Discretization of Linear Continuous-Time State-Space Models

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Recap: State-Space Models

- Nonlinear continuous-time state-space model:

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t) \\
\boldsymbol{y}_n &= g(\boldsymbol{x}_n) + \boldsymbol{r}_n
\end{aligned}
$$

- Linear discrete-time state-space model:

$$
\begin{aligned}
\boldsymbol{x}_n &= \boldsymbol{F}\boldsymbol{x}_{n-1} + \boldsymbol{B}_q\boldsymbol{q}_n \\
\boldsymbol{y}_n &= \boldsymbol{G}\boldsymbol{x}_n + \boldsymbol{r}_n
\end{aligned}
$$

- Nonlinear discrete-time state-space model:

$$
\begin{aligned}
\boldsymbol{x}_n &= f(\boldsymbol{x}_{n-1}) + \boldsymbol{B}_q(\boldsymbol{x}_{n-1})\boldsymbol{q}_n \\
\boldsymbol{y}_n &= g(\boldsymbol{x}_n) + \boldsymbol{r}_n
\end{aligned}
$$

Recap: Quasi-Constant Turn Model in 2D

- A vehicle at $\boldsymbol{p}(t)$, speed $v(t)$, and heading $\varphi(t)$
- State vector: $\boldsymbol{x} = \begin{bmatrix} p^x(t) & p^y(t) & v(t) & \varphi(t) \end{bmatrix}^\mathsf{T}$
- Dynamic model

$$\begin{bmatrix} \dot{p}^x(t) \\ \dot{p}^y(t) \\ \dot{v}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} v(t)\cos(\varphi(t)) \\ v(t)\sin(\varphi(t)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{w}(t)$$

Intended Learning Outcomes

After this lecture, you will be able to:

- explain why continuous-time dynamic models need to be discretized in practice
- construct discrete-time dynamic models from linear ODE and SDE state-space models

Discretization of Continuous-Time Models: Why?

- Sensor fusion is implemented in digital computers
- Data is only processed at $t_1, t_2, \ldots, t_n$
- Discretized continuous-time models are closely related to discrete-time models
- Example: Vehicle tracking

Discretization of continuous-time models is equivalent to solving the ODE/SDE model between t_{n-1} and t_n

Solving Linear First Order ODEs (1/2)

- Goal: Solve the first order ODE

$$\dot{x}(t) = ax(t) + bu(t),$$

  on the interval $(t_{n-1}, t_n]$.

- Ansatz: Multiply by the integrating factor e^{-at}

$$e^{-at}\dot{x}(t) = e^{-at}ax(t) + e^{-at}bu(t)$$

Solving Linear First Order ODEs (2/2)

- Separable (in t) ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-at}x(t) = e^{-at}bu(t)$$

- Solution:

$$e^{-at_n}x(t_n) - e^{-at_{n-1}}x(t_{n-1}) = \int_{t_{n-1}}^{t_n} e^{-at}bu(t)\mathrm{d}t$$

- Rearranged:

$$x(t_n) = e^{a\Delta t}x(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{a(t_n-t)}bu(t)\mathrm{d}t$$

Vector-Valued Linear First Order ODE

- General linear dynamic model:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u\boldsymbol{u}(t)$$

- This is a vector-valued first order ODE

What is the integrating factor for vector-valued first order ODEs?

Matrix Exponential

- Definition of the exponential:

$$e^{a} = \sum_{k=0}^{\infty} \frac{1}{k} a^{k}$$

- Definition of the matrix exponential:

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{A}^{k}$$

- Derivative of matrix exponential w.r.t. scalar x:

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{\boldsymbol{A}x} = e^{\boldsymbol{A}x} \boldsymbol{A}$$

- Matrix exponential of $\boldsymbol{A}^{\mathsf{T}}$:

$$(e^{\boldsymbol{A}})^{\mathsf{T}} = e^{\boldsymbol{A}^{\mathsf{T}}}$$

Solving Linear First Order Vector ODEs (1/2)

- General linear dynamic model:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u\boldsymbol{u}(t)$$

- Multiplication by the integrating factor $e^{-\boldsymbol{A}t}$:

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) = e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) + e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)$$

- Rearranging:

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}_u\boldsymbol{u}(t)$$

- Substituting $\frac{\mathrm{d}}{\mathrm{d}t}e^{-\boldsymbol{A}t}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}}\boldsymbol{A}\boldsymbol{x}(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{-\boldsymbol{A}t}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}_u\boldsymbol{u}(t)$$

Solving Linear First Order Vector ODEs (2/2)

- Separable (in t) ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{-\boldsymbol{A}t}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}_u\boldsymbol{u}(t)$$

- Integration w.r.t. t:

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} \mathrm{d}\left[e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right] &= \int_{t_{n-1}}^{t_n} e^{-\boldsymbol{A}t}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t \\
\left[e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right]_{t=t_{n-1}}^{t_n} &= \int_{t_{n-1}}^{t_n} e^{-\boldsymbol{A}t}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t \\
e^{-\boldsymbol{A}t_n}\boldsymbol{x}(t_n) - e^{-\boldsymbol{A}t_{n-1}}\boldsymbol{x}(t_{n-1}) &= \int_{t_{n-1}}^{t_n} e^{-\boldsymbol{A}t}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t
\end{aligned}$$

- Rearranging:

$$\boldsymbol{x}(t_n) = e^{\boldsymbol{A}(t_n - t_{n-1})}\boldsymbol{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t,$$

Zero-Order-Hold Inputs

- Solution:

$$\boldsymbol{x}(t_n) = e^{\boldsymbol{A}(t_n - t_{n-1})}\boldsymbol{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t,$$

- The input $\boldsymbol{u}(t)$ is often constant between sampling instants (zero-order-hold; ZOH)
- Then:

$$\int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t = \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u\mathrm{d}t\boldsymbol{u}_{n-1}$$

Discretized Deterministic Linear Dynamic Model

- Linear continuous-time dynamic model:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u \boldsymbol{u}(t)$$

- Discretized dynamic model:

$$\boldsymbol{x}_n = \boldsymbol{F}_n \boldsymbol{x}_{n-1} + \boldsymbol{L}_n \boldsymbol{u}_{n-1}.$$

where

$$\boldsymbol{F}_n \triangleq e^{\boldsymbol{A}(t_n - t_{n-1})}$$
$$\boldsymbol{L}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)} \boldsymbol{B}_u \mathrm{d}t$$

The discretized dynamic model is completely equivalent to the continuous-time model

Example: Deterministic 1D Motion Model (1/4)

- Dynamic model:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

- Recall:

$$\boldsymbol{F}_n = e^{\boldsymbol{A}\Delta t} = \sum_{j=0}^{\infty} \frac{1}{j!} \boldsymbol{A}^j (\Delta t)^j$$

Example: Deterministic 1D Motion Model (2/4)

- Powers of $\boldsymbol{A}$:

$$
\begin{aligned}
\boldsymbol{A}^0 &= \boldsymbol{I} \\
\boldsymbol{A}^1 &= \boldsymbol{A} \\
\boldsymbol{A}^j &= \boldsymbol{0}\ j \geq 2
\end{aligned}
$$

- Hence:

$$
\begin{aligned}
\boldsymbol{F}_n &= \sum_{j=0}^{\infty} \frac{1}{j!} \boldsymbol{A}^j (\Delta t)^j = \frac{1}{0!} \boldsymbol{I} (\Delta t)^0 + \frac{1}{1!} \boldsymbol{A} \Delta t \\
&= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Example: Deterministic 1D Motion Model (3/4)

- Input matrix:

$$\boldsymbol{L}_n = \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)} \boldsymbol{B}_u \mathrm{d}t$$

where:

$$e^{\boldsymbol{A}(t_n - t)} = \begin{bmatrix} 1 & t_n - t \\ 0 & 1 \end{bmatrix}, \; \boldsymbol{B}_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example: Deterministic 1D Motion Model (4/4)

- Continuous-time model:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

- Discretized model:

$$\boldsymbol{x}_n = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \boldsymbol{x}_{n-1} + \begin{bmatrix} \frac{(\Delta t)^2}{2} \\ \Delta t \end{bmatrix} \boldsymbol{u}_{n-1}$$

Stochastic Linear Dynamic Model

- Stochastic linear dynamic model:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_w\boldsymbol{w}(t),$$

- The only difference to the deterministic model is the input $\boldsymbol{u}(t)$ ($\boldsymbol{w}(t)$)
- $\boldsymbol{w}(t)$ is a white stochastic process
- Auto-correlation function:

$$R_{ww}(\tau) = \mathrm{E}\{\boldsymbol{w}(t+\tau)\boldsymbol{w}(t)\} = \boldsymbol{\Sigma}_w\delta(\tau)$$

- Hence:

$$\boldsymbol{x}(t_n) = e^{\boldsymbol{A}(t_n - t_{n-1})}\boldsymbol{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_w\boldsymbol{w}(t)\mathrm{d}t$$

Integration of the Stochastic Process

- Model:

$$\boldsymbol{x}(t_n) = e^{\boldsymbol{A}(t_n - t_{n-1})}\boldsymbol{x}(t_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_w \boldsymbol{w}(t)\mathrm{d}t$$

- $\boldsymbol{w}(t)$ is stochastic; not ZOH and not even integrable (with standard tools)
- Define a random variable as the process noise:

$$\boldsymbol{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_w \boldsymbol{w}(t)\mathrm{d}t$$

- Then:

$$\boldsymbol{x}_n = \boldsymbol{F}_n \boldsymbol{x}_{n-1} + \boldsymbol{q}_n$$

Mean of the Process Noise

- Process noise:

$$
\boldsymbol{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)} \boldsymbol{B}_w \boldsymbol{w}(t) \mathrm{d}t.
$$

Covariance of the Process Noise (1/2)

- Process noise:

$$\boldsymbol{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)} \boldsymbol{B}_w \boldsymbol{w}(t) \mathrm{d}t$$

- Covariance:

$$\mathrm{Cov}\{\boldsymbol{q}_n\} = \mathrm{E}\{(\boldsymbol{q}_n - \mathrm{E}\{\boldsymbol{q}_n\})(\boldsymbol{q}_n - \mathrm{E}\{\boldsymbol{q}_n\})^\mathsf{T}\}$$

Covariance of the Process Noise (2/2)

- Process noise:

$$\boldsymbol{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)} \boldsymbol{B}_w \boldsymbol{w}(t) \mathrm{d}t$$

- Covariance:

$$\mathrm{Cov}\{\boldsymbol{q}_n\} = \iint_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)} \boldsymbol{B}_w \, \mathrm{E}\left\{\boldsymbol{w}(t)\boldsymbol{w}(\tau)^\mathsf{T}\right\} \boldsymbol{B}_w^\mathsf{T} (e^{\boldsymbol{A}(t_n - \tau)})^\mathsf{T} \mathrm{d}\tau \mathrm{d}t$$

Properties of the Process Noise

- Process noise:

$$\boldsymbol{q}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)} \boldsymbol{B}_w \boldsymbol{w}(t) \mathrm{d}t$$

- Mean and covariance:

$$\begin{aligned}
\mathrm{E}\{\boldsymbol{q}_n\} &= 0 \\
\mathrm{Cov}\{\boldsymbol{q}_n\} &= \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - \tau)} \boldsymbol{B}_w \boldsymbol{\Sigma}_w \boldsymbol{B}_w^{\mathsf{T}} e^{\boldsymbol{A}^{\mathsf{T}}(t_n - \tau)} \mathrm{d}\tau \triangleq \boldsymbol{Q}_n
\end{aligned}$$

- Distribution:

$$\boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{Q}_n)$$

Discretized Stochastic Linear Dynamic Model

- Discretized stochastic dynamic model:

$$\boldsymbol{x}_n = \boldsymbol{F}_n \boldsymbol{x}_{n-1} + \boldsymbol{q}_n$$

where:

$$\begin{aligned}
\boldsymbol{F}_n &\triangleq e^{\boldsymbol{A}(t_n - t_{n-1})} \\
\boldsymbol{q}_n &\sim \mathcal{N}(0, \boldsymbol{Q}_n) \\
\boldsymbol{Q}_n &= \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - \tau)} \boldsymbol{B}_w \boldsymbol{\Sigma}_w \boldsymbol{B}_w^{\mathsf{T}} e^{\boldsymbol{A}^{\mathsf{T}}(t_n - \tau)} \mathrm{d}\tau
\end{aligned}$$

Example: 1D Wiener Velocity Model (1/3)

- Dynamic model:

$$\begin{bmatrix}\dot{p}(t) \\ \dot{v}(t)\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix} \begin{bmatrix}p(t) \\ v(t)\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix} w(t)$$

  with white noise process $w(t)$ and $R_{ww}(\tau) = \sigma_w^2 \delta(\tau)$
- Process noise covariance:

$$\boldsymbol{Q}_n = \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n-\tau)} \boldsymbol{B}_w \boldsymbol{\Sigma}_w \boldsymbol{B}_w^{\mathsf{T}} e^{\boldsymbol{A}^{\mathsf{T}}(t_n-\tau)} \mathrm{d}\tau$$

- Recall:

$$e^{\boldsymbol{A}(t_n-\tau)} = \begin{bmatrix}1 & t_n - \tau \\ 0 & 1\end{bmatrix}$$

$$e^{\boldsymbol{A}(t_n-\tau)} \boldsymbol{B}_w = \begin{bmatrix}1 & t_n - \tau \\ 0 & 1\end{bmatrix} \begin{bmatrix}0 \\ 1\end{bmatrix} = \begin{bmatrix}t_n - \tau \\ 1\end{bmatrix}$$

Example: 1D Wiener Velocity Model (2/3)

- Process noise covariance:

$$Q_n = \int_{t_{n-1}}^{t_n} e^{A(t_n-\tau)} B_w \Sigma_w B_w^{\mathsf{T}} e^{A^{\mathsf{T}}(t_n-\tau)} \mathrm{d}\tau$$

Example: 1D Wiener Velocity Model (3/3)

- Dynamic model:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

- Discretized model:

$$\begin{bmatrix} p_n \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{n-1} \\ v_{n-1} \end{bmatrix} + \boldsymbol{q}_n$$

with $\boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{Q}_n)$ and

$$\boldsymbol{Q}_n = \sigma_w^2 \begin{bmatrix} \frac{(\Delta t)^3}{3} & \frac{(\Delta t)^2}{2} \\ \frac{(\Delta t)^2}{2} & \Delta t \end{bmatrix}$$

Summary (1/2)

- The discretization of the linear ODE model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u\boldsymbol{u}(t)$$

  is

$$\boldsymbol{x}_n = \boldsymbol{F}_n\boldsymbol{x}_{n-1} + \boldsymbol{L}_n\boldsymbol{u}_{n-1}$$

$$\boldsymbol{F}_n \triangleq e^{\boldsymbol{A}(t_n - t_{n-1})},\ \boldsymbol{L}_n \triangleq \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u \mathrm{d}t$$

- The discretization of the linear SDE model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_w\boldsymbol{w}(t)$$

  is

$$\boldsymbol{x}_n = \boldsymbol{F}_n\boldsymbol{x}_{n-1} + \boldsymbol{q}_n,\ \boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{Q}_n)$$

$$\boldsymbol{Q}_n = \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - \tau)}\boldsymbol{B}_w\boldsymbol{\Sigma}_w\boldsymbol{B}_w^{\mathsf{T}}e^{\boldsymbol{A}^{\mathsf{T}}(t_n - \tau)}\mathrm{d}\tau$$

Summary (2/2)

- Discretized 1D-Wiener velocity model:

$$\begin{bmatrix} p_n \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{n-1} \\ v_{n-1} \end{bmatrix} + \boldsymbol{q}_n$$

with $\boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{Q})$ and

$$\boldsymbol{Q}_n = \begin{bmatrix} \frac{(\Delta t)^3}{3} & \frac{(\Delta t)^2}{2} \\ \frac{(\Delta t)^2}{2} & \Delta t \end{bmatrix}$$

Announcements

- Mid-term feedback open until November 2, 2018 (MyCourses $\Rightarrow$ Feedback)
- Intermediate report can now be submitted (MyCourses $\Rightarrow$ Project)
  - Deadline: November 2, 2018, 23:55
  - Review period: November 5, 2018 – November 9, 2018
  - Please report any issues with submission/peer review