of the earth F_g and the propulsion force F_p. The position $\boldsymbol{p}(t)$ of the craft is defined according to the coordinate system shown in Figure 4.3, with the origin at the center of the earth. The gravitational acceleration of an object at a distance $|\boldsymbol{p}(t)|$ from the earth's center is approximately

$$g \approx g_0 \left(\frac{r_e}{|\boldsymbol{p}(t)|} \right)^2,$$

where $g_0 = 9.81\,\text{m/s}^2$ is the gravitational acceleration on the earth's surface and $r_e = 6371\,\text{km}$ is the mean earth radius. The gravitational pull is in the opposite direction of $\boldsymbol{p}(t)$ and hence, we can write it on vector form as

$$\begin{aligned}\boldsymbol{F}_g &= -mg_0 \left(\frac{r_e}{|\boldsymbol{p}(t)|}\right)^2 \frac{\boldsymbol{p}(t)}{|\boldsymbol{p}(t)|} \\ &= -mg_0 r_e^2 \frac{\boldsymbol{p}(t)}{|\boldsymbol{p}(t)|^3}.\end{aligned}$$

The propulsion force is perpendicular to the direction of the position vector $\boldsymbol{p}(t)$. Hence, we can write

$$\boldsymbol{F}_p = F_p \frac{1}{|\boldsymbol{p}(t)|} \begin{bmatrix} -p^y(t) \\ p^x(t) \end{bmatrix}$$

When tracking the spacecraft from the ground, for example using radar, the magnitude of the propulsion force F_p is unknown. But since we know that the engines are only used to make small flight path corrections to conserve fuel, we can model this as a stochastic process, that is, we can assume that $F_p = w(t)$ with some spectral density σ_w^2.

Using Newton's second law it follows that the motion of the spacecraft is governed by the vector-valued differential equation

$$m\boldsymbol{a}(t) = -mg_0 r_e^2 \frac{\boldsymbol{p}(t)}{|\boldsymbol{p}(t)|^3} + \frac{1}{|\boldsymbol{p}(t)|} \begin{bmatrix} -p^y(t) \\ p^x(t) \end{bmatrix} w(t). \tag{4.20}$$

The highest order of the derivative here is the acceleration on the left hand side. Hence, a suitable state vector includes the position (zeroth derivative) and the speed (first derivative), that is,

$$\boldsymbol{x}(t) = \begin{bmatrix} p^x(t) & p^y(t) & v^x(t) & v^y(t) \end{bmatrix}^\mathsf{T}.$$

However, when trying to rewrite (4.20) into the form of a state-space representation (4.19), we notice that this is not possible since the right hand side of (4.20) is not linear in $\boldsymbol{p}(t)$. Nevertheless, we can still write it as a nonlinear equation system in terms of $\boldsymbol{x}(t)$ as

$$\begin{aligned}\begin{bmatrix} v^x(t) \\ v^y(t) \\ a^x(t) \\ a^y(t) \end{bmatrix} &= \begin{bmatrix} v^x(t) \\ v^y(t) \\ -g_0 r_e^2 \frac{p^x(t)}{|\boldsymbol{p}(t)|^3} \\ -g_0 r_e^2 \frac{p^y(t)}{|\boldsymbol{p}(t)|^3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{p^y(t)}{m|\boldsymbol{p}(t)|} \\ \frac{p^x}{m|\boldsymbol{p}(t)|} \end{bmatrix} w(t) \\ &= \begin{bmatrix} f_1(\boldsymbol{x}(t)) \\ f_2(\boldsymbol{x}(t)) \\ f_3(\boldsymbol{x}(t)) \\ f_4(\boldsymbol{x}(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{p^y(t)}{m|\boldsymbol{p}(t)|} \\ \frac{p^x}{m|\boldsymbol{p}(t)|} \end{bmatrix} w(t),\end{aligned} \tag{4.21}$$

where the $f_i(\boldsymbol{x}(t))$ are nonlinear functions of $\boldsymbol{x}(t)$.

This idea can be generalized to any arbitrary nonlinear dynamic system that is described by a nonlinear ODE or SDE, including coupled systems. Given the state vector $\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_{d_x}(t) \end{bmatrix}^\mathsf{T}$ of dimension d_x, we can write the ODE/SDE as a vector-valued, nonlinear equation system of the form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{d_x}(t) \end{bmatrix} = \begin{bmatrix} f_1(\boldsymbol{x}(t)) \\ f_2(\boldsymbol{x}(t)) \\ \vdots \\ f_{d_x}(\boldsymbol{x}(t)) \end{bmatrix} + \begin{bmatrix} b_{11}(\boldsymbol{x}(t)) & \dots & b_{1d_w}(\boldsymbol{x}(t)) \\ b_{21}(\boldsymbol{x}(t)) & & \vdots \\ \vdots & \ddots & \\ b_{d_x1}(\boldsymbol{x}(t)) & \dots & b_{d_xd_w}(\boldsymbol{x}(t)) \end{bmatrix} \boldsymbol{w}(t). \tag{4.22}$$

More compactly, this can be written as

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t), \tag{4.23}$$

where $f(\boldsymbol{x}(t))$ is a vector-valued nonlinear function of the state, and $\boldsymbol{B}_w(\boldsymbol{x}(t))$ is a matrix that depends on the state $\boldsymbol{x}(t)$.

In order to estimated the state $\boldsymbol{x}(t)$ based on observations $\boldsymbol{y}_n$, we again need to relate the measurements to the state. This is achieved by combining the dynamic model (4.23) with a measurement model as introduced in Chapters 2–3. The most general model arises if we chose a nonlinear observation model of the form (3.1) where the state $\boldsymbol{x}_n \triangleq \boldsymbol{x}(t_n)$ at time t_n takes the place of the parameters $\boldsymbol{\theta}$. This yields the general *nonlinear state-space model*

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t), \tag{4.24a}$$

$$\boldsymbol{y}_n = g(\boldsymbol{x}_n) + \boldsymbol{r}_n, \tag{4.24b}$$

where $f(\boldsymbol{x}(t))$ is the nonlinear *state transition function*, $\boldsymbol{w}(t)$ is the driving stochastic process, $g(\boldsymbol{x}_n)$ is the measurement model, and $\boldsymbol{r}_n$ is the measurement noise.

4.2 Discrete-Time State-Space Models

4.2.1 Deterministic Linear State-Space Models

As discussed in the previous section, ODEs and SDEs are a natural way for modeling the behavior of dynamic systems. Transforming them to state-space form furthermore provides a way of describing how the state of a system evolves over time.

However, not all dynamic systems can be modeled by using differential equations. For some systems, the state is only defined at some discrete points in time $t_1, t_2, \dots$, that is, only in *discrete-time*. In this case, another approach than differential equations is needed to model the dynamic behavior. The discrete-time equivalent to differential equations are *difference equations* that describe the relationship between the value of a variable at time t_n to its previous values at times $t_{n-1}, t_{n-2}, \dots$.

The process of obtaining a state-space representation from differential equations is closely related to the approach used for differential equations. Consider a d_xth order equation system with discrete-time, deterministic inputs $u_{1,n} \triangleq u_1(t_n), u_{2,n} \triangleq u_2(t_n), \ldots, u_{d_u,n} \triangleq u_{d_u}(t_n)$ of the form

$$\begin{aligned} x_{1,n} &= a_{11}x_{1,n-1} + \cdots + a_{1d_x}x_{d_x,n-1} + b_{11}u_{1,n} + \cdots + b_{1d_u}u_{d_u,n} \\ x_{2,n} &= a_{21}x_{1,n-1} + \cdots + a_{2d_x}x_{d_x,n-1} + b_{21}u_{1,n} + \cdots + b_{2d_u}u_{d_u,n} \\ &\vdots \\ x_{d_x,n} &= a_{d_x1}x_{1,n-1} + \cdots + a_{d_xd_x}x_{d_x,n-1} + b_{d_x1}u_{1,n} + \cdots + b_{d_xd_u}u_{d_u,n} \end{aligned}$$

where $x_{i,n} \triangleq x_i(t_n)$ is the ith variable at time t_n. This can be rewritten in matrix form according to

$$\begin{bmatrix} x_{1,n} \\ \vdots \\ x_{d_x,n} \end{bmatrix} = \begin{bmatrix} a_{11} & \ldots & a_{1d_x} \\ \vdots & \ddots & \vdots \\ a_{d_x1} & \ldots & a_{d_xd_x} \end{bmatrix} \begin{bmatrix} x_{1,n-1} \\ \vdots \\ x_{d_x,n-1} \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1d_u} \\ \vdots & \ddots & \vdots \\ b_{d_x1} & \ldots & b_{d_xd_u} \end{bmatrix} \begin{bmatrix} u_{1,n} \\ \vdots \\ u_{d_x,n} \end{bmatrix}.$$

More compactly, this can also be written as

$$\boldsymbol{x}_n = \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_u\boldsymbol{u}_n, \tag{4.25}$$

with

$$\boldsymbol{x}_n = \begin{bmatrix} x_{1,n} \\ \vdots \\ x_{d_x,n} \end{bmatrix},\ \boldsymbol{F} = \begin{bmatrix} a_{11} & \ldots & a_{1d_x} \\ \vdots & \ddots & \vdots \\ a_{d_x1} & \ldots & a_{d_xd_x} \end{bmatrix},\ \boldsymbol{B}_u = \begin{bmatrix} b_{11} & \ldots & b_{1d_u} \\ \vdots & \ddots & \vdots \\ b_{d_x1} & \ldots & b_{d_xd_u} \end{bmatrix},\ \boldsymbol{u}_n = \begin{bmatrix} u_{1,n} \\ \vdots \\ u_{d_u,n} \end{bmatrix}.$$

Equation (4.25) is the general dynamic model for linear discrete-time systems with deterministic inputs $\boldsymbol{u}_n$.

To convert the Lth order difference equation (with single input u_n)

$$z_n = c_1z_{n-1} + c_2z_{n-2} + \cdots + c_Lz_{n-L} + d_1u_n \tag{4.26}$$

to the form (4.25), we proceed as follows. First, we have to choose the state variables $\boldsymbol{x}_n$. This is easiest done at time $n-1$ and for the model (4.26), we can choose

$$x_{1,n-1} = z_{n-1},\ x_{2,n-1} = z_{n-2},\ \ldots,\ x_{d_x,n-1} = z_{n-L}.$$

Then we obtain that

$$\begin{aligned} x_{1,n} &= z_n = c_1z_{n-1} + c_2z_{n-2} + \cdots + c_Lz_{n-L} + d_1u_n \\ &= c_1x_{1,n-1} + c_2x_{2,n-1} + \cdots + c_Lx_{d_x,n-1} + d_1u_n, \\ x_{2,n} &= z_{n-1} \\ &= x_{1,n-1}, \\ &\vdots \\ x_{d_x,n} &= z_{n-L+1} \\ &= x_{d_x+1,n-1}, \end{aligned}$$

where we have expressed the state at time n solely in terms of the state at previous times as well as the input u_n. Rewriting this on matrix form finally yields

$$\begin{bmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{d_x,n} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \dots & c_L \\ 1 & 0 & & \vdots \\ \vdots & \ddots & & \\ 0 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,n-1} \\ x_{2,n-1} \\ \vdots \\ x_{d_x,n-1} \end{bmatrix} + \begin{bmatrix} d_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_n,$$

which is of the form (4.25).

Equation (4.25) only models the dynamic behavior of the state. As for the continuous case, only noisy measurements of the dynamic process are available for estimation. Combining the dynamic model with a measurement model thus again yields the complete discrete-time state-space model. Assuming a linear measurement model of the form (2.7) then yields the linear discrete-time model

$$\boldsymbol{x}_n = \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_u\boldsymbol{u}_n, \tag{4.27a}$$
$$\boldsymbol{y}_n = \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n. \tag{4.27b}$$

4.2.2 Stochastic Linear Dynamic Models

As for the continuous case, the deterministic discrete-time dynamic model can not take into account random effects, uncertainty, and unknown inputs. To account for such effects, we need an approach that is similar to the random process input in the continuous case. For the discrete-time case, we can model this using a *random variable* as the input to the dynamic model (compared to a stochastic process for the continuous case). We denote this random input, commonly called the *process noise*, as $\boldsymbol{q}_n$. Then, replacing the deterministic input $\boldsymbol{u}_n$ with $\boldsymbol{q}_n$ yields the stochastic discrete-time dynamic model

$$\boldsymbol{x}_n = \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_q\boldsymbol{q}_n. \tag{4.28}$$

The random variable $\boldsymbol{q}_n$ follows some probability density function such that

$$\boldsymbol{q}_n \sim p(\boldsymbol{q}_n).$$

Here, we assume that $\boldsymbol{q}_n$ is zero-mean, that is $\mathrm{E}\{\boldsymbol{q}_n\} = 0$, and that it has covariance $\mathrm{Cov}\{\boldsymbol{q}_n\} = \boldsymbol{Q}_n$. Furthermore, we also assume that the cross-covariance between two random inputs $\boldsymbol{q}_m$ and $\boldsymbol{q}_n$ is $\mathrm{Cov}\{\boldsymbol{q}_m, \boldsymbol{q}_n\} = 0$ for $m \neq n$.

Together with a linear sensor model, we then obtain the stochastic linear discrete-time state-space model given by

$$\boldsymbol{x}_n = \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_q\boldsymbol{q}_n, \tag{4.29a}$$
$$\boldsymbol{y}_n = \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n. \tag{4.29b}$$

4.2.3 Nonlinear Dynamic Model

Similar to differential equations, difference equations may be nonlinear. In this case, we start from the nonlinear equation system with the random variables $q_{1,n}, q_{2,n}, \dots, q_{d_q,n}$ as the inputs given by

$$\begin{aligned}
x_{1,n} &= f_1(x_{1,n-1}, x_{2,n-1}, \dots, x_{d_x,n-1}) + b_{11}q_{1,n} + \dots + b_{1d_q}q_{d_q,n}, \\
x_{2,n} &= f_2(x_{1,n-1}, x_{2,n-1}, \dots, x_{d_x,n-1}) + b_{21}q_{1,n} + \dots + b_{2d_q}q_{d_q,n}, \\
&\vdots \\
x_{d_x,n} &= f_{d_x}(x_{1,n-1}, x_{2,n-1}, \dots, x_{d_x,n-1}) + b_{d_x1}q_{1,n} + \dots + b_{d_xd_q}q_{d_q,n}.
\end{aligned}$$

This can directly be rewritten into vector form, which yields

$$\begin{bmatrix} x_{1,n} \\ \vdots \\ x_{d_x,n} \end{bmatrix} = \begin{bmatrix} f_1(x_{1,n-1}, x_{2,n-1}, \dots, x_{d_x,n-1}) \\ \vdots \\ f_{d_x}(x_{1,n-1}, x_{2,n-1}, \dots, x_{d_x,n-1}) \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1d_u} \\ \vdots & \ddots & \vdots \\ b_{d_x1} & \dots & b_{d_xd_u} \end{bmatrix} \begin{bmatrix} q_{1,n} \\ \vdots \\ q_{d_u,n} \end{bmatrix},$$

or more compactly

$$\boldsymbol{x}_n = f(\boldsymbol{x}_{n-1}) + \boldsymbol{B}_q\boldsymbol{q}_n. \tag{4.30}$$

The nonlinear dynamic model (4.30) together with the general nonlinear measurement model (3.1) yields the nonlinear discrete-time state-space model

$$\boldsymbol{x}_n = f(\boldsymbol{x}_{n-1}) + \boldsymbol{B}_q\boldsymbol{q}_n, \tag{4.31a}$$

$$\boldsymbol{y}_n = g(\boldsymbol{x}_n) + \boldsymbol{r}_n, \tag{4.31b}$$

where $\boldsymbol{q}_n \sim p(\boldsymbol{q}_n)$, $\mathrm{E}\{\boldsymbol{q}_n\} = 0$, $\mathrm{Cov}\{\boldsymbol{q}_n\} = \boldsymbol{Q}_n$, and $\boldsymbol{r}_n \sim p(\boldsymbol{r}_n)$, $\mathrm{E}\{\boldsymbol{r}_n\} = 0$, $\mathrm{Cov}\{\boldsymbol{r}_n\} = \boldsymbol{R}_n$.

4.3 Discretization of Linear Dynamic Models

In practice, modern sensor fusion systems are implemented in a digital computer. Hence, dealing with continuous dynamic models, that are defined on continuous t is not possible. Instead, we have to discretize the continuous-time model to obtain an (approximately) equivalent discrete-time dynamic model. In other words, we have to solve the differential equation on the interval between times t_{n-1} and t_n, that is, we have to integrate the differential equation. For linear dynamic models, this can be achieved exactly and we develop the necessary tools in this section. For most nonlinear models, exact discretization is not possible and we have to resort to approximate methods. This is the subject of Section 4.4.

4.3.1 Deterministic Linear Dynamic Models

Scalar Model. To derive the necessary expressions for discretizing the linear dynamic model, we first review how to solve a first order scalar case. Consider the non-homogeneous first order ODE

$$\dot{x}(t) = ax(t) + bu(t), \tag{4.32}$$

which we want to solve on the interval between the two sampling points t_n and t_{n-1}. To do this, we introduce the *integrating factor* $e^{-ax(t)}$ and note that it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-at}x(t) = e^{-at}\dot{x}(t) - e^{-at}ax(t). \tag{4.33}$$

Rearranging (4.32) by moving the term $ax(t)$ to the left hand side and multiplying by the integrating factor yields

$$e^{-at}\dot{x}(t) - e^{-at}ax(t) = e^{-at}bu(t)$$

and thus, using (4.33), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-at}x(t) = e^{-at}bu(t). \tag{4.34}$$

Equation (4.34) is separable in t. Hence, the ODE can directly be integrated from t_{n-1} to t_n according to

$$\int_{t_{n-1}}^{t_n} \mathrm{d}\left[e^{-at}x(t)\right] = \int_{t_{n-1}}^{t_n} e^{-at}bu(t)\mathrm{d}t,$$

which yields

$$\begin{aligned}
\left[e^{-at}x(t)\right]_{t=t_{n-1}}^{t_n} &= \int_{t_{n-1}}^{t_n} e^{-at}bu(t)\mathrm{d}t, \\
e^{-at_n}x(t_n) - e^{-at_{n-1}}x(t_{n-1}) &= \int_{t_{n-1}}^{t_n} e^{-at}bu(t)\mathrm{d}t.
\end{aligned}$$

Now we can solve for $x(t_n)$ by rearranging and multiplying by the factor e^{at_n}, which yields

$$\begin{aligned}
x(t_n) &= e^{at_n - at_{n-1}}x(t_{n-1}) + e^{at_n}\int_{t_{n-1}}^{t_n} e^{-at}bu(t)\mathrm{d}t, \\
&= e^{a(t_n - t_{n-1})}x(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{a(t_n - t)}bu(t)\mathrm{d}t.
\end{aligned}$$

Finally, letting $\Delta t \triangleq t_n - t_{n-1}$ and using the notation $x_n \triangleq x(t_n)$, we obtain

$$x_n = e^{a\Delta t}x_{n-1} + \int_{t_{n-1}}^{t_n} e^{a(t_n - t)}bu(t)\mathrm{d}t. \tag{4.35}$$

Note that the second term on the right hand side of (4.35) is the convolution between the factor $e^{a(t_n - t)}$ and the input $u(t)$ on the interval between t_{n-1} and t_n.

General Case. Based on the steps used for solving the scalar ODE (4.32), the general case can now be solved. First, recall that the general dynamic model on state-space form was given by the vector valued first order ODE system (4.15), that is,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u\boldsymbol{u}(t). \tag{4.36}$$

The key here is to note that (4.36) is a first order vector ODE just like (4.32). Hence, we should be able to solve it using an integrating factor similar to the one for the scalar case. Indeed, such an integrating factor can be found through the *matrix exponential.* The matrix exponential for a square matrix $\boldsymbol{A}$ can be defined in the same way as the regular exponential, that is, as an infinite sum of the form

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{A}^k.$$

Similar to the (scalar) exponential, it holds that the derivative with respect to a scalar x of $e^{\boldsymbol{A}x}$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{\boldsymbol{A}x} = e^{\boldsymbol{A}x}\boldsymbol{A}. \tag{4.37}$$

Another useful property of the matrix exponential is that of its transpose, which is given by

$$(e^{\boldsymbol{A}})^\mathsf{T} = e^{\boldsymbol{A}^\mathsf{T}}. \tag{4.38}$$

With this in mind, we can solve the general case following the same steps as for the scalar case. First, we multiply (4.36) by the integrating factor $e^{-\boldsymbol{A}t}$ and rearrange it to obtain

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}_u\boldsymbol{u}(t)$$

Next, similar to (4.33) (product rule), the left hand side is

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-\boldsymbol{A}t}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}}\boldsymbol{A}\boldsymbol{x}(t),$$

and thus, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-\boldsymbol{A}t}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}_u\boldsymbol{u}(t).$$

This is again separable in t. Direct integration from t_{n-1} to t_n then leads to

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} \mathrm{d}\left[e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right] &= \int_{t_{n-1}}^{t_n} e^{-\boldsymbol{A}t}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t. \\
\left[e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right]_{t=t_{n-1}}^{t_n} &= \int_{t_{n-1}}^{t_n} e^{-\boldsymbol{A}t}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t. \\
e^{-\boldsymbol{A}t_n}\boldsymbol{x}(t_n) - e^{-\boldsymbol{A}t_{n-1}}\boldsymbol{x}(t_{n-1}) &= \int_{t_{n-1}}^{t_n} e^{-\boldsymbol{A}t}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t.
\end{aligned}$$

Solving for $\boldsymbol{x}(t_n)$ by rearranging and multiplying both sides by $e^{\boldsymbol{A}t_n}$ yields

$$\boldsymbol{x}(t_n) = e^{\boldsymbol{A}(t_n - t_{n-1})}\boldsymbol{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t,$$

or

$$\boldsymbol{x}_n = e^{\boldsymbol{A}\Delta t}\boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t. \tag{4.39}$$

In this case, the factor in front of x_{n-1} (i.e., the state at time t_{n-1}) is a matrix exponential that again depends on the *time difference* Δt between the two discrete times t_{n-1} and t_n. Note that Δt can be arbitrary, meaning that there is no need for uniform sampling between points t_{n-2}, t_{n-1}, t_n, and so forth. Furthermore, the second term in (4.39) is again the convolution between the deterministic input $u(t)$ and the matrix exponential on the same interval.

If the input $\boldsymbol{u}(t)$ is a zeroth-order-hold (ZOH) signal, which is common in control or robot navigation, the second term can further be simplified. In that case, $\boldsymbol{u}(t)$ is constant on the interval between t_{n-1} and t_n and we can denote it as $\boldsymbol{u}_{n-1}$. Thus, we then have that

$$\begin{aligned}\int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t, &= \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u\boldsymbol{u}_{n-1}\mathrm{d}t, \\ &= \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u\mathrm{d}t\boldsymbol{u}_{n-1},\end{aligned}$$

where $\boldsymbol{B}_u$ may or may not depend on t.

Defining

$$\boldsymbol{F}_n \triangleq e^{\boldsymbol{A}(t_n - t_{n-1})}, \tag{4.40a}$$

$$\boldsymbol{L}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u\mathrm{d}t, \tag{4.40b}$$

we can rewrite (4.39) as

$$\boldsymbol{x}_n = \boldsymbol{F}_n\boldsymbol{x}_{n-1} + \boldsymbol{L}_n\boldsymbol{u}_{n-1}. \tag{4.41}$$

Equation (4.41) is an exact solution to (4.36) and thus it is an exact discretization and equivalent representation of the continuous-time model. Furthermore, it also has the same form as the discrete-time state-space model (4.27).

4.3.2 Stochastic Linear Dynamic Models

Discretization of the stochastic linear dynamic model in (4.19) and given by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_w\boldsymbol{w}(t), \tag{4.42}$$

with $R_{ww}(\tau) = \mathrm{E}\{\boldsymbol{w}(t+\tau)\boldsymbol{w}(t)\} = \boldsymbol{\Sigma}_w\delta(\tau)$, follows the same steps as the discretization of the deterministic model. This yields

$$\boldsymbol{x}(t_n) = e^{\boldsymbol{A}(t_n-t_{n-1})}\boldsymbol{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-t)}\boldsymbol{B}_w\boldsymbol{w}(t)\mathrm{d}t.$$

Now note that special attention has to be paid to the second term on the right hand side. The noise process $\boldsymbol{w}(t)$ does not have the properties required for regular integration rules to apply (Øksendal, 2010; Särkkä and Solin, 2018).

However, we can define a random variable $\boldsymbol{q}_n$ as

$$\boldsymbol{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-t)}\boldsymbol{B}_w\boldsymbol{w}(t)\mathrm{d}t.$$

Next, we can calculate the properties of $\boldsymbol{q}_n$. The mean is given by

$$\begin{aligned}
\mathrm{E}\{\boldsymbol{q}_n\} &= \mathrm{E}\left\{\int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-t)}\boldsymbol{B}_w\boldsymbol{w}(t)\mathrm{d}t\right\} \\
&= \int_{t_{n-1}}^{t_n} \mathrm{E}\left\{e^{\boldsymbol{A}(t_n-t)}\boldsymbol{B}_w\boldsymbol{w}(t)\right\}\mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-t)}\boldsymbol{B}_w\,\mathrm{E}\left\{\boldsymbol{w}(t)\right\}\mathrm{d}t \\
&= 0.
\end{aligned}$$

Similarly, we can calculate the covariance matrix

$$\begin{aligned}
\mathrm{Cov}\{\boldsymbol{q}_n\} &= \mathrm{E}\{(\boldsymbol{q}_n - \mathrm{E}\{\boldsymbol{q}_n\})(\boldsymbol{q}_n - \mathrm{E}\{\boldsymbol{q}_n\})^\mathsf{T}\} \\
&= \mathrm{E}\{\boldsymbol{q}_n\boldsymbol{q}_n^\mathsf{T}\} \\
&= \mathrm{E}\left\{\left(\int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-t)}\boldsymbol{B}_w\boldsymbol{w}(t)\mathrm{d}t\right)\left(\int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-\tau)}\boldsymbol{B}_w\boldsymbol{w}(\tau)\mathrm{d}\tau\right)^\mathsf{T}\right\} \\
&= \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-t)}\boldsymbol{B}_w\,\mathrm{E}\left\{\boldsymbol{w}(t)\boldsymbol{w}(\tau)^\mathsf{T}\right\}\boldsymbol{B}_w^\mathsf{T}(e^{\boldsymbol{A}(t_n-\tau)})^\mathsf{T}\mathrm{d}\tau\mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-t)}\boldsymbol{B}_w R_{ww}(t-\tau)\boldsymbol{B}_w^\mathsf{T}(e^{\boldsymbol{A}(t_n-\tau)})^\mathsf{T}\mathrm{d}\tau\mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-t)}\boldsymbol{B}_w\boldsymbol{\Sigma}_w\delta(t-\tau)\boldsymbol{B}_w^\mathsf{T}(e^{\boldsymbol{A}(t_n-\tau)})^\mathsf{T}\mathrm{d}\tau\mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-\tau)}\boldsymbol{B}_w\boldsymbol{\Sigma}_w\boldsymbol{B}_w^\mathsf{T}e^{\boldsymbol{A}^\mathsf{T}(t_n-\tau)}\mathrm{d}\tau \\
&\triangleq \boldsymbol{Q}_n.
\end{aligned}$$

Hence, the mean and covariance of the random variable $\boldsymbol{q}_n$ given that $\boldsymbol{w}(t)$ is a zero-mean white noise process with autocorrelation function $R_{ww}(\tau) = \boldsymbol{\Sigma}_w \delta(\tau)$ are given by

$$\mathrm{E}\{\boldsymbol{q}_n\} = 0, \tag{4.43a}$$

$$\mathrm{Cov}\{\boldsymbol{q}_n\} = \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-\tau)} \boldsymbol{B}_w \boldsymbol{\Sigma}_w \boldsymbol{B}_w^{\mathsf{T}} e^{\boldsymbol{A}^{\mathsf{T}}(t_n-\tau)} \mathrm{d}\tau \triangleq \boldsymbol{Q}_n. \tag{4.43b}$$

Furthermore, it turns out that in this case, the process noise $\boldsymbol{q}_n$ is actually a Gaussian random variable, that is, it holds that

$$\boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{Q}_n). \tag{4.44}$$

In summary, the discretized stochastic dynamic model (4.42) is

$$\boldsymbol{x}_n = \boldsymbol{F}_n \boldsymbol{x}_{n-1} + \boldsymbol{q}_n \tag{4.45}$$

where

$$\boldsymbol{F}_n \triangleq e^{\boldsymbol{A}(t_n - t_{n-1})}, \tag{4.46a}$$

$$\boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{Q}_n), \tag{4.46b}$$

with $\boldsymbol{Q}_n$ as in (4.43). Again, the discretized model (4.45)–(4.46) is an exact discretization of the continuous-time model (4.42) and thus completely equivalent.

4.4 Discretization of Nonlinear Dynamic Models

Unlike the linear models discussed in Section 4.3, discretization of nonlinear dynamic models is generally harder. In particular, closed-form solutions to the discretization problem can only be found in a few special cases. In most other cases, approximations have to be used. In this section three approaches for approximate discretization are discussed.

4.4.1 Discretization of Linearized Nonlinear Models

The first approach is based on a Taylor series expansion of the nonlinear function. Truncating the Taylor series expansion of the nonlinear function $f(\boldsymbol{x}(t))$ around the state $\boldsymbol{x}_{n-1}$ at the linear term yields the following approximation of the (deterministic) nonlinear dynamic model:

$$\dot{\boldsymbol{x}}(t) \approx f(\boldsymbol{x}_{n-1}) + \boldsymbol{A}_x(\boldsymbol{x}(t) - \boldsymbol{x}_{n-1}) + \boldsymbol{B}_u \boldsymbol{u}(t),$$

where $\boldsymbol{A}_x$ is the Jacobian of $f(\boldsymbol{x}(t))$ with respect to $\boldsymbol{x}(t)$, evaluated at $\boldsymbol{x}(t) = \boldsymbol{x}_{n-1}$. Rearranging the terms on the right hand side yields

$$\dot{\boldsymbol{x}}(t) \approx \boldsymbol{A}_x \boldsymbol{x}(t) + f(\boldsymbol{x}_{n-1}) - \boldsymbol{A}_x \boldsymbol{x}_{n-1} + \boldsymbol{B}_u \boldsymbol{u}(t),$$

which is linear in $\boldsymbol{x}(t)$, and $f(\boldsymbol{x}_{n-1})$ and $\boldsymbol{A}_x x_{n-1}$ are constants. Thus, we can use the same approach as for linear models in Section 4.3, that is, we can use the integrating factor $e^{-\boldsymbol{A}_x t}$. This yields

$$\begin{aligned}\boldsymbol{x}_n \approx e^{\boldsymbol{A}_x \Delta t}\boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\mathrm{d}t f(\boldsymbol{x}_{n-1}) - \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\mathrm{d}t \boldsymbol{A}_x \boldsymbol{x}_{n-1} \\ + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\boldsymbol{B}_u \boldsymbol{u}(t)\mathrm{d}t.\end{aligned}$$

Noting that the third term is the integral of the derivative of the matrix exponential with respect to t (see (4.37)), we have that

$$\begin{aligned}\boldsymbol{x}_n &\approx e^{\boldsymbol{A}_x \Delta t}\boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\mathrm{d}t f(\boldsymbol{x}_{n-1}) + \left[e^{\boldsymbol{A}_x(t_n - t)}\right]_{t=t_{n-1}}^{t_n} \boldsymbol{x}_{n-1} \\ &\quad + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\boldsymbol{B}_u \boldsymbol{u}(t)\mathrm{d}t \\ &= e^{\boldsymbol{A}_x \Delta t}\boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\mathrm{d}t f(\boldsymbol{x}_{n-1}) + (\boldsymbol{I} - e^{\boldsymbol{A}_x(t_n - t_{n-1})})\boldsymbol{x}_{n-1} \\ &\quad + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\boldsymbol{B}_u \boldsymbol{u}(t)\mathrm{d}t,\end{aligned}$$

and finally,

$$\boldsymbol{x}_n \approx \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\mathrm{d}t f(\boldsymbol{x}_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\boldsymbol{B}_u \boldsymbol{u}(t)\mathrm{d}t. \tag{4.47}$$

As usual, the drawback of this Taylor-series-based solution is that the linearization is only local and large Δts as well as highly nonlinear functions may be problematic. On the other hand, as it can be seen from (4.47), the approximation only affects the deterministic term of the model, whereas the input is calculated in the same way as for the linear model (but with the Jacobian $\boldsymbol{A}_x$). Hence, for the stochastic dynamic model, the result immediately follows from (4.47) to be

$$\boldsymbol{x}_n \approx \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\mathrm{d}t f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n, \tag{4.48}$$

with

$$\begin{aligned}\boldsymbol{q}_n &\sim \mathcal{N}(0, \boldsymbol{Q}_n), \\ \boldsymbol{Q}_n &\approx \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - \tau)}\boldsymbol{B}_w \boldsymbol{\Sigma}_w \boldsymbol{B}_w^\mathsf{T} e^{\boldsymbol{A}_x^\mathsf{T}(t_n - \tau)}\mathrm{d}\tau.\end{aligned}$$

4.4.2 Euler Discretization of Deterministic Nonlinear Models

The second approach is based on the so-called Euler approximation. We know that the solution for $\boldsymbol{x}_n$ can be written as

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} f(\boldsymbol{x}(t))\mathrm{d}t + \int_{t_{n-1}}^{t_n} \boldsymbol{B}_u \boldsymbol{u}(t)\mathrm{d}t,$$

and the problem is to calculate the integrals in the second and third terms on the right hand side. However, for sufficiently small $\Delta t = t_n - t_{n-1}$, we can approximate the integral by using the rectangle approximation. This means that we can approximate the integral as

$$\begin{aligned}\int_{t_{n-1}}^{t_n} f(\boldsymbol{x}(t))\mathrm{d}t &\approx f(\boldsymbol{x}_{n-1})(t_n - t_{n-1}) \\ &= \Delta t f(\boldsymbol{x}_{n-1}),\end{aligned}$$

and similar for the second integral with the input $\boldsymbol{u}(t)$.

This then yields the Euler discretization of the deterministic, nonlinear dynamic model given by

$$\boldsymbol{x}_n \approx \boldsymbol{x}_{n-1} + \Delta t f(\boldsymbol{x}_{n-1}) + \Delta t \boldsymbol{B}_u \boldsymbol{u}_{n-1}. \tag{4.49}$$

4.4.3 Euler–Maruyama Discretization of Stochastic Nonlinear Models

The Euler discretization is quite simple to implement. However, it does not readily work for stochastic dynamic models. In this case, we are interested in solving the integral equation

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} f(\boldsymbol{x}(t))\mathrm{d}t + \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w \boldsymbol{w}(t)\mathrm{d}t.$$

We can still use the rectangle approximation for the integral involving the deterministic part (second term on the right hand side), but for the stochastic part, we are again lacking the integration rules. Thus, we can not use the rectangle approximation in this case. However, we can again define the resulting random variable as

$$\boldsymbol{q}_n \triangleq \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w \boldsymbol{w}(t)\mathrm{d}t$$

and investigate its properties instead.

First, the mean is given by

$$\begin{aligned}\mathrm{E}\{\boldsymbol{q}_n\} &= \mathrm{E}\left\{\int_{t_{n-1}}^{t_n} \boldsymbol{B}_w \boldsymbol{w}(t)\mathrm{d}t\right\} \\ &= \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w \,\mathrm{E}\left\{\boldsymbol{w}(t)\right\}\mathrm{d}t \\ &= 0,\end{aligned}$$

where we have made use of the assumption that $\boldsymbol{w}(t)$ is a zero-mean process. Second, the covariance can be found similar to the linear case as follows.

$$\begin{aligned}
\operatorname{Cov}\{\boldsymbol{q}_n\} &= \mathrm{E}\left\{\left(\int_{t_{n-1}}^{t_n} \boldsymbol{B}_w \boldsymbol{w}(t)\mathrm{d}t\right)\left(\int_{t_{n-1}}^{t_n} \boldsymbol{B}_w \boldsymbol{w}(\tau)\mathrm{d}\tau\right)^{\mathsf{T}}\right\} \\
&= \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} \boldsymbol{B}_w \,\mathrm{E}\{\boldsymbol{w}(t)\boldsymbol{w}(\tau)^{\mathsf{T}}\}\boldsymbol{B}_w^{\mathsf{T}}\mathrm{d}\tau\mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} \boldsymbol{B}_w \boldsymbol{\Sigma}_w \delta(t-\tau)\boldsymbol{B}_w^{\mathsf{T}}\mathrm{d}\tau\mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w \boldsymbol{\Sigma}_w \boldsymbol{B}_w^{\mathsf{T}}\mathrm{d}\tau.
\end{aligned}$$

At this point, note that the remaining integral does not involve any random process anymore. Hence, we can again use the rectangle approximation of the integral, which yields

$$\begin{aligned}
\operatorname{Cov}\{\boldsymbol{q}_n\} &\approx (\boldsymbol{B}_w \boldsymbol{\Sigma}_w \boldsymbol{B}_w^{\mathsf{T}})(t_n - t_{n-1}) \\
&= \Delta t \boldsymbol{B}_w \boldsymbol{\Sigma}_w \boldsymbol{B}_w^{\mathsf{T}} \\
&\triangleq \boldsymbol{Q}_n.
\end{aligned}$$

This yields the *Euler–Maruyama* discretization of the stochastic nonlinear dynamic model given by

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \Delta t f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n, \tag{4.50}$$

with

$$\begin{aligned}
\boldsymbol{q}_n &\sim \mathcal{N}(0, \boldsymbol{Q}_n), \\
\boldsymbol{Q}_n &= \Delta t \boldsymbol{B}_w \boldsymbol{\Sigma}_w \boldsymbol{B}_w^{\mathsf{T}}.
\end{aligned}$$