Discretization of Nonlinear Continuous-Time State-Space Models

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November 7, 2018

Recap: Discretization of Linear Dynamic Models

- Data is processed in a computer at $t_1, t_2, \ldots, t_n$
- Discretization on $(t_{n-1}, t_n]$ corresponds to solving the integral on that interval
- The discretization of the linear SDE model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_u\boldsymbol{u}(t) + \boldsymbol{B}_w\boldsymbol{w}(t)$$

  is

$$\begin{aligned}
\boldsymbol{x}_n &= e^{\boldsymbol{A}(t_n - t_{n-1})}\boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - t)}\boldsymbol{B}_u \mathrm{d}t\,\boldsymbol{u}_{n-1} + \boldsymbol{q}_n \\
\boldsymbol{q}_n &\sim \mathcal{N}(0, \boldsymbol{Q}_n) \\
\boldsymbol{Q}_n &= \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}(t_n - \tau)}\boldsymbol{B}_w\boldsymbol{\Sigma}_w\boldsymbol{B}_w^{\mathsf{T}} e^{\boldsymbol{A}^{\mathsf{T}}(t_n - \tau)}\mathrm{d}\tau
\end{aligned}$$

- The discrete-time dynamic models are equivalent to the continuous-time models

Intended Learning Outcomes

After this lecture, you will be able to:

- discuss the differences between discretization of linear and nonlinear dynamic models
- explain two methods for discretizing nonlinear dynamic models: Discretization of linearized models and Euler–Maruyama discretization

Discretization of Nonlinear Dynamic Models

- Objective: Discretization of nonlinear models

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_u(\boldsymbol{x}(t))\boldsymbol{u}(t)$$

  and

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)$$

- Problem: In most cases, no exact approach exists
- A few possible approaches:
  - Linearization of the nonlinear model followed by discretization
  - Approximation of the derivative (integral)
  - Exact integration (of at least the dynamics)
  - & many more…

Linearization of Nonlinear Models

- Nonlinear dynamic model:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_u(\boldsymbol{x}(t))\boldsymbol{u}(t)$$

- 1st order Taylor series approximation of $f(\boldsymbol{x}(t))$ around $\boldsymbol{x}(t) = \boldsymbol{x}(t_{n-1})$:

$$f(\boldsymbol{x}(t)) \approx f(\boldsymbol{x}_{n-1}) + \boldsymbol{A}_x(\boldsymbol{x}(t) - \boldsymbol{x}_{n-1})$$

- Approximation of the ODE:

$$\dot{\boldsymbol{x}}(t) \approx f(\boldsymbol{x}_{n-1}) + \boldsymbol{A}_x(\boldsymbol{x}(t) - \boldsymbol{x}_{n-1}) + \boldsymbol{B}_u\boldsymbol{u}(t)$$

Discretization of Linearized Models (1/2)

- Approximation of the ODE:

$$\dot{\boldsymbol{x}}(t) \approx f(\boldsymbol{x}_{n-1}) + \boldsymbol{A}_x(\boldsymbol{x}(t) - \boldsymbol{x}_{n-1}) + \boldsymbol{B}_u\boldsymbol{u}(t)$$

- Rewritten approximation of the ODE

$$\dot{\boldsymbol{x}}(t) \approx \boldsymbol{A}_x\boldsymbol{x}(t) + f(\boldsymbol{x}_{n-1}) - \boldsymbol{A}_x\boldsymbol{x}_{n-1} + \boldsymbol{B}_u\boldsymbol{u}(t)$$

- Solution of the approximation:

$$\begin{aligned}\boldsymbol{x}_n \approx e^{\boldsymbol{A}_x\Delta t}\boldsymbol{x}_{n-1} &+ \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n-t)}\mathrm{d}t f(\boldsymbol{x}_{n-1}) \\ &- \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n-t)}\mathrm{d}t\boldsymbol{A}_x\boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n-t)}\boldsymbol{B}_u\boldsymbol{u}(t)\mathrm{d}t\end{aligned}$$

Discretization of Linearized Models (2/2)

- Solution of the approximation:

$$
\begin{aligned}
\boldsymbol{x}_n \approx e^{\boldsymbol{A}_x \Delta t} \boldsymbol{x}_{n-1} &+ \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x (t_n - t)} \mathrm{d}t f(\boldsymbol{x}_{n-1}) \\
&- \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x (t_n - t)} \mathrm{d}t \boldsymbol{A}_x \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x (t_n - t)} \boldsymbol{B}_u \boldsymbol{u}(t) \mathrm{d}t
\end{aligned}
$$

- Simplified solution:

$$
\boldsymbol{x}_n \approx \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x (t_n - t)} \mathrm{d}t f(\boldsymbol{x}_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x (t_n - t)} \boldsymbol{B}_u \boldsymbol{u}(t) \mathrm{d}t
$$

Discretization of Linearized Models (Stochastic)

- Stochastic nonlinear model:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)$$

- Discretization is the same as for the ODE model:

$$\begin{aligned}\boldsymbol{x}_n &\approx \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\mathrm{d}t f(\boldsymbol{x}_{n-1}) + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\boldsymbol{B}_w\boldsymbol{w}(t)\mathrm{d}t \\ &= \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\mathrm{d}t f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n\end{aligned}$$

with

$$\begin{aligned}\boldsymbol{q}_n &\sim \mathcal{N}(0, \boldsymbol{Q}_n), \\ \boldsymbol{Q}_n &\approx \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - \tau)}\boldsymbol{B}_w\boldsymbol{\Sigma}_w\boldsymbol{B}_w^{\mathsf{T}}e^{\boldsymbol{A}_x^{\mathsf{T}}(t_n - \tau)}\mathrm{d}\tau\end{aligned}$$

Properties of the Discretization

- Stochastic nonlinear model:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)$$

- Linearized model:

$$\dot{\boldsymbol{x}}(t) \approx f(\boldsymbol{x}_{n-1}) + \boldsymbol{A}_x(x(t) - x_{n-1}) + \boldsymbol{B}_w\boldsymbol{w}(t)$$

- Discretized model:

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)} \mathrm{d}t f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n$$

- Integration is exact, model is not
- Discretization is not exact
- Linearization is local, may cause problems

Example: Quasi-Constant Turn Model (1/5)

- Model:

$$\begin{bmatrix} \dot{p}^x(t) \\ \dot{p}^y(t) \\ \dot{v}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} v(t)\cos(\varphi(t)) \\ v(t)\sin(\varphi(t)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{w}(t)$$

- Jacobian of $f(\boldsymbol{x}(t))$:

$$\begin{aligned} \boldsymbol{A}_x &= \begin{bmatrix} 0 & 0 & \cos(\varphi(t)) & -v(t)\sin(\varphi(t)) \\ 0 & 0 & \sin(\varphi(t)) & v(t)\cos(\varphi(t)) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \cos(\varphi_{n-1}) & -v_{n-1}\sin(\varphi_{n-1}) \\ 0 & 0 & \sin(\varphi_{n-1}) & v_{n-1}\cos(\varphi_{n-1}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Example: Quasi-Constant Turn Model (2/5)

- Powers of $\boldsymbol{A}_x$:

$$\begin{aligned}
\boldsymbol{A}_x^0 &= \boldsymbol{I} \\
\boldsymbol{A}_x^1 &= \boldsymbol{A}_x \\
\boldsymbol{A}_x^2 &= \boldsymbol{0}
\end{aligned}$$

- Matrix exponential:

$$\begin{aligned}
e^{\boldsymbol{A}_x(t_n - t)} &= \boldsymbol{I} + \boldsymbol{A}_x(t_n - t) \\
&= \begin{bmatrix} 1 & 0 & \cos(\varphi_{n-1})(t_n - t) & -v_{n-1}\sin(\varphi_{n-1})(t_n - t) \\ 0 & 1 & \sin(\varphi_{n-1})(t_n - t) & v_{n-1}\cos(\varphi_{n-1})(t_n - t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Example: Quasi-Constant Turn Model (3/5)

- Integral:

$$
\int_{t_{n-1}}^{t_n} \begin{bmatrix} 1 & 0 & \cos(\varphi_{n-1})(t_n - t) & -v_{n-1}\sin(\varphi_{n-1})(t_n - t) \\ 0 & 1 & \sin(\varphi_{n-1})(t_n - t) & v_{n-1}\cos(\varphi_{n-1})(t_n - t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{d}t
$$

$$
= \begin{bmatrix} t & 0 & -\frac{(t_n-t)^2}{2}\cos(\varphi_{n-1}) & \frac{(t_n-t)^2}{2}v_{n-1}\sin(\varphi_{n-1}) \\ 0 & t & -\frac{(t_n-t)^2}{2}\sin(\varphi_{n-1}) & -\frac{(t_n-t)^2}{2}v_{n-1}\cos(\varphi_{n-1}) \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{bmatrix}_{t=t_{n-1}}^{t_n}
$$

$$
= \begin{bmatrix} \Delta t & 0 & \frac{(\Delta t)^2}{2}\cos(\varphi_{n-1}) & -\frac{(\Delta t)^2}{2}v_{n-1}\sin(\varphi_{n-1}) \\ 0 & \Delta t & \frac{(\Delta t)^2}{2}\sin(\varphi_{n-1}) & \frac{(\Delta t)^2}{2}v_{n-1}\cos(\varphi_{n-1}) \\ 0 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & \Delta t \end{bmatrix}
$$

Example: Quasi-Constant Turn Model (4/5)

- Discretized model:

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)} \mathrm{d}t f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n$$

- Second term:

$$\begin{bmatrix} \Delta t & 0 & \frac{(\Delta t)^2}{2}\cos(\varphi_{n-1}) & -\frac{(\Delta t)^2}{2} v_{n-1}\sin(\varphi_{n-1}) \\ 0 & \Delta t & \frac{(\Delta t)^2}{2}\sin(\varphi_{n-1}) & \frac{(\Delta t)^2}{2} v_{n-1}\cos(\varphi_{n-1}) \\ 0 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & \Delta t \end{bmatrix} \begin{bmatrix} v_{n-1}\cos(\varphi_{n-1}) \\ v_{n-1}\sin(\varphi_{n-1}) \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \Delta t v_{n-1}\cos(\varphi_{n-1}) \\ \Delta t v_{n-1}\sin(\varphi_{n-1}) \\ 0 \\ 0 \end{bmatrix}$$

Example: Quasi-Constant Turn Model (5/5)

- Discretized model:

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)} \mathrm{d}t \, f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n$$

- Discretization of Linearized Model:

$$\begin{bmatrix} p_n^x \\ p_n^y \\ v_n \\ \varphi_n \end{bmatrix} = \begin{bmatrix} p_{n-1}^x \\ p_{n-1}^y \\ v_{n-1} \\ \varphi_{n-1} \end{bmatrix} + \begin{bmatrix} \Delta t v_{n-1} \cos(\varphi_{n-1}) \\ \Delta t v_{n-1} \sin(\varphi_{n-1}) \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{q}_n$$

- What about $\boldsymbol{Q}_n$?

$$\boldsymbol{Q}_n \approx \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - \tau)} \boldsymbol{B}_w \boldsymbol{\Sigma}_w \boldsymbol{B}_w^\mathsf{T} e^{\boldsymbol{A}_x^\mathsf{T}(t_n - \tau)} \mathrm{d}\tau$$

Rectangle Integral Approximation

Idea: Approximate the integral rather than the model

Euler Approximation

- Dynamic model:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_u(\boldsymbol{x}(t))\boldsymbol{u}(t)$$

- Integral equation:

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} f(\boldsymbol{x}(t))\mathrm{d}t + \int_{t_{n-1}}^{t_n} \boldsymbol{B}_u(\boldsymbol{x}(t))\boldsymbol{u}(t)\mathrm{d}t$$

- Euler approximation:

$$\boldsymbol{x}_n \approx \boldsymbol{x}_{n-1} + \Delta t f(\boldsymbol{x}_{n-1}) + \Delta t \boldsymbol{B}_u(\boldsymbol{x}_{n-1})\boldsymbol{u}_{n-1}.$$

Task: Euler–Maruyama Discretization

Task

Using the right rectangle approximation, find the Euler–Maruyama discretization of the model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)$$

Hint

- Define the integral over the stochastic process $\boldsymbol{w}(t)$ as a random variable $\boldsymbol{q}_n$ and find its properties

Vote for one of the following solutions at `https://presemo.aalto.fi/sensorfusion`:

A $\boldsymbol{x}_n \approx \boldsymbol{x}_{n-1} + \Delta t \boldsymbol{f}(\boldsymbol{x}_{n-1}) + \Delta t \boldsymbol{B}_w(\boldsymbol{x}_{n-1})\boldsymbol{q}_n, \ \boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{\Sigma}_w)$

B $\boldsymbol{x}_n \approx \boldsymbol{x}_{n-1} + \Delta t \boldsymbol{f}(\boldsymbol{x}_{n-1}) + \frac{\Delta t}{2} \boldsymbol{B}_w(\boldsymbol{x}_{n-1})\boldsymbol{q}_n, \ \boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{\Sigma}_w)$

C $\boldsymbol{x}_n \approx \boldsymbol{x}_{n-1} + \Delta t \boldsymbol{f}(\boldsymbol{x}_{n-1}) + \sqrt{\Delta t} \boldsymbol{B}_w(\boldsymbol{x}_{n-1})\boldsymbol{q}_n, \ \boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{\Sigma}_w)$

Euler–Maruyama Discretization (1)

- Stochastic dynamic model:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)$$

- Integral representation:

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} f(\boldsymbol{x}(t))\mathrm{d}t + \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)\mathrm{d}t$$

- Process noise definition:

$$\boldsymbol{q}_n \triangleq \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)\mathrm{d}t$$

Mean of the Process Noise

- Process noise:

$$\boldsymbol{q}_n \triangleq \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)\mathrm{d}t$$

- Mean:

$$\begin{aligned}
\mathrm{E}\{\boldsymbol{q}_n\} &= \mathrm{E}\left\{\int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)\mathrm{d}t\right\} \\
&= \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t))\,\mathrm{E}\left\{\boldsymbol{w}(t)\right\}\mathrm{d}t \\
&= 0
\end{aligned}$$

Covariance of the Process Noise (1/2)

- Process noise:

$$\boldsymbol{q}_n \triangleq \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)\mathrm{d}t$$

- Covariance:

$$\begin{aligned}
\mathrm{Cov}\{\boldsymbol{q}_n\} &= \mathrm{E}\left\{\left(\int_{t_{n-1}}^{t_n} \boldsymbol{B}_w\boldsymbol{w}(t)\mathrm{d}t\right)\left(\int_{t_{n-1}}^{t_n} \boldsymbol{B}_w\boldsymbol{w}(\tau)\mathrm{d}\tau\right)^{\mathsf{T}}\right\} \\
&= \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t))\,\mathrm{E}\{\boldsymbol{w}(t)\boldsymbol{w}(\tau)^{\mathsf{T}}\}\boldsymbol{B}_w(\boldsymbol{x}(t))^{\mathsf{T}}\mathrm{d}\tau\mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{\Sigma}_w\delta(t-\tau)\boldsymbol{B}_w(\boldsymbol{x}(t)))^{\mathsf{T}}\mathrm{d}\tau\mathrm{d}t \\
&= \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t)))\boldsymbol{\Sigma}_w\boldsymbol{B}_w^{\mathsf{T}}(\boldsymbol{x}(t))\mathrm{d}\tau
\end{aligned}$$

Covariance of the Process Noise (2/2)

- Covariance:

$$\mathrm{Cov}\{\boldsymbol{q}_n\} = \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{\Sigma}_w\boldsymbol{B}_w(\boldsymbol{x}(t))^{\mathsf{T}}\mathrm{d}\tau$$

- Rectangle approximation of the integral:

$$\begin{aligned}
\mathrm{Cov}\{\boldsymbol{q}_n\} &= \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t)))\boldsymbol{\Sigma}_w\boldsymbol{B}_w^{\mathsf{T}}(\boldsymbol{x}(t))\mathrm{d}\tau \\
&\approx \boldsymbol{B}_w(\boldsymbol{x}_{n-1})\boldsymbol{\Sigma}_w\boldsymbol{B}_w(\boldsymbol{x}_{n-1})^{\mathsf{T}}(t_n - t_{n-1}) \\
&= \Delta t\boldsymbol{B}_w(\boldsymbol{x}_{n-1})\boldsymbol{\Sigma}_w\boldsymbol{B}_w(\boldsymbol{x}_{n-1})^{\mathsf{T}} \\
&\triangleq \boldsymbol{Q}_n
\end{aligned}$$

Euler–Maruyama Discretization (2)

- Dynamic model:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)$$

- Euler–Maruyama discretization:

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \Delta t f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n$$

  with $\boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{Q}_n),\ \boldsymbol{Q}_n \approx \Delta t \boldsymbol{B}_w(\boldsymbol{x}_{n-1})\boldsymbol{\Sigma}_w \boldsymbol{B}_w(\boldsymbol{x}_{n-1})^\mathsf{T}$

- …or equivalently:

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \Delta t f(\boldsymbol{x}_{n-1}) + \sqrt{\Delta t}\boldsymbol{B}_w(\boldsymbol{x}_{n-1})\boldsymbol{q}_n$$

  with $\boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{\Sigma}_w)$

- Discretization is not exact

Summary (1/2)

- Nonlinear dynamic model:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)$$

- General solution:

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} f(\boldsymbol{x}_{n-1})\mathrm{d}t + \int_{t_{n-1}}^{t_n} \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t)\mathrm{d}t$$

- Discretization of the linearized model:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= f(\boldsymbol{x}(t)) + \boldsymbol{B}_w\boldsymbol{w}(t) \\ &\approx f(\boldsymbol{x}_{n-1}) + \boldsymbol{A}_x(\boldsymbol{x}(t) - \boldsymbol{x}_{n-1}) + \boldsymbol{B}_w\boldsymbol{w}(t)\end{aligned}$$

$$\Downarrow$$

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n - t)}\mathrm{d}t f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n$$

with

$$\boldsymbol{q}_n \sim \mathcal{N}(0, \boldsymbol{Q}_n),\ \boldsymbol{Q}_n \approx \int_{t_{n-1}}^{t_n} e^{\boldsymbol{A}_x(t_n-\tau)}\boldsymbol{B}_w\boldsymbol{\Sigma}_w\boldsymbol{B}_w^{\mathsf{T}}e^{\boldsymbol{A}_x^{\mathsf{T}}(t_n-\tau)}\mathrm{d}\tau$$

Summary (2/2)

- Euler–Maruyama discretization:

$$
\begin{gathered}
\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) + \boldsymbol{B}_w(\boldsymbol{x}(t))\boldsymbol{w}(t) \\
\Downarrow \\
\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \Delta t f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n
\end{gathered}
$$

  with

$$
\begin{aligned}
\boldsymbol{q}_n &\sim \mathcal{N}(0, \boldsymbol{Q}_n), \\
\boldsymbol{Q}_n &\approx \Delta t \boldsymbol{B}_w(\boldsymbol{x}_{n-1})\boldsymbol{\Sigma}_w \boldsymbol{B}_w(\boldsymbol{x}_{n-1})^\mathsf{T}.
\end{aligned}
$$

Announcements

- Peer-review of intermediate report due Friday, November 5, 2018
- Mid-term survey results are in