Continuous-Time State-Space Models

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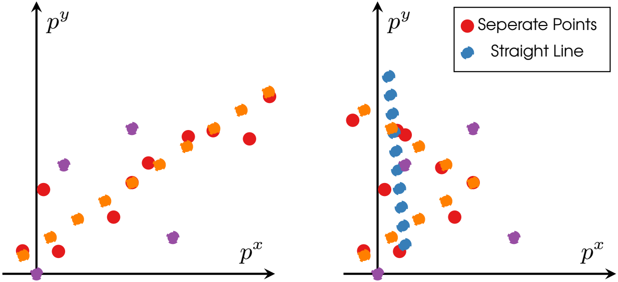
Intended Learning Outcomes

After this lecture, you will be able to:

- Describe the idea of state-space modeling,
- explain the process of constructing state-space models,
- distinguish deterministic and stochastic linear state-space models.

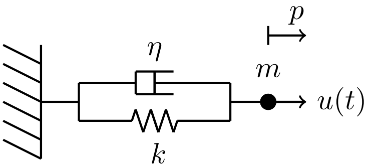
Recap: Time-Varying (Dynamic) Problems

- Most sensor fusion problems involve dynamically changing parameters
- Sensors sample (measure) periodically
- Parameters are correlated in time
- Example: Localizing a moving target



Differential equations can be used to describe dynamic systems!

Recap: Spring-Damper System



- Second order ordinary differential equation:

$$ma(t) = -kp(t) - \eta v(t) + u(t)$$

- Equation system representation:

$$\begin{bmatrix} v(t) \\ a(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

- First order ODE equation system:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

Example: A Coffee Cup's Cooling (1/2)

- Newton's law of cooling for the coffee cup:

$$\frac{\mathrm{d}T_{\mathsf{c}}(t)}{\mathrm{d}t} = -k_1(T_{\mathsf{c}}(t) - T_{\mathsf{r}}(t)),$$

- Newton's law of cooling for the room:

$$\frac{\mathrm{d}T_{\mathsf{r}}(t)}{\mathrm{d}t} = -k_2(T_{\mathsf{r}}(t) - T_{\mathsf{a}}(t)) + h(t),$$

- Equation system:

$$\begin{aligned}
\frac{\mathrm{d}T_{\mathsf{r}}(t)}{\mathrm{d}t} &= -k_2(T_{\mathsf{r}}(t) - T_{\mathsf{a}}(t)) + h(t) \\
\frac{\mathrm{d}T_{\mathsf{c}}(t)}{\mathrm{d}t} &= -k_1(T_{\mathsf{c}}(t) - T_{\mathsf{r}}(t))
\end{aligned}$$

Example: A Coffee Cup's Cooling (2/2)

- Equation system:

$$\frac{\mathrm{d}T_\mathsf{r}(t)}{\mathrm{d}t} = -k_2(T_\mathsf{r}(t) - T_\mathsf{a}(t)) + h(t)$$
$$\frac{\mathrm{d}T_\mathsf{c}(t)}{\mathrm{d}t} = -k_1(T_\mathsf{c}(t) - T_\mathsf{r}(t))$$

- On matrix form:

$$\begin{bmatrix} \frac{\mathrm{d}T_\mathsf{r}(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}T_\mathsf{c}(t)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -k_2 & 0 \\ k_1 & -k_1 \end{bmatrix} \begin{bmatrix} T_\mathsf{r}(t) \\ T_\mathsf{c}(t) \end{bmatrix} + \begin{bmatrix} k_2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_\mathsf{a}(t) \\ h(t) \end{bmatrix}$$

- Compact notation:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u\boldsymbol{u}(t)$$

A Linear System of Differential Equations (1/2)

$$
\begin{aligned}
\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1d_x}x_{d_x}(t) \\
&\quad + b_{11}u_1(t) + b_{12}u_2(t) + \cdots + b_{1d_u}u_{d_u}(t) \\
\dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2d_x}x_{d_x}(t) \\
&\quad + b_{21}u_1(t) + b_{22}u_2(t) + \cdots + b_{2d_u}u_{d_u}(t) \\
&\vdots \\
\dot{x}_{d_x}(t) &= a_{d_x1}x_1(t) + a_{d_x2}x_2(t) + \cdots + a_{d_xd_x}x_{d_x}(t) \\
&\quad + b_{d_x1}u_1(t) + b_{d_x2}u_2(t) + \cdots + b_{d_xd_u}u_{d_u}(t)
\end{aligned}
$$

A Linear System of Differential Equations (2/2)

- On matrix form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_{d_x}(t) \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1d_x} \\ \vdots & \ddots & \vdots \\ a_{d_x 1} & \dots & a_{d_x d_x} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_{d_x}(t) \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1d_u} \\ \vdots & \ddots & \vdots \\ b_{d_x 1} & \dots & b_{d_x d_u} \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{d_u}(t) \end{bmatrix}$$

- Compact notation:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u \boldsymbol{u}(t)$$

- This is called the state-space form of the differential equation system, $\boldsymbol{x}(t)$ is the state of the system

Transforming ODEs to State-Space Form (1/2)

- Lth order ODE in $z(t)$

$$\frac{\mathrm{d}^L z(t)}{\mathrm{d}t^L} = c_0 z(t) + c_2 \frac{\mathrm{d}z(t)}{\mathrm{d}t} + \cdots + c_{L-1} \frac{\mathrm{d}^{L-1} z(t)}{\mathrm{d}t^{L-1}} + d_1 u(t)$$

- Choose:

$$x_1(t) = z(t),\ x_2(t) = \frac{\mathrm{d}z(t)}{\mathrm{d}t},\ \ldots,\ x_{d_x}(t) = \frac{\mathrm{d}^{L-1} z(t)}{\mathrm{d}t^{L-1}}$$

- Then:

$$\begin{aligned}
\dot{x}_1(t) &= \frac{\mathrm{d}z(t)}{\mathrm{d}t} = x_2(t) \\
\dot{x}_2(t) &= \frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} = x_3(t) \\
&\ \ \vdots \\
\dot{x}_{d_x}(t) &= \frac{\mathrm{d}^L z(t)}{\mathrm{d}t^L} = c_0 z(t) + c_2 \frac{\mathrm{d}z(t)}{\mathrm{d}t} + \cdots + c_{L-1} \frac{\mathrm{d}^{L-1} z(t)}{\mathrm{d}t^{L-1}} + d_1 u(t)
\end{aligned}$$

Transforming ODEs to State-Space Form (2/2)

- Lth order equation system:

$$\dot{x}_1(t) = \frac{\mathrm{d}z(t)}{\mathrm{d}t} = x_2(t)$$

$$\vdots$$

$$\dot{x}_{d_x}(t) = \frac{\mathrm{d}^L z(t)}{\mathrm{d}t^L} = c_0 z(t) + c_2 \frac{\mathrm{d}z(t)}{\mathrm{d}t} + \cdots + c_{L-1} \frac{\mathrm{d}^{L-1} z(t)}{\mathrm{d}t^{L-1}} + d_1 u(t)$$

- Matrix form:

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{d_x}(t) \end{bmatrix}}_{\triangleq \dot{\boldsymbol{x}}(t)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ c_0 & c_1 & & \dots & c_{L-1} \end{bmatrix}}_{\triangleq \boldsymbol{A}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{d_x}(t) \end{bmatrix}}_{\triangleq \boldsymbol{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ d_1 \end{bmatrix}}_{\triangleq \boldsymbol{B}_u} u(t).$$

Deterministic Linear State-Space Model

- The dynamic model describes the evolution of the *state*:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

- The measurement model relates the *state* $\boldsymbol{x}_n = \boldsymbol{x}(t_n)$ at t_n to the *measurement* $\boldsymbol{y}_n$
- The linear measurement model is

$$\boldsymbol{y}_n = \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n,$$

- The deterministic linear state-space model combines the linear dynamic and measurement models

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u\boldsymbol{u}(t), \\ \boldsymbol{y}_n &= \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n.\end{aligned}$$

Example: A Robot Navigating in 2D (1)

Uncertainty in Dynamic Models

- The deterministic input $u(t)$ might not be known
- The model does not capture every aspect of the process
- Solution: Add a stochastic process $w(t)$ as an input
- Example: Stochastic differential equation (SDE) of order L:

$$\frac{\mathrm{d}^L z(t)}{\mathrm{d}t^L} = c_0 z(t) + c_1 \frac{\mathrm{d}z(t)}{\mathrm{d}t} + \cdots + c_{L-1} \frac{\mathrm{d}^{L-1} z(t)}{\mathrm{d}t^{L-1}} + d_1 w(t)$$

Input Process $w(t)$

- Assumed to be *zero-mean* and *stationary*
- Characterized by its autocorrelation function...

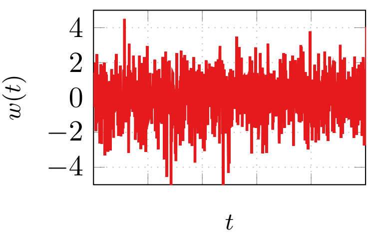
$$R_{ww}(\tau) = \mathrm{E}\{w(t+\tau)w(t)\}$$

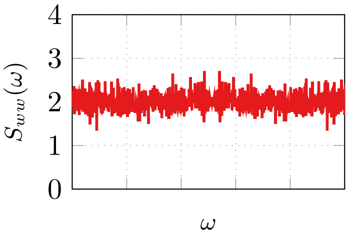
- ...or its power spectral density

$$S_{ww}(\omega) = \int R_{ww}(\tau)e^{-\mathrm{i}\,\omega\tau}\mathrm{d}\tau$$

White Processes

- "White noise" — equal contributions of each frequency
- Autocorrelation function: $R_{ww}(\tau) = \sigma_w^2 \delta(\tau)$
- Power spectral density: $S_{ww} = \sigma_w^2$
- Many forms of colored noise are filtered versions of white noise





Stochastic Linear State-Space Model

- Derivation of the dynamic model follows the same steps as for the deterministic case
- The stochastic process $\boldsymbol{w}(t)$ takes the place of the deterministic input $\boldsymbol{u}(t)$
- A system can have both deterministic and stochastic inputs
- Linear dynamic model:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_w\boldsymbol{w}(t)$$

- Linear stochastic state-space model:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_w\boldsymbol{w}(t) \\ \boldsymbol{y}_n &= \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n \end{aligned}$$

Example: A Robot Navigating in 2D (2)

- Deterministic dynamic model:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/m \end{bmatrix} \begin{bmatrix} F_p^x \\ F_p^y \end{bmatrix}$$

- $\boldsymbol{u}(t)$ might be unknown when localizing the robot
- Assume stochastic processes as the input:

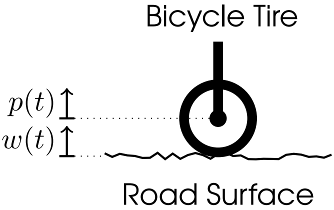
$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

- This is the Wiener velocity model in 2D

Exercise: Bicycle Tire Model

Task:

- Find the second order SDE for the spring-damper equivalent model of a bicycle tire shown below
- Assume that the road surface can be modeled as a stochastic process
- Transform the SDE to state-space form and determine $\boldsymbol{x}(t)$, $\boldsymbol{A}$, and $\boldsymbol{B}_w$.



Summary

- Higher order ODEs and SDEs can be transformed to a first-order vector-valued equation system
- The deterministic linear state-space model is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_w \boldsymbol{w}(t)$$
$$\boldsymbol{y}_n = \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n$$

- The stochastic linear state-space model with stochastic input process $\boldsymbol{w}(t)$ is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_w \boldsymbol{w}(t)$$
$$\boldsymbol{y}_n = \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n$$

- The 2D Wiener velocity model is

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$