GIS-E3010

Least-Squares Methods in Geoscience

Lecture 3/2018

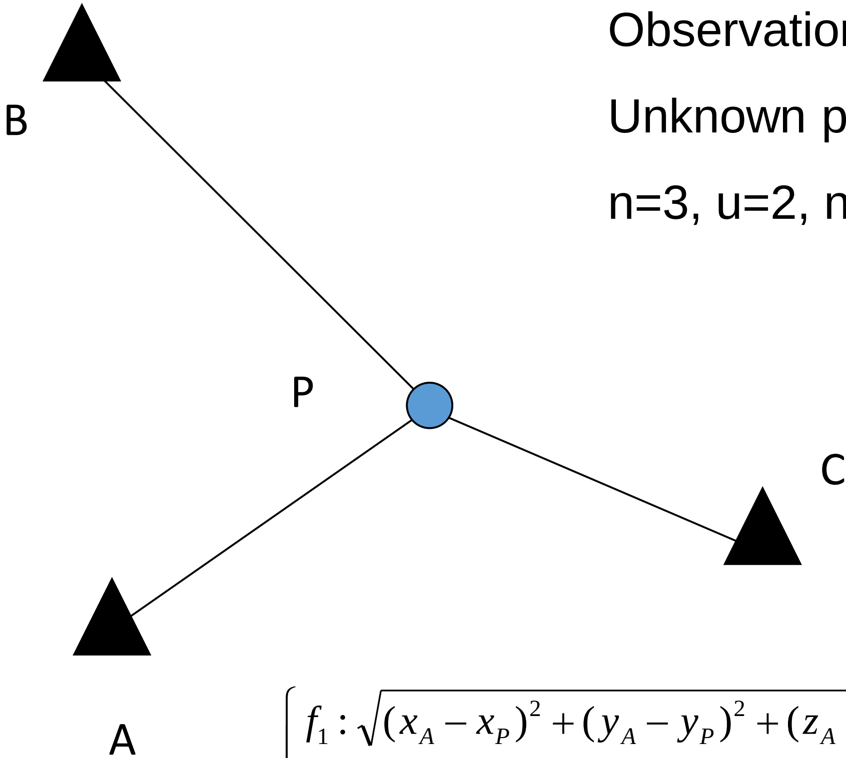
Variance propagation in general

Variance propagation in LS adjustment

Error ellipsoids

Precision

Non-linear functional models, trilateration



Observations:distances s

Unknown parameters: x,y

n=3, u=2, n-u=1

- Linearize
- A-matrix?
- y-vector?

$$\begin{cases} f_1 : \sqrt{(x_A - x_P)^2 + (y_A - y_P)^2 + (z_A - z_P)^2} - s_{PA} = 0 \\ f_2 : \sqrt{(x_B - x_P)^2 + (y_B - y_P)^2 (z_B - z_P)^2} - s_{PB} = 0 \\ f_3 : \sqrt{(x_C - x_P)^2 + (y_C - y_P)^2 (z_C - z_P)^2} - s_{PC} = 0 \end{cases}$$

Variance, covariance, Covariance matrix

The variance, standard deviation, error-ellipsoids are measures of precision

$$\sigma_{x_i}^2 = E\left((x_i - \mu_{x_i})^2\right)$$

$$\sigma_x = \sqrt{\sigma_x^2}$$

$$\sigma_{x_i x_j} = E\left((x_i - \mu_{x_i})(x_j - \mu_{x_j})\right)$$

$$\Sigma_x = E\left((X - M_x)(X - M_x)^T\right)$$

$$\Sigma_x = \begin{pmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} & \cdots & \sigma_{x_1 x_n} \\ \sigma_{x_1 x_2} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{x_1 x_n} & \cdots & \cdots & \sigma_{x_2}^2 \end{pmatrix}$$

Cofactor matrix, Weight matrix, Covariance matrix

$Q_x \quad Q_v \quad Q_{\hat{\ell}} \quad Q_\ell$

Cofactor matrices for parameters, residuals, adjusted observations, observations

$$\Sigma = \sigma_0^2 Q$$

Covariance matrix

$$P = \sigma_0^2 \Sigma_\ell^{-1} = Q_\ell^{-1}$$

Weight matrix

Variance propagation

$$Y = A_0 + AX$$

Y is linear combination of X

$$\begin{cases} y_1 = a_{0_1} + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 = a_{0_2} + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ y_c = a_{0_c} + a_{c1}x_1 + a_{c2}x_2 + \cdots + a_{cn}x_n \end{cases}$$

$$E(Y) = A_0 + AE(X)$$

Expectation of Y

We know the covariance matrix of X

$$\Sigma_x = E\left((X - E(X))(X - E(X))^T\right)$$

$$\Sigma_x = \begin{pmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_n} \\ \sigma_{x_1x_2} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{x_1x_n} & \cdots & \cdots & \sigma_{x_2}^2 \end{pmatrix}$$

How we obtain the covariance matrix of Y?

Example: we have measured angles and distances and we know the the precision of the instrument. What is the precision of the measured point coordinates?

Variance propagation law

$$E(\Sigma_Y) = E\left((Y - E(Y))(Y - E(Y))^T\right) =$$

$$E\left((Y - A_0 - AE(X))(Y - A_0 - AE(X))^T\right) =$$

$$E\left((A_0 + AX - A_0 - AE(X))(A_0 + AX - A_0 - AE(X))^T\right) =$$

$$E\left((AX - AE(X))(AX - AE(X))^T\right) =$$

$$AE\left((X - E(X))(X - E(X))^T\right)A^T =$$

$$AE\left(\Sigma_x\right)A^T$$

Examples

$$\Sigma_x = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} 3.0869 & 1.3226 \\ 1.3226 & 1.8432 \end{pmatrix}$$

$$y_1 = x_2 - x_1$$
$$y_2 = x_2 + x_1$$

Calculate

- Standard deviation of x_1 and x_2
- Standard deviations of y_1 and y_2
- Covariance matrix of y
- Correlation of y_1 and y_2

$$x = s \cdot \cos(\alpha)$$
$$y = s \cdot \sin(\alpha)$$

$$\Sigma_{\alpha,s} = \begin{pmatrix} \sigma_\alpha^2 & \sigma_{\alpha s} \\ \sigma_{as} & \sigma_s^2 \end{pmatrix} = \begin{pmatrix} 2.46\text{d}-8 & 0 \\ 0 & 25\text{d}-6 \end{pmatrix}$$

$$\alpha = \frac{\pi}{6} \text{ [rad]}$$

$$s = 20 \text{ m}$$

- Standard deviation of α and s
- Standard deviations of x and y
- Covariance matrix of x and y

In the case of non-linear equations

$$Y = F(X)$$

$$\begin{cases} y_1 &= f_1(x_1, x_2, \ldots, x_n) \\ y_2 &= f_2(x_1, x_2, \ldots, x_n) \\ \vdots & \qquad \vdots \\ y_c &= f_c(x_1, x_2, \ldots, x_n) \end{cases}$$

We linearize using Taylor theorem

$$Y = F(X_0) + J(X - X_0)$$

$$\begin{cases} y_1 &= f_1(x_{1_0}, x_{20}, \ldots, x_{n_0}) + \frac{\partial f_1}{\partial x_1}(x_1 - x_{1_0}) + \cdots + \frac{\partial f_1}{\partial x_n}(x_n - x_{n_0}) \\ y_2 &= f_2(x_{1_0}, x_{20}, \ldots, x_{n_0}) + \frac{\partial f_2}{\partial x_1}(x_1 - x_{1_0}) + \cdots + \frac{\partial f_2}{\partial x_n}(x_n - x_{n_0}) \\ \vdots & \qquad \vdots \qquad\qquad\qquad \vdots \\ y_c &= f_c(x_{1_0}, x_{20}, \ldots, x_{n_0}) + \frac{\partial f_c}{\partial x_1}(x_1 - x_{1_0}) + \cdots + \frac{\partial f_c}{\partial x_n}(x_n - x_{n_0}) \end{cases}$$

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_c}{\partial x_1} & \frac{\partial f_c}{\partial x_2} & \cdots & \frac{\partial f_c}{\partial x_n} \end{pmatrix}$$

$$\Sigma_y = J \Sigma_x J^T$$

Variance propagation in least squares process: Observation equation model

Covariance matrix of adjusted parameters

$$x - x_0 = (A^T PA)^{-1} A^T Py$$

$\Sigma_y = BC_\ell B^T = \Sigma_\ell$, kun $\quad B = -I$

$$J = (A^T PA)^{-1} A^T P$$

$$J\Sigma_\ell J^T = (A^T PA)^{-1} A^T P \sigma_0^2 Q_\ell PA (A^T PA)^{-1} = \sigma_0^2 (A^T PA)^{-1}$$

$$\Sigma_x = \sigma_0^2 N^{-1} = \sigma_0^2 Q_x$$

Note! This can be calculated before measurements, if we know the measurement method and instruments (P) and the structure of network (A)

Variance propagation in least squares process: observation equation model

Covariance and cofactor matrix of adjusted observations:

$$\hat{y} = A\hat{x}, \qquad \Sigma_x = \sigma_0^2 N^{-1} = \sigma_0^2 Q_x$$

$$\Sigma_{\hat{\ell}} = A\Sigma_{\hat{x}}A^T \qquad Q_{\hat{\ell}} = AQ_{\hat{x}}A^T$$

Covariance matrix of adjusted observations:

$$v = \hat{\ell} - \ell$$

$$\Sigma_v = \Sigma_\ell - \Sigma_{\hat{\ell}} \qquad \mathrm{Q}_v = \mathrm{Q}_\ell - Q_{\hat{\ell}}$$

Note! Theses can be calculated before measurements, if we know the measurement method and instruments (P) and the structure of network (A)

Axes of hyper-ellipsoid

$$P\left[(x-\hat{x})^T \Sigma_{\hat{x}}^{-1} (x-\hat{x}) \leq uF_{\alpha,u,r} \right]$$

$$d = \sqrt{(x-\hat{x})^T \Sigma_{\hat{x}}^{-1} (x-\hat{x})}$$

The **Mahalanobis distance** is a measure of the distance between a point P and a distribution D

$$R^T \Sigma_{\hat{x}}^{-1} R = \Lambda^{-1} \qquad R^T (x-\hat{x}) = z$$

Ortogonal transformation: Eigenvalues and eigen vectors to $\Sigma_{\hat{x}}$

$$P\left[(x-\hat{x})^T RR^T C_{\hat{x}}^{-1} RR^T (x-\hat{x}) \leq uF_{\alpha,u,r} \right] = 1-\alpha$$

$$P\left[z^T \Lambda^{-1} z \leq uF_{\alpha,u,r} \right] = 1-\alpha$$

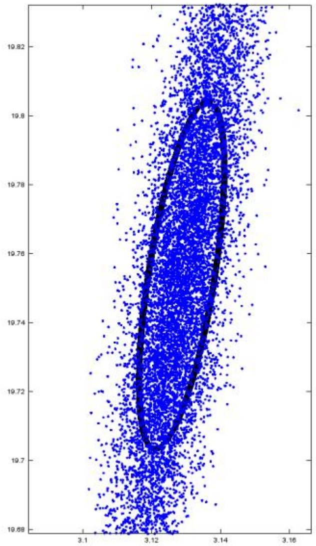
$$P\left[\frac{z_1^2}{\sqrt{\lambda_1}^2} + \frac{z_2^2}{\sqrt{\lambda_2}^2} + \cdots + \frac{z_u^2}{\sqrt{\lambda_u}^2} \leq uF_{\alpha,u,r} \right] = 1-\alpha$$

λ:s are variances of z (eigen values) and squares of the semi axes of hyper-ellipsoid

Scaling the standard error ellipsoids

$$P\left[\frac{z_1^2}{\left(\sqrt{\lambda_1}\sqrt{uF_{\alpha,u,r}}\right)^2}+\frac{z_2^2}{\left(\sqrt{\lambda_2}\sqrt{uF_{\alpha,u,r}}\right)^2}+\cdots+\frac{z_u^2}{\left(\sqrt{\lambda_u}\sqrt{uF_{\alpha,u,r}}\right)^2}\leq 1\right]=1-\alpha$$

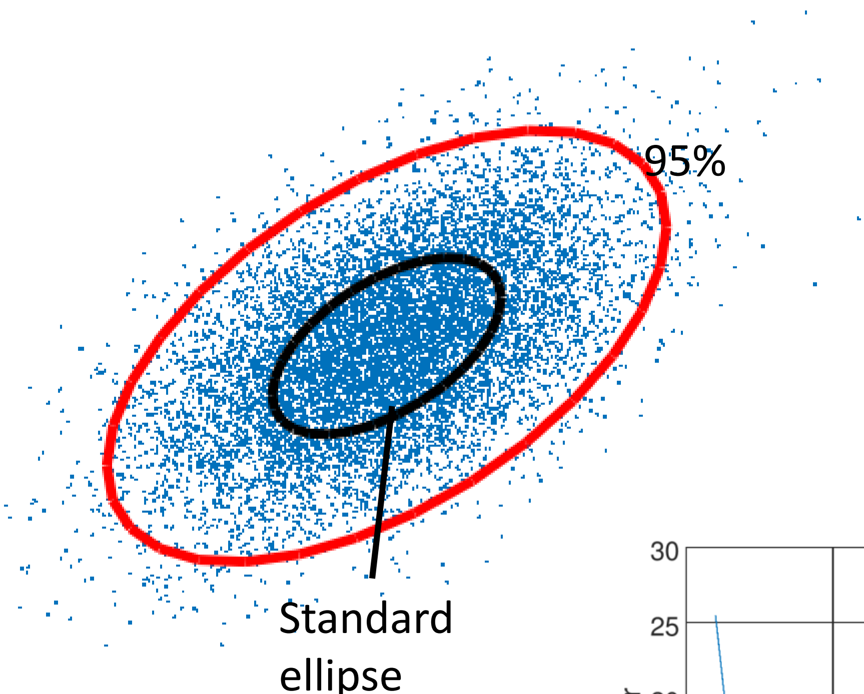
The size of the error ellisoid depends on the number of parameters u, redundance of the adjustment r and the chosen probability. The scaling factor is

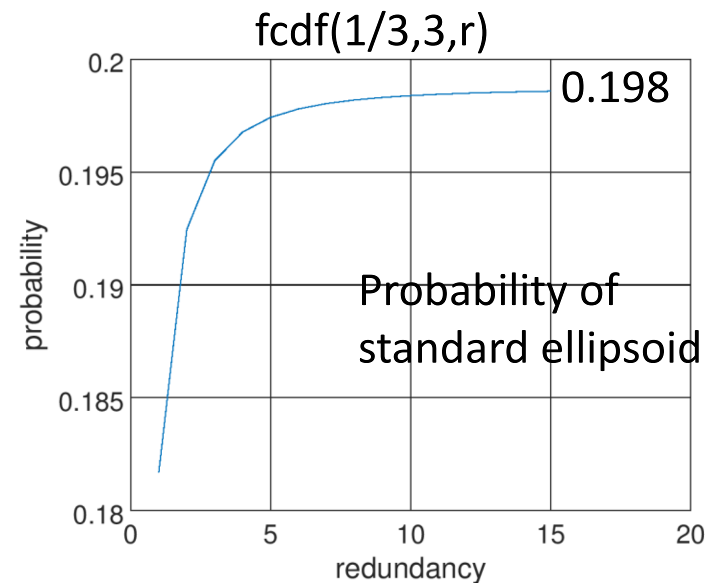


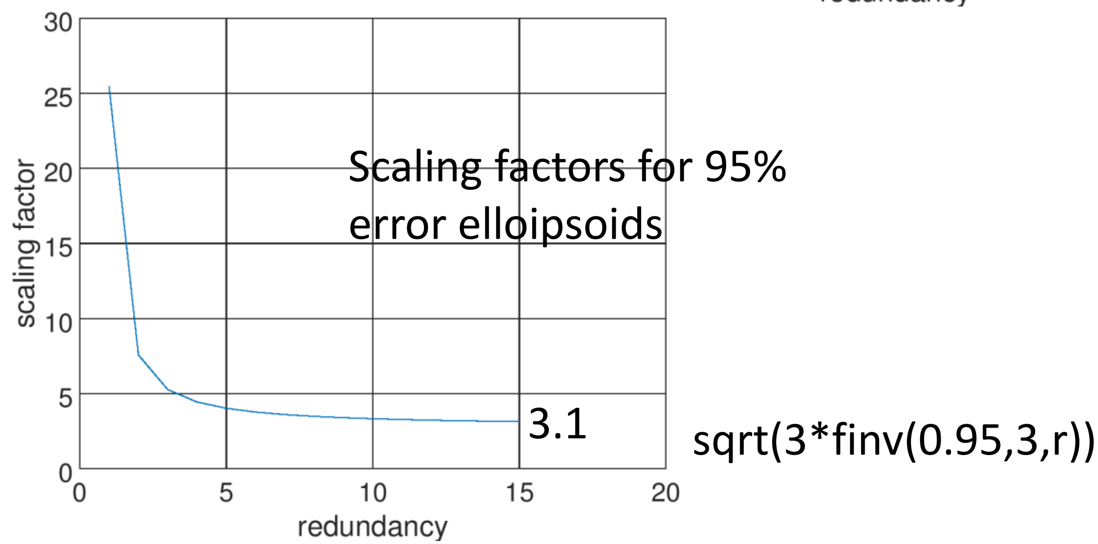
$$\sqrt{uF_{\alpha,u,r}}$$

If scaling factor is 1, we have standard error ellipsoids with semiaxes $\sqrt{\lambda_i}$

Confidence regions





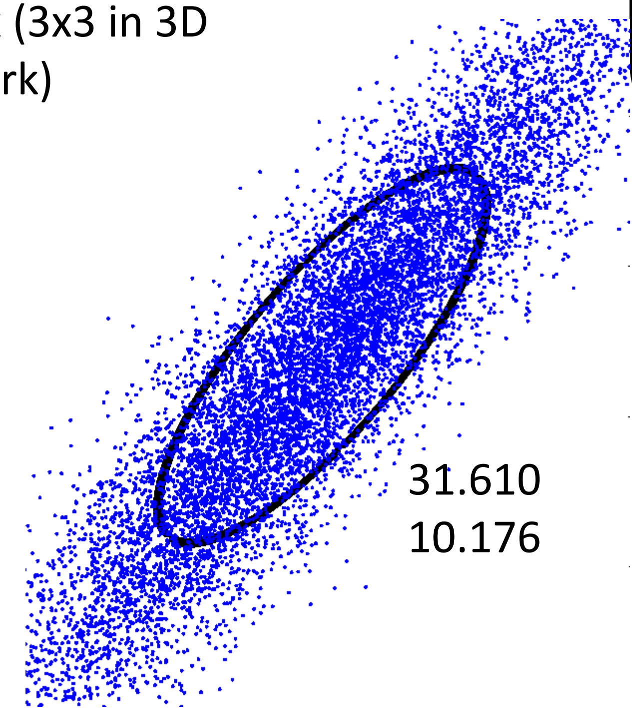


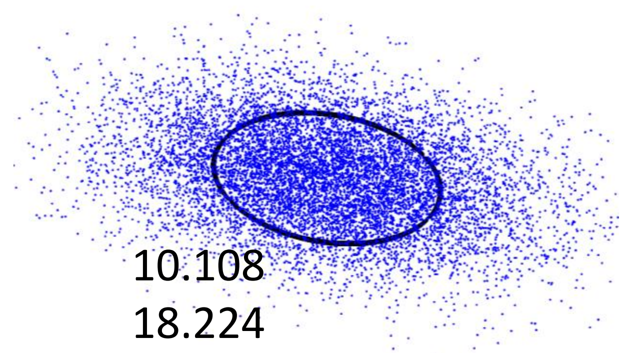
Calculating error ellipses

In network point error ellipses are calculated to corresponding part of the covariance matrix (3x3 in 3D network)

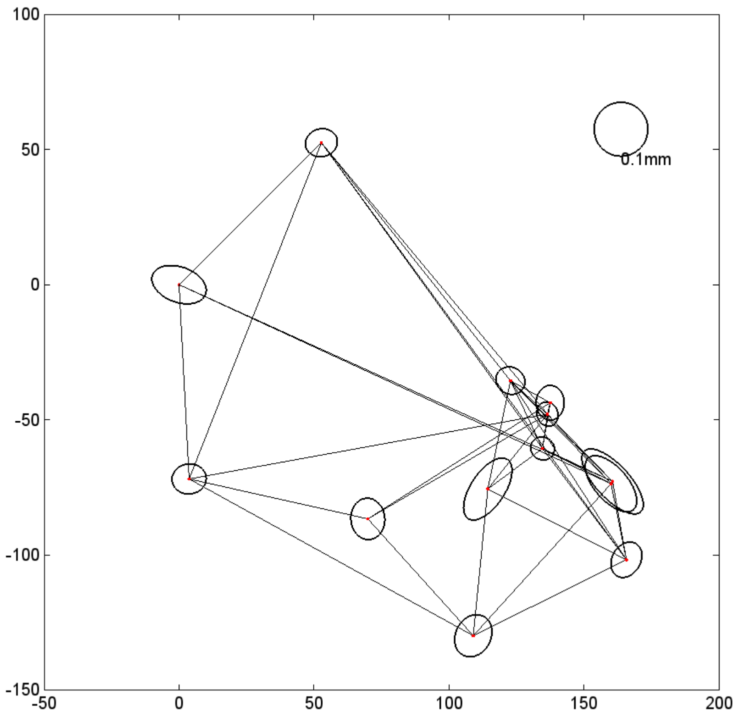
$$\Sigma_x = \begin{pmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} & \cdots & \sigma_{x_1 x_n} \\ \sigma_{x_1 x_2} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{x_1 x_n} & \cdots & \cdots & \sigma_{x_2}^2 \end{pmatrix}$$

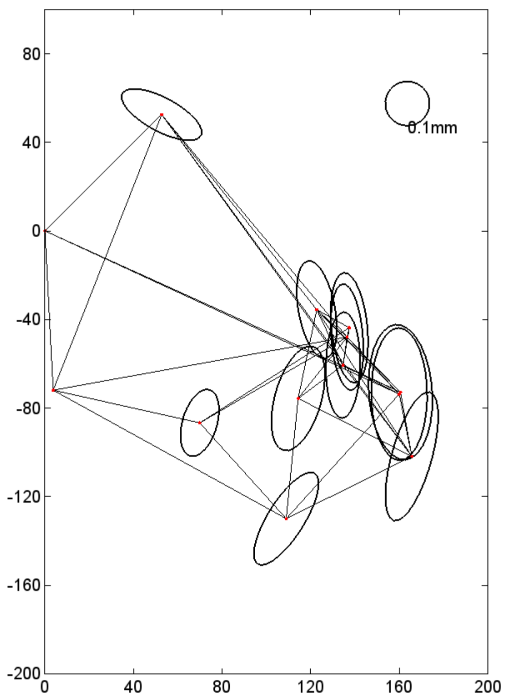
Calculate eigen values and eigen vectors for the part of the covariance matrix



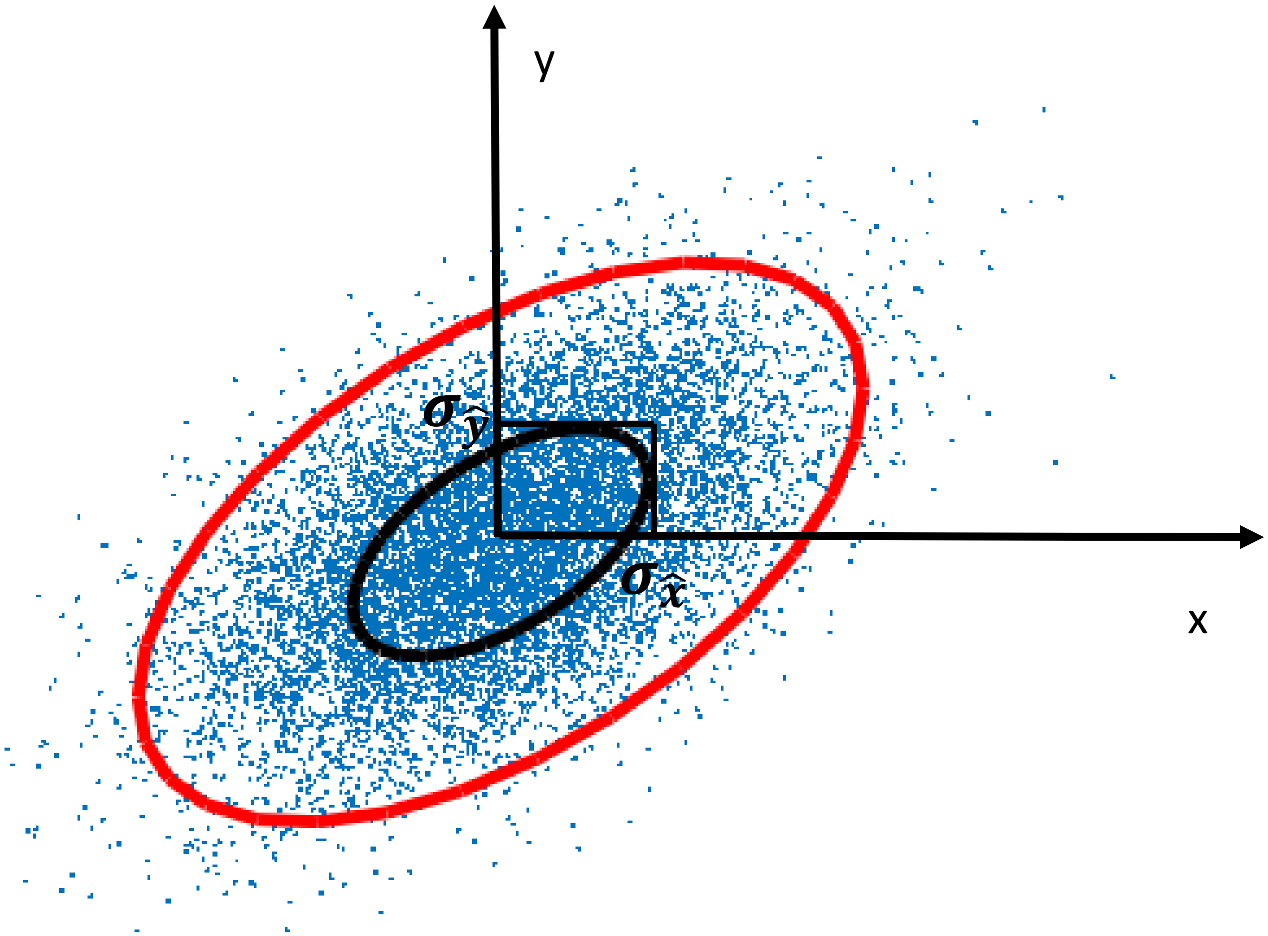


The size and direction of ellipses depend on the reference





Standard deviations



Relative error ellipses (ellipsoids) are error ellipses for coordinate difference DX

$$\Sigma_{\Delta X} = D\Sigma_X D^T \qquad D = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$