Nonlinear Continuous-Time Models and Discrete-Time Models

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October 17, 2018

Recap: Linear Continuous-Time Models

- The deterministic linear dynamic model is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u\boldsymbol{u}(t)$$

- Stochastic processes can account for uncertainty in dynamic models
- A white noise process has autocorrelation function $R_{ww}(\tau)$ and power spectral density $S_{ww}(\omega)$

$$R_{ww}(\tau) = \mathrm{E}\{w(t+\tau)w(t)\} = \sigma_w^2\delta(\tau)$$
$$S_{ww}(\omega) = \int R_{ww}(\tau)e^{-\,\mathrm{i}\,\omega\tau}\mathrm{d}\tau = \sigma_w^2$$

- The stochastic linear dynamic model is

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_w\boldsymbol{w}(t)$$

Recap: Wiener Velocity Model in 2D

- A robot moving in 2D where the control actions are unknown can be described by the Wiener velocity model
- The model is given by

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

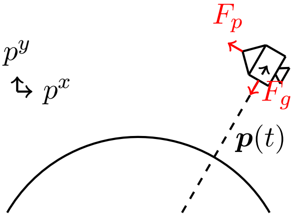
where $\boldsymbol{x}(t) = \begin{bmatrix} p^x(t) & p^y(t) & v^x(t) & v^y(t) \end{bmatrix}^\mathsf{T}$

Intended Learning Outcomes

After this lecture, you will be able to:

- Identify and construct nonlinear continuous-time state-space models, and
- recognize and explain discrete-time state-space models.

Example: Dynamic Model for a Spacecraft (1/2)



- Gravitational acceleration:

$$g \approx g_0 \left(\frac{r_e}{|\boldsymbol{p}(t)|} \right)^2,$$

Example: Dynamic Model for a Spacecraft (2/2)

- Gravitational pull: $\boldsymbol{F}_g = -mg_0r_e^2\frac{\boldsymbol{p}(t)}{|\boldsymbol{p}(t)|^3}$
- Propulsion: $\boldsymbol{F}_p = F_p\frac{1}{|\boldsymbol{p}(t)|}\begin{bmatrix}-p^y(t) \\ p^x(t)\end{bmatrix}$
- Differential equation:

$$m\boldsymbol{a}(t) = -mg_0r_e^2\frac{\boldsymbol{p}(t)}{|\boldsymbol{p}(t)|^3} + \frac{1}{|\boldsymbol{p}(t)|}\begin{bmatrix}-p^y(t) \\ p^x(t)\end{bmatrix}u(t).$$

- State vector:

$$\boldsymbol{x}(t) = \begin{bmatrix}p^x(t) & p^y(t) & v^x(t) & v^y(t)\end{bmatrix}^\mathsf{T}.$$

Can not be written as $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u\boldsymbol{u}(t)$.

Nonlinear Differential Equation Systems

- Nonlinear ordinary differential equation system (b_{ij} may depend on $x_{mn}(t)$):

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t), x_2(t), \dots, x_{d_x}(t)) + b_{11}u_1(t) + \dots b_{1d_u}u_{d_u}(t) \\
\dot{x}_2(t) &= f_2(x_1(t), x_2(t), \dots, x_{d_x}(t)) + b_{21}u_1(t) + \dots b_{2d_u}u_{d_u}(t) \\
&\vdots \\
\dot{x}_{d_x}(t) &= f_{d_x}(x_1(t), x_2(t), \dots, x_{d_x}(t)) + b_{d_x1}u_1(t) + \dots b_{d_xd_u}u_{d_u}(t)
\end{aligned}
$$

- State vector: $\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_{d_x}(t) \end{bmatrix}^\mathsf{T}$
- In vector form:

$$
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_{d_x}(t) \end{bmatrix} = \begin{bmatrix} f_1(\boldsymbol{x}(t)) \\ f_2(\boldsymbol{x}(t)) \\ \vdots \\ f_{d_x}(\boldsymbol{x}(t)) \end{bmatrix} + \begin{bmatrix} b_{11}(\boldsymbol{x}(t)) & \dots & b_{1d_u}(\boldsymbol{x}(t)) \\ b_{21}(\boldsymbol{x}(t)) & & \vdots \\ \vdots & \ddots & \\ b_{d_x1}(\boldsymbol{x}(t)) & \dots & b_{d_xd_u}(\boldsymbol{x}(t)) \end{bmatrix} \boldsymbol{u}(t).
$$

Nonlinear Continuous-Time State-Space Models

- Deterministic nonlinear dynamic model:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_u(\boldsymbol{x}(t))\boldsymbol{u}(t)$$

- Stochastic nonlinear dynamic model:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)$$

- Nonlinear measurement model:

$$\boldsymbol{y}_n = g(\boldsymbol{x}_n) + \boldsymbol{r}_n$$

- Stochastic nonlinear state-space model:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t) \\ \boldsymbol{y}_n &= g(\boldsymbol{x}_n) + \boldsymbol{r}_n\end{aligned}$$

Example: Dynamic Model for a Spacecraft (2)

- Differential equation:

$$m\boldsymbol{a}(t) = -mg_0 r_e^2 \frac{\boldsymbol{p}(t)}{|\boldsymbol{p}(t)|^3} + \frac{1}{|\boldsymbol{p}(t)|} \begin{bmatrix} -p^y(t) \\ p^x(t) \end{bmatrix} w(t).$$

- State vector:

$$\boldsymbol{x}(t) = \begin{bmatrix} p^x(t) & p^y(t) & v^x(t) & v^y(t) \end{bmatrix}^\mathsf{T}.$$

- Vector form:

$$\begin{aligned} \begin{bmatrix} v^x(t) \\ v^y(t) \\ a^x(t) \\ a^y(t) \end{bmatrix} &= \begin{bmatrix} v^x(t) \\ v^y(t) \\ -g_0 r_e^2 \frac{p^x(t)}{|\boldsymbol{p}(t)|^3} \\ -g_0 r_e^2 \frac{p^y(t)}{|\boldsymbol{p}(t)|^3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{p^y(t)}{m|\boldsymbol{p}(t)|} \\ \frac{p^x}{m|\boldsymbol{p}(t)|} \end{bmatrix} w(t) \\ &= \begin{bmatrix} f_1(\boldsymbol{x}(t)) \\ f_2(\boldsymbol{x}(t)) \\ f_3(\boldsymbol{x}(t)) \\ f_4(\boldsymbol{x}(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{p^y(t)}{m|\boldsymbol{p}(t)|} \\ \frac{p^x}{m|\boldsymbol{p}(t)|} \end{bmatrix} w(t), \end{aligned}$$

Example: Vehicle Navigation

Discrete-Time Processes and Difference Equations

- Some processes are only defined at discrete time points $t_1, t_2, \dots$
- The discrete-time equivalent of differential equations are difference equations
- The difference of two discrete points in time takes the role of the derivative

Vector Form of Difference Equation Systems

- Equation system of d_x linear difference equations:

$$
\begin{aligned}
x_{1,n} &= a_{11}x_{1,n-1} + \cdots + a_{1d_x}x_{d_x,n-1} + b_{11}u_{1,n} + \cdots + b_{1d_u}u_{d_u,n} \\
x_{2,n} &= a_{21}x_{1,n-1} + \cdots + a_{2d_x}x_{d_x,n-1} + b_{21}u_{1,n} + \cdots + b_{2d_u}u_{d_u,n} \\
&\vdots \\
x_{d_x,n} &= a_{d_x1}x_{1,n-1} + \cdots + a_{d_xd_x}x_{d_x,n-1} + b_{d_x1}u_{1,n} + \cdots + b_{d_xd_u}u_{d_u,n}
\end{aligned}
$$

- Vector form:

$$
\begin{bmatrix} x_{1,n} \\ \vdots \\ x_{d_x,n} \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1d_x} \\ \vdots & \ddots & \vdots \\ a_{d_x1} & \dots & a_{d_xd_x} \end{bmatrix} \begin{bmatrix} x_{1,n-1} \\ \vdots \\ x_{d_x,n-1} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1d_u} \\ \vdots & \ddots & \vdots \\ b_{d_x1} & \dots & b_{d_xd_u} \end{bmatrix} \begin{bmatrix} u_{1,n} \\ \vdots \\ u_{d_x,n} \end{bmatrix}
$$

- Compact notation:

$$
\boldsymbol{x}_n = \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_u\boldsymbol{u}_n
$$

Deterministic Discrete-Time State-Space Model

- Linear discrete-time dynamic model:

$$\boldsymbol{x}_n = \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_u\boldsymbol{u}_n$$

- Deterministic, linear discrete-time state-space model:

$$\begin{aligned}\boldsymbol{x}_n &= \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_u\boldsymbol{u}_n \\ \boldsymbol{y}_n &= \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n.\end{aligned}$$

with $\mathrm{E}\{\boldsymbol{r}_n\} = 0$, $\mathrm{Cov}\{\boldsymbol{r}_n\} = \boldsymbol{R}_n$, $\mathrm{Cov}\{\boldsymbol{r}_n, \boldsymbol{r}_m\} = 0$ $(n \neq m)$

Conversion of Lth Order Difference Equation (1/2)

- Lth order difference equation (with single input u_n):

$$z_n = c_1 z_{n-1} + c_2 z_{n-2} + \cdots + c_L z_{n-L} + d_1 u_n$$

- It is easier to choose $\boldsymbol{x}_{n-1}$ on the RHS
  (c.f. continuous case)
- A possible choice:

$$x_{1,n-1} = z_{n-1},\ x_{2,n-1} = z_{n-2},\ \ldots,\ x_{d_x,n-1} = z_{n-L}.$$

Conversion of Lth Order Difference Equation (2/2)

- Difference equation system:

$$\begin{aligned} x_{1,n} &= c_1 x_{1,n-1} + c_2 x_{2,n-1} + \cdots + c_L x_{d_x,n-1} + d_1 u_n \\ x_{2,n} &= z_{n-1} = x_{1,n-1} \\ &\vdots \\ x_{d_x,n} &= z_{n-L+1} = x_{d_x+1,n-1} \end{aligned}$$

- Vector form:

$$\begin{bmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{d_x,n} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \dots & c_L \\ 1 & 0 & & \vdots \\ \vdots & \ddots & & \\ 0 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,n-1} \\ x_{2,n-1} \\ \vdots \\ x_{d_x,n-1} \end{bmatrix} + \begin{bmatrix} d_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_n,$$

Stochastic Linear State-Space Model (1/2)

- Dynamics are not entirely deterministic and inputs may not always be known
- Let the process noise $\boldsymbol{q}_n$ (random variable) take the place of the input $\boldsymbol{u}_n$ (or in addition to $\boldsymbol{u}_n$)
- Stochastic linear discrete-time dynamic model:

$$\boldsymbol{x}_n = \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_q\boldsymbol{q}_n$$

Stochastic Linear State-Space Model (2/2)

- Stochastic linear discrete-time dynamic model:

$$\boldsymbol{x}_n = \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_q\boldsymbol{q}_n$$

- The process noise follows

$$\boldsymbol{q}_n \sim p(\boldsymbol{q}_n)$$

  with $\mathrm{E}\{\boldsymbol{q}_n\} = 0$, $\mathrm{Cov}\{\boldsymbol{q}_n\} = \boldsymbol{Q}_n$, and $\mathrm{Cov}\{\boldsymbol{q}_m, \boldsymbol{q}_n\} = 0$ $(m \neq n)$

- Stochastic linear discrete-time state-space model:

$$\begin{aligned} \boldsymbol{x}_n &= \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_q\boldsymbol{q}_n \\ \boldsymbol{y}_n &= \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n \end{aligned}$$

Nonlinear Discrete-Time Dynamic Model

- Difference equations may also be nonlinear
- Nonlinear difference equation system (with process noise inputs):

$$
\begin{aligned}
x_{1,n} &= f_1(x_{1,n-1}, x_{2,n-1}, \dots, x_{d_x,n-1}) + b_{11} q_{1,n} + \cdots + b_{1d_q} q_{d_q,n} \\
x_{2,n} &= f_2(x_{1,n-1}, x_{2,n-1}, \dots, x_{d_x,n-1}) + b_{21} q_{1,n} + \cdots + b_{2d_q} q_{d_q,n} \\
&\vdots \\
x_{d_x,n} &= f_{d_x}(x_{1,n-1}, x_{2,n-1}, \dots, x_{d_x,n-1}) + b_{d_x 1} q_{1,n} + \cdots + b_{d_x d_q} q_{d_q,n}
\end{aligned}
$$

- Vector form:

$$
\begin{bmatrix} x_{1,n} \\ \vdots \\ x_{d_x,n} \end{bmatrix} = \begin{bmatrix} f_1(x_{1,n-1}, x_{2,n-1}, \dots, x_{d_x,n-1}) \\ \vdots \\ f_{d_x}(x_{1,n-1}, x_{2,n-1}, \dots, x_{d_x,n-1}) \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1d_u} \\ \vdots & \ddots & \vdots \\ b_{d_x 1} & \dots & b_{d_x d_u} \end{bmatrix} \begin{bmatrix} q_{1,n} \\ \vdots \\ q_{d_u,n} \end{bmatrix}
$$

Nonlinear Discrete-Time State-Space Model

- Compact notation of the dynamic model:

$$\boldsymbol{x}_n = f(\boldsymbol{x}_{n-1}) + \boldsymbol{B}_q \boldsymbol{q}_n$$

- Nonlinear discrete-time state-space model:

$$\begin{aligned}
\boldsymbol{x}_n &= f(\boldsymbol{x}_{n-1}) + \boldsymbol{B}_q \boldsymbol{q}_n \\
\boldsymbol{y}_n &= g(\boldsymbol{x}_n) + \boldsymbol{r}_n
\end{aligned}$$

  where:
  - $\boldsymbol{q}_n \sim p(\boldsymbol{q}_n)$, $\mathrm{E}\{\boldsymbol{q}_n\} = 0$, $\mathrm{Cov}\{\boldsymbol{q}_n\} = \boldsymbol{Q}_n$
  - $\boldsymbol{r}_n \sim p(\boldsymbol{r}_n)$, $\mathrm{E}\{\boldsymbol{r}_n\} = 0$, $\mathrm{Cov}\{\boldsymbol{r}_n\} = \boldsymbol{R}_n$

Summary

- Nonlinear continuous-time state-space model:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t) \\ \boldsymbol{y}_n &= g(\boldsymbol{x}_n) + \boldsymbol{r}_n\end{aligned}$$

- Linear discrete-time state-space model:

$$\begin{aligned}\boldsymbol{x}_n &= \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_q\boldsymbol{q}_n \\ \boldsymbol{y}_n &= \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n\end{aligned}$$

- Nonlinear discrete-time state-space model:

$$\begin{aligned}\boldsymbol{x}_n &= f(\boldsymbol{x}_{n-1}) + \boldsymbol{B}_q(\boldsymbol{x}_{n-1})\boldsymbol{q}_n \\ \boldsymbol{y}_n &= g(\boldsymbol{x}_n) + \boldsymbol{r}_n\end{aligned}$$

Announcements

- No lecture and exercises next week
- Project work should be well under way now
- Remember to book time for experiments
- Mid-term feedback will be available next week, please participate