Extended and Unscented Kalman Filtering

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Recap: Filtering and the Kalman Filter

- The filtering approach iterates between two steps:
  1. Prediction: $\hat{\boldsymbol{x}}_{n-1|n-1}, \boldsymbol{P}_{n-1|n-1} \Rightarrow \hat{x}_{n|n-1}, \boldsymbol{P}_{n|n-1}$
  2. Measurement update: $\hat{x}_{n|n-1}, \boldsymbol{P}_{n|n-1} \Rightarrow \hat{x}_{n|n}, \boldsymbol{P}_{n|n}$
- The Kalman filter is the optimal filter for linear state-space models

$$
\begin{aligned}
\boldsymbol{x}_n &= \boldsymbol{F}_n \boldsymbol{x}_{n-1} + \boldsymbol{q}_n \\
\boldsymbol{y}_n &= \boldsymbol{G}_n \boldsymbol{x}_n + \boldsymbol{r}_n
\end{aligned}
$$

1. Prediction:

$$
\begin{aligned}
\hat{\boldsymbol{x}}_{n|n-1} &= \boldsymbol{F}_n \hat{\boldsymbol{x}}_{n-1|n-1} \\
\boldsymbol{P}_{n|n-1} &= \boldsymbol{F}_n \boldsymbol{P}_{n-1|n-1} \boldsymbol{F}_n^{\mathsf{T}} + \boldsymbol{Q}_n
\end{aligned}
$$

2. Measurement update:

$$
\begin{aligned}
\boldsymbol{K}_n &= \boldsymbol{P}_{n|n-1} \boldsymbol{G}_n^{\mathsf{T}} (\boldsymbol{G}_n \boldsymbol{P}_{n|n-1} \boldsymbol{G}_n + \boldsymbol{R}_n)^{-1} \\
\hat{\boldsymbol{x}}_{n|n} &= \hat{\boldsymbol{x}}_{n|n-1} + \boldsymbol{K}_n (\boldsymbol{y}_n - \boldsymbol{G}_n \hat{\boldsymbol{x}}_{n|n-1}) \\
\boldsymbol{P}_{n|n} &= \boldsymbol{P}_{n|n-1} - \boldsymbol{K}_n (\boldsymbol{G}_n \boldsymbol{P}_{n|n-1} \boldsymbol{G}_n + \boldsymbol{R}_n) \boldsymbol{K}_n^{\mathsf{T}}
\end{aligned}
$$

Intended Learning Outcomes

After this lecture, you will be able to:

- recognize the challenges for filtering in nonlinear state-space models,
- describe and employ the extended and unscented Kalman filters for nonlinear state-space models,

Discrete-Time Nonlinear State-Space Model

- Discrete-time nonlinear state-space model:

$$\begin{aligned} \boldsymbol{x}_n &= f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n \\ \boldsymbol{y}_n &= g(\boldsymbol{x}_n) + \boldsymbol{r}_n \end{aligned}$$

- Process and measurement noises ($\boldsymbol{q}_n$ and $\boldsymbol{r}_n$):

$$\begin{aligned} \mathrm{E}\{\boldsymbol{q}_n\} = 0,\ \mathrm{Cov}\{\boldsymbol{q}_n\} = \boldsymbol{Q}_n \\ \mathrm{E}\{\boldsymbol{r}_n\} = 0,\ \mathrm{Cov}\{\boldsymbol{r}_n\} = \boldsymbol{R}_n \end{aligned}$$

- Initial conditions:

$$\mathrm{E}\{\boldsymbol{x}_0\} = \boldsymbol{m}_0,\ \mathrm{Cov}\{\boldsymbol{x}_0\} = \boldsymbol{P}_0$$

Filtering for Nonlinear Models

- For most nonlinear models, exact prediction and/or update steps can not be found
- Example: Prediction for general nonlinear model

$$\hat{\boldsymbol{x}}_{n|n-1} = \mathrm{E}\{\boldsymbol{x}_n \mid \boldsymbol{y}_{1:n-1}\}$$

Approximations to the exact solutions are required!

Linearized Model: Prediction (1/2)

- State estimate from t_{n-1}: $\hat{\boldsymbol{x}}_{n-1|n-1}$, $\boldsymbol{P}_{n-1|n-1}$
- Linearization around $\hat{\boldsymbol{x}}_{n-1|n-1}$ (dynamic model):

$$\begin{aligned}\boldsymbol{x}_n &= f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n \\ &\approx f(\hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{F}_x(\boldsymbol{x}_{n-1} - \hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{q}_n\end{aligned}$$

- Predicted mean:

$$\begin{aligned}\hat{\boldsymbol{x}}_{n|n-1} &= \mathrm{E}\{\boldsymbol{x}_n \mid \boldsymbol{y}_{1:n-1}\} \\ &\approx \mathrm{E}\{f(\hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{F}_x(\boldsymbol{x}_{n-1} - \hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{q}_n \mid \boldsymbol{y}_{1:n-1}\} \\ &= f(\hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{F}_x \,\mathrm{E}\{\boldsymbol{x}_{n-1} \mid \boldsymbol{y}_{1:n-1}\} - \boldsymbol{F}_x\hat{\boldsymbol{x}}_{n-1|n-1} \\ &= f(\hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{F}_x\hat{\boldsymbol{x}}_{n-1|n-1} - \boldsymbol{F}_x\hat{\boldsymbol{x}}_{n-1|n-1} \\ &= f(\hat{\boldsymbol{x}}_{n-1|n-1})\end{aligned}$$

Linearized Model: Prediction (2/2)

- State estimate from t_{n-1}: $\hat{\boldsymbol{x}}_{n-1|n-1}$, $\boldsymbol{P}_{n-1|n-1}$
- Linearization around $\hat{\boldsymbol{x}}_{n-1|n-1}$ (dynamic model):

$$\boldsymbol{x}_n \approx f(\hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{F}_x(\boldsymbol{x}_{n-1} - \hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{q}_n,$$

- Covariance:

$$\begin{aligned}
\boldsymbol{P}_{n|n-1} &= \mathrm{E}\{(\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1})(\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1})^{\mathsf{T}} \mid \boldsymbol{y}_{1:n-1}\} \\
&\approx \mathrm{E}\{[f(\hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{F}_x(\boldsymbol{x}_{n-1} - \hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{q}_n - f(\hat{\boldsymbol{x}}_{n-1|n-1})] \\
&\qquad \times [\dots]^{\mathsf{T}} \mid \boldsymbol{y}_{1:n-1}\} \\
&= \mathrm{E}\{[\boldsymbol{F}_x(\boldsymbol{x}_{n-1} - \hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{q}_n][\dots]^{\mathsf{T}} \mid \boldsymbol{y}_{1:n-1}\} \\
&= \boldsymbol{F}_x \,\mathrm{E}\{(\boldsymbol{x}_{n-1} - \hat{\boldsymbol{x}}_{n-1|n-1})(\boldsymbol{x}_{n-1} - \hat{\boldsymbol{x}}_{n-1|n-1})^{\mathsf{T}} \mid \boldsymbol{y}_{1:n-1}\}\boldsymbol{F}_x^{\mathsf{T}} \\
&\qquad + \mathrm{E}\{\boldsymbol{q}_n\boldsymbol{q}_n^{\mathsf{T}} \mid \boldsymbol{y}_{1:n-1}\} \\
&= \boldsymbol{F}_x\boldsymbol{P}_{n-1|n-1}\boldsymbol{F}_x^{\mathsf{T}} + \boldsymbol{Q}_n
\end{aligned}$$

Linearized Model: Measurement Update (1/3)

- Prediction from t_{n-1} to t_n: $\hat{\boldsymbol{x}}_{n|n-1}$, $\boldsymbol{P}_{n|n-1}$
- Linearization around $\hat{\boldsymbol{x}}_{n|n-1}$:

$$\begin{aligned}\boldsymbol{y}_n &= g(\boldsymbol{x}_n) + \boldsymbol{r}_n \\ &\approx g(\hat{\boldsymbol{x}}_{n|n-1}) + \boldsymbol{G}_x(\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1}) + \boldsymbol{r}_n\end{aligned}$$

- Regularized linear least squares:

$$\begin{aligned}J_{\mathsf{ReLS}}(\boldsymbol{x}_n) = (\boldsymbol{y}_n - g(\hat{\boldsymbol{x}}_{n|n-1}) - \boldsymbol{G}_x(\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1}))^{\mathsf{T}}\boldsymbol{R}_n^{-1} \\ \times (\boldsymbol{y}_n - g(\hat{\boldsymbol{x}}_{n|n-1}) - \boldsymbol{G}_x(\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1})) \\ + (\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1})^{\mathsf{T}}\boldsymbol{P}_{n|n-1}^{-1}(\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1}) \\ \hat{\boldsymbol{x}}_{n|n} = \operatorname*{argmin}_{\boldsymbol{x}_n} J_{\mathsf{ReLS}}(\boldsymbol{x}_n)\end{aligned}$$

Linearized Model: Measurement Update (2/3)

- Regularized linear least squares:

$$
\begin{aligned}
J_{\text{ReLS}}(\boldsymbol{x}_n) = (\boldsymbol{y}_n - g(\hat{\boldsymbol{x}}_{n|n-1}) - \boldsymbol{G}_x(\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1}))^{\mathsf{T}} \boldsymbol{R}_n^{-1} \\
\times (\boldsymbol{y}_n - g(\hat{\boldsymbol{x}}_{n|n-1}) - \boldsymbol{G}_x(\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1})) \\
+ (\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1})^{\mathsf{T}} \boldsymbol{P}_{n|n-1}^{-1} (\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1})
\end{aligned}
$$

- Change of variables: $\boldsymbol{z}_n = \boldsymbol{y}_n - g(\hat{\boldsymbol{x}}_{n|n-1}) + \boldsymbol{G}_x \hat{\boldsymbol{x}}_{n|n-1}$:

$$
\begin{aligned}
J_{\text{ReLS}}(\boldsymbol{x}_n) = (\boldsymbol{z}_n - \boldsymbol{G}_x \boldsymbol{x}_n)^{\mathsf{T}} \boldsymbol{R}_n^{-1} (\boldsymbol{z}_n - \boldsymbol{G}_x \boldsymbol{x}_n) \\
+ (\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1})^{\mathsf{T}} \boldsymbol{P}_{n|n-1}^{-1} (\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1})
\end{aligned}
$$

- Solution (see Chapters 2.4, 5.2):

$$
\begin{aligned}
\hat{\boldsymbol{x}}_{n|n} &= \hat{\boldsymbol{x}}_{n|n-1} + \boldsymbol{K}_n (\boldsymbol{z}_n - \boldsymbol{G}_x \hat{\boldsymbol{x}}_{n|n-1}) \\
\boldsymbol{K}_n &= \boldsymbol{P}_{n|n-1} \boldsymbol{G}_x^{\mathsf{T}} (\boldsymbol{G}_x \boldsymbol{P}_{n|n-1} \boldsymbol{G}_x^{\mathsf{T}} + \boldsymbol{R}_n)^{-1} \\
\boldsymbol{P}_{n|n} &\approx \boldsymbol{P}_{n|n-1} - \boldsymbol{K}_n (\boldsymbol{G}_x \boldsymbol{P}_{n|n-1} \boldsymbol{G}_x^{\mathsf{T}} + \boldsymbol{R}_n) \boldsymbol{K}_n^{\mathsf{T}}
\end{aligned}
$$

Linearized Model: Measurement Update (3/3)

- Measurement update:

$$\hat{\boldsymbol{x}}_{n|n} = \hat{\boldsymbol{x}}_{n|n-1} + \boldsymbol{K}_n(\boldsymbol{z}_n - \boldsymbol{G}_x\hat{\boldsymbol{x}}_{n|n-1})$$

- Substitution of $\boldsymbol{z}_n = \boldsymbol{y}_n - g(\hat{\boldsymbol{x}}_{n|n-1}) + \boldsymbol{G}_x\hat{\boldsymbol{x}}_{n|n-1}$:

$$\begin{aligned}\hat{\boldsymbol{x}}_{n|n} &= \hat{\boldsymbol{x}}_{n|n-1} + \boldsymbol{K}_n(\boldsymbol{y}_n - g(\hat{\boldsymbol{x}}_{n|n-1}) + \boldsymbol{G}_x\hat{\boldsymbol{x}}_{n|n-1} - \boldsymbol{G}_x\hat{\boldsymbol{x}}_{n|n-1}) \\ &= \hat{\boldsymbol{x}}_{n|n-1} + \boldsymbol{K}_n(\boldsymbol{y}_n - g(\hat{\boldsymbol{x}}_{n|n-1}))\end{aligned}$$

Linearized Model: Summary

- Model approximation:

$$\begin{aligned}\boldsymbol{x}_n &\approx f(\hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{F}_x(\boldsymbol{x}_{n-1} - \hat{\boldsymbol{x}}_{n-1|n-1}) + \boldsymbol{q}_n \\ \boldsymbol{y}_n &\approx g(\hat{\boldsymbol{x}}_{n|n-1}) + \boldsymbol{G}_x(\boldsymbol{x}_n - \hat{\boldsymbol{x}}_{n|n-1}) + \boldsymbol{r}_n\end{aligned}$$

- Prediction:

$$\begin{aligned}\hat{\boldsymbol{x}}_{n|n-1} &= f(\hat{\boldsymbol{x}}_{n-1|n-1}), \\ \boldsymbol{P}_{n|n-1} &= \boldsymbol{F}_x \boldsymbol{P}_{n-1|n-1} \boldsymbol{F}_x^{\mathsf{T}} + \boldsymbol{Q}_n,\end{aligned}$$

- Measurement update:

$$\begin{aligned}\boldsymbol{K}_n &= \boldsymbol{P}_{n|n-1} \boldsymbol{G}_x^{\mathsf{T}} (\boldsymbol{G}_x \boldsymbol{P}_{n|n-1} \boldsymbol{G}_x^{\mathsf{T}} + \boldsymbol{R}_n)^{-1}, \\ \hat{\boldsymbol{x}}_{n|n} &= \hat{\boldsymbol{x}}_{n|n-1} + \boldsymbol{K}_n(\boldsymbol{y}_n - g(\hat{\boldsymbol{x}}_{n|n-1})), \\ \boldsymbol{P}_{n|n} &= \boldsymbol{P}_{n|n-1} - \boldsymbol{K}_n(\boldsymbol{G}_x \boldsymbol{P}_{n|n-1} \boldsymbol{G}_x^{\mathsf{T}} + \boldsymbol{R}_n) \boldsymbol{K}_n^{\mathsf{T}}.\end{aligned}$$

Extended Kalman Filter

Algorithm 1 Extended Kalman Filter

Initialize $\hat{\boldsymbol{x}}_{0|0} = \boldsymbol{m}_0$, $\boldsymbol{P}_{0|0} = \boldsymbol{P}_0$

for $n = 1, 2, \ldots$ **do**

  Prediction (time update):

$$
\begin{aligned}
\hat{\boldsymbol{x}}_{n|n-1} &= f(\hat{\boldsymbol{x}}_{n-1|n-1}) \\
\boldsymbol{P}_{n|n-1} &= \boldsymbol{F}_x \boldsymbol{P}_{n-1|n-1} \boldsymbol{F}_x^\mathsf{T} + \boldsymbol{Q}_n
\end{aligned}
$$

  Measurement update:

$$
\begin{aligned}
\boldsymbol{K}_n &= \boldsymbol{P}_{n|n-1} \boldsymbol{G}_x^\mathsf{T} (\boldsymbol{G}_x \boldsymbol{P}_{n|n-1} \boldsymbol{G}_x + \boldsymbol{R}_n)^{-1} \\
\hat{\boldsymbol{x}}_{n|n} &= \hat{\boldsymbol{x}}_{n|n-1} + \boldsymbol{K}_n (\boldsymbol{y}_n - g(\hat{\boldsymbol{x}}_{n|n-1})) \\
\boldsymbol{P}_{n|n} &= \boldsymbol{P}_{n|n-1} - \boldsymbol{K}_n (\boldsymbol{G}_x \boldsymbol{P}_{n|n-1} \boldsymbol{G}_x + \boldsymbol{R}_n) \boldsymbol{K}_n^\mathsf{T}
\end{aligned}
$$

end for

Example: Object Tracking (1/3)

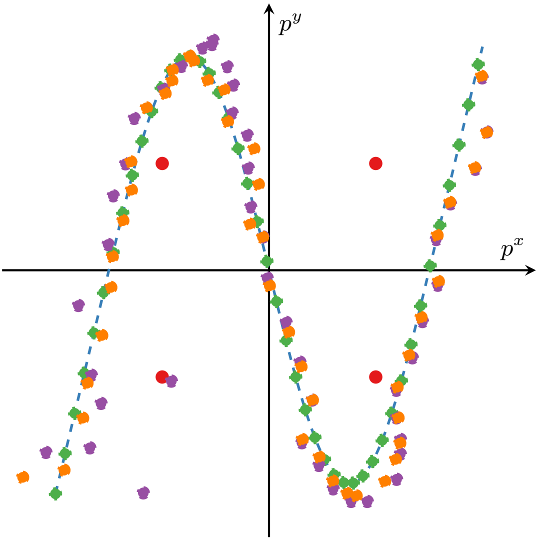
- Quasi-constant turn model:

$$\begin{bmatrix} \dot{p}^x(t) \\ \dot{p}^y(t) \\ \dot{v}(t) \\ \dot{\varphi}(t) \end{bmatrix} = \begin{bmatrix} v(t)\cos(\varphi(t)) \\ v(t)\sin(\varphi(t)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{w}(t)$$

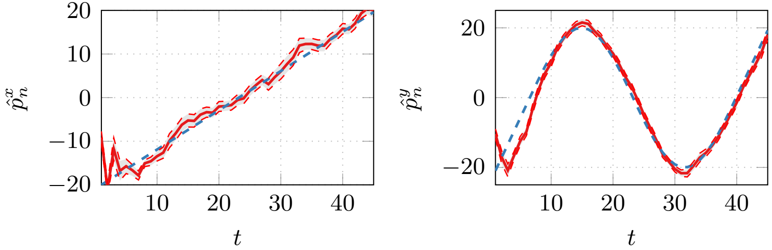
- Range (distance) measurements:

$$\boldsymbol{y}_n = \begin{bmatrix} |\boldsymbol{p}_n - \boldsymbol{p}_1^s| \\ |\boldsymbol{p}_n - \boldsymbol{p}_2^s| \\ \vdots \\ |\boldsymbol{p}_n - \boldsymbol{p}_K^s| \end{bmatrix} + \boldsymbol{r}_n$$

Example: Object Tracking (2/3)



Example: Object Tracking (3/3)

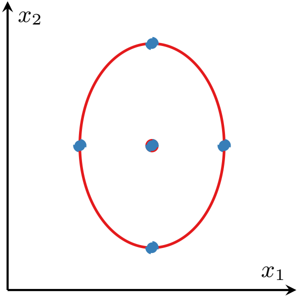


Position RMSE: 3.83 m

Nonlinear Transformations of Random Points (1/3)

- Given: Random variable $\boldsymbol{x}$ with mean $\boldsymbol{m}$ and covariance $\boldsymbol{P}$
- Choose points $\boldsymbol{x}^j$ and weights w_m^j, w_P^j such that:

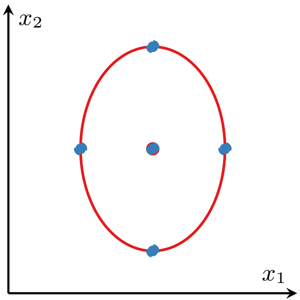
$$\boldsymbol{m} = \sum_{j=0}^{J-1} w_m^j \boldsymbol{x}^j, \boldsymbol{P} = \sum_{j=0}^{J-1} w_P^j (\boldsymbol{x}^j - \boldsymbol{m})(\boldsymbol{x}^j - \boldsymbol{m})^\mathsf{T},$$

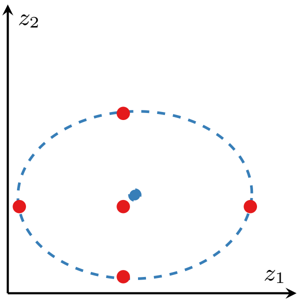


Nonlinear Transformations of Random Points (2/3)

- Given: Points $\boldsymbol{x}^j$ and weights w_m^j, w_P^j
- Nonlinear transformation: $\boldsymbol{z} = h(\boldsymbol{x})$
- Transformed points:

$$\boldsymbol{z}^j = h(\boldsymbol{x}^j)$$





Nonlinear Transformations of Random Points (3/3)

- Given: Points $\boldsymbol{x}^j$ and weights w_m^j, w_P^j
- Nonlinear transformation: $\boldsymbol{z} = h(\boldsymbol{x})$
- Transformed points:

$$\boldsymbol{z}^j = h(\boldsymbol{x}^j)$$

- Moments of the transformed variable

$$\mathrm{E}\{\boldsymbol{z}\} \approx \sum_{j=1}^{J} w_m^j \boldsymbol{z}^j$$

$$\mathrm{Cov}\{\boldsymbol{z}\} \approx \sum_{j=1}^{J} w_P^j (\boldsymbol{z}^j - \mathrm{E}\{\boldsymbol{z}\})(\boldsymbol{z}^j - \mathrm{E}\{\boldsymbol{z}\})^\mathsf{T}$$

$$\mathrm{Cov}\{\boldsymbol{x}, \boldsymbol{z}\} \approx \sum_{j=1}^{J} w_P^j (\boldsymbol{x}^j - \boldsymbol{m})(\boldsymbol{z}^j - \mathrm{E}\{\boldsymbol{z}\})^\mathsf{T}$$

Unscented Transform

- Unscented Transform: One way of choosing $\boldsymbol{x}^j$, w_m^j and w_P^j, uses $2L+1$ points
- Location of the sigma-points:

$$
\begin{aligned}
\boldsymbol{x}^0 &= \boldsymbol{m} \\
\boldsymbol{x}^j &= \boldsymbol{m} + \sqrt{L+\lambda}[\sqrt{\boldsymbol{P}}]_j, & j &= 1, \dots, L \\
\boldsymbol{x}^j &= \boldsymbol{m} - \sqrt{L+\lambda}[\sqrt{\boldsymbol{P}}]_{(j-L)}, & j &= L+1, \dots, 2L
\end{aligned}
$$

- Weights of the sigma-points:

$$
\begin{aligned}
w_m^0 &= \frac{\lambda}{L+\lambda} \\
w_P^0 &= \frac{\lambda}{L+\lambda} + (1-\alpha^2+\beta) \\
w_m^j = w_P^j &= \frac{1}{2(L+\lambda)}, & j &= 1, \dots, 2L
\end{aligned}
$$

Unscented Transform: Prediction (1/2)

- Dynamic model:

$$\boldsymbol{x}_n = f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n$$

- Sigma-points with $\boldsymbol{m} = \hat{\boldsymbol{x}}_{n-1|n-1}$, $\boldsymbol{P} = \boldsymbol{P}_{n-1|n-1}$:

$$\begin{aligned}
\boldsymbol{x}_{n-1}^0 &= \hat{\boldsymbol{x}}_{n-1|n-1} \\
\boldsymbol{x}_{n-1}^j &= \hat{\boldsymbol{x}}_{n-1|n-1} + \sqrt{L+\lambda}\left[\sqrt{\boldsymbol{P}_{n-1|n-1}}\right]_j, \qquad j = 1, \dots, L \\
\boldsymbol{x}_{n-1}^j &= \hat{\boldsymbol{x}}_{n-1|n-1} - \sqrt{L+\lambda}\left[\sqrt{\boldsymbol{P}_{n-1|n-1}}\right]_{(j-L)}, \; j = L+1, \dots, 2L
\end{aligned}$$

Unscented Transform: Prediction (2/2)

- Dynamic model:

$$\boldsymbol{x}_n = f(\boldsymbol{x}_{n-1}) + \boldsymbol{q}_n$$

- Transformed points:

$$\boldsymbol{x}_n^j = f(\boldsymbol{x}_{n-1}^j), \qquad j = 0, \dots, 2L$$

- Moments of the prediction:

$$\hat{\boldsymbol{x}}_{n|n-1} = \sum_{j=0}^{2L} w_m^j \boldsymbol{x}_n^j$$

$$\boldsymbol{P}_{n|n-1} = \sum_{j=0}^{2L} w_c^j (\boldsymbol{x}_n^j - \hat{\boldsymbol{x}}_{n|n-1})(\boldsymbol{x}_n^j - \hat{\boldsymbol{x}}_{n|n-1})^{\mathsf{T}} + \boldsymbol{Q}_n$$

Unscented Transform: Measurement Update (1/2)

- Measurement model:

$$\boldsymbol{y}_n = g(\boldsymbol{x}_n) + \boldsymbol{r}_n$$

- Recall: Alternative form of measurement update:

$$\begin{aligned}
\boldsymbol{K}_n &= \mathrm{Cov}\{\boldsymbol{x}_n, \boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\} \, \mathrm{Cov}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\}^{-1}, \\
\hat{\boldsymbol{x}}_{n|n} &= \hat{\boldsymbol{x}}_{n|n-1} + \boldsymbol{K}_n(\boldsymbol{y}_n - \mathrm{E}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\}), \\
\boldsymbol{P}_{n|n} &= \boldsymbol{P}_{n|n-1} - \boldsymbol{K}_n \, \mathrm{Cov}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\} \boldsymbol{K}_n^{\mathsf{T}}.
\end{aligned}$$

- We can calculate $\mathrm{E}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\}$, $\mathrm{Cov}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\}$, and $\mathrm{Cov}\{\boldsymbol{x}_n, \boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\}$ using the unscented transform
- Sigma-points based on $\hat{\boldsymbol{x}}_{n|n-1}$, $\boldsymbol{P}_{n|n-1}$:

$$\begin{aligned}
\boldsymbol{x}_n^0 &= \hat{\boldsymbol{x}}_{n|n-1} \\
\boldsymbol{x}_n^j &= \hat{\boldsymbol{x}}_{n|n-1} + \sqrt{L+\lambda}\left[\sqrt{\boldsymbol{P}_{n|n-1}}\right]_j, \qquad j = 1, \dots, L \\
\boldsymbol{x}_n^j &= \hat{\boldsymbol{x}}_{n|n-1} - \sqrt{L+\lambda}\left[\sqrt{\boldsymbol{P}_{n|n-1}}\right]_{(j-L)}, \; j = L+1, \dots, 2L
\end{aligned}$$

Unscented Transform: Measurement Update (2/2)

- Measurement model:

$$\boldsymbol{y}_n = g(\boldsymbol{x}_n) + \boldsymbol{r}_n$$

- Transformed sigma-points:

$$\boldsymbol{y}_n^j = g(\boldsymbol{x}_n^j), \qquad j = 0, \dots, 2L$$

- Moments of the predicted $\boldsymbol{y}_n$:

$$\mathrm{E}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\} = \sum_{j=0}^{2L} w_m^j \boldsymbol{y}_n^j$$

$$\begin{aligned}\mathrm{Cov}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\} = \sum_{j=0}^{2L} w_P^j (\boldsymbol{y}_n^j - \mathrm{E}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\}) \\ \times (\boldsymbol{y}_n^j - \mathrm{E}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\})^\mathsf{T} + \boldsymbol{R}_n\end{aligned}$$

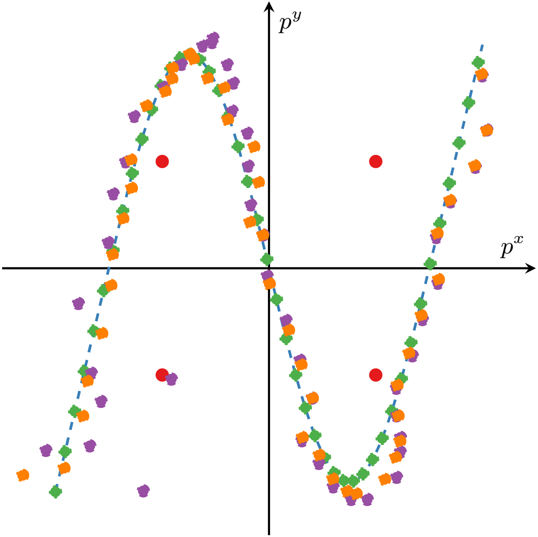
$$\mathrm{Cov}\{\boldsymbol{x}_n, \boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\} = \sum_{j=0}^{2L} w_P^j (\boldsymbol{x}_n^j - \hat{\boldsymbol{x}}_{n|n-1})(\boldsymbol{y}_n^j - \mathrm{E}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\})^\mathsf{T}$$

Unscented Kalman Filter

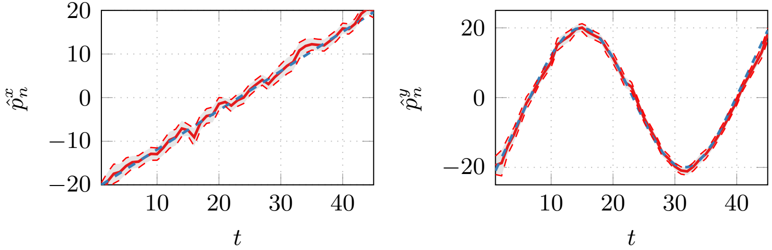
- Prediction:
  - Calculate the sigma-points using $\hat{\boldsymbol{x}}_{n-1|n-1}$ and $\boldsymbol{P}_{n-1|n-1}$
  - Propagate the sigma-points $\boldsymbol{x}_n^j = f(\boldsymbol{x}_{n-1}^j)$
  - Calculate the mean and covariance $\hat{\boldsymbol{x}}_{n|n-1}$, $\boldsymbol{P}_{n|n-1}$
- Measurement update:
  - Calculate the sigma-points using $\hat{\boldsymbol{x}}_{n|n-1}$ and $\boldsymbol{P}_{n|n-1}$
  - Propagate the sigma-points $\boldsymbol{y}_n^j = g(\boldsymbol{x}_n^j)$
  - Calculate the mean and covariance $\mathrm{E}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\}$, $\mathrm{Cov}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\}$, $\mathrm{Cov}\{\boldsymbol{x}_n, \boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\}$
  - Perform the Kalman filter measurement update:

$$
\begin{aligned}
\boldsymbol{K}_n &= \mathrm{Cov}\{\boldsymbol{x}_n, \boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\} \, \mathrm{Cov}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\}^{-1}, \\
\hat{\boldsymbol{x}}_{n|n} &= \hat{\boldsymbol{x}}_{n|n-1} + \boldsymbol{K}_n(\boldsymbol{y}_n - \mathrm{E}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\}), \\
\boldsymbol{P}_{n|n} &= \boldsymbol{P}_{n|n-1} - \boldsymbol{K}_n \, \mathrm{Cov}\{\boldsymbol{y}_n \mid \boldsymbol{y}_{1:n-1}\} \boldsymbol{K}_n^{\mathsf{T}}.
\end{aligned}
$$

Example: Object Tracking (1/2)



Example: Object Tracking (2/2)



Position RMSE: 1.45 m

Unscented Transform: Choice of Parameters

- The parameter λ is actually:

$$\lambda = \alpha^2(L + \kappa) - L$$

- α, β, and κ are tuning parameters
- κ is usually set to 0
- α controls the spread of the sigma-points:

$$\sqrt{L + \lambda} = \sqrt{L + \alpha^2(L + \kappa) - L} = \alpha\sqrt{L}.$$

- Suggestions vary, e.g., $\alpha = 1 \times 10^{-3}$
- β only affects the covariance weight, a good starting point is $\beta = 2$

Summary

- Nonlinear state-space models require approximative solutions
- The extended Kalman filter uses a linearization of the dynamic and measurement models
- The unscented Kalman filter uses a set of deterministic sigma-points (samples) to calculate the means and covariances