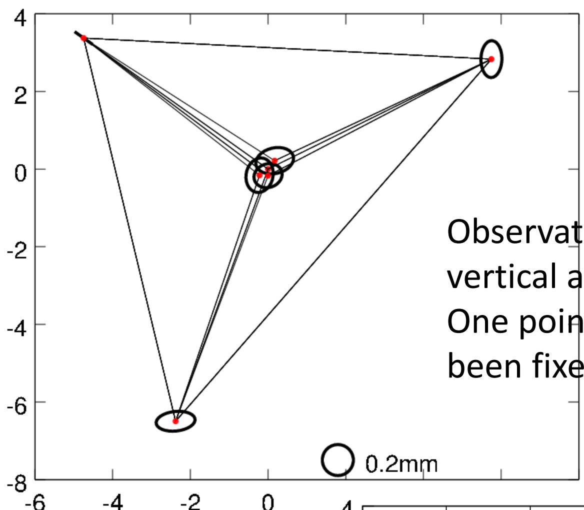
GIS-E3010
Least-Squares Methods in Geoscience
Lecture 4/2018

Datum problem

Iterative least squares process

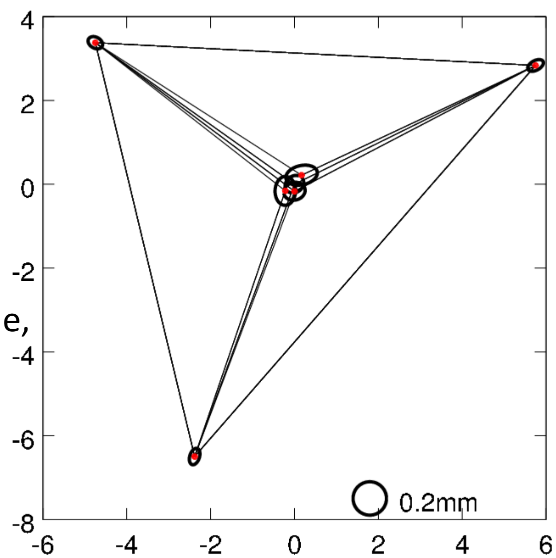
1. **Functional model**: in the case of observation equation model we have one equation for each observation . Express each observation as a function of unknown parameters
2. Initial values for parameters. Approximate values are necessary for linearization
3. Number of rows and columns of A-matrix
4. Linearization: partial derivtives, Jacobian matrix , design matrix **A**
5. **y-vector**: observed minus calculated (with approximate values)
6. **Numerical values for the elements of A-matrix** using approximate values of unknown parameters (number of columns equals to number of unknown parameters, number of rows equals to number of observations)
7. Stochastic model: **weighting**; $P = m_0^2 \Sigma^{-1}$
8. Normal equations
9. Solve normal equations for the corrections to approximate values
10. Correct initial values (new approximate values)
11. Iteration : go back to 6. (Gauss-Newton -iteration)

Datum-problem,
how to connect the network to the reference frame
georeferencing



Observations: horizontal and vertical angles and distances, One point and azimuth has been fixed

- Where the network is? (**position, translation**)
- What is the attitude of the network in the reference frame (**orientation**)
- Size of the network vertices (**scale**)
- Have we observed position, scale or orientation
- Datum defect? Rank of A, rank of N



Observations same, inner constraints

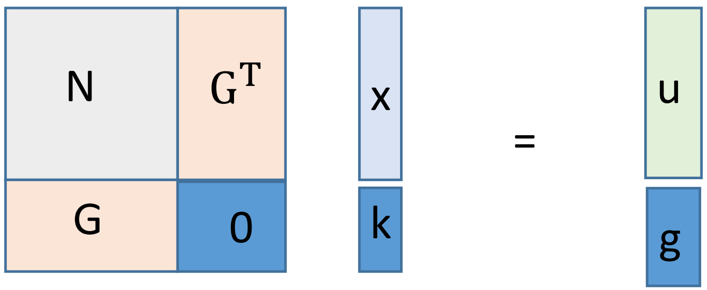
dimension	**Datum-parameters**	**Maximum datum defect**
1	translation	1
2	translation(2), orientation (1), scale(1)	4
3	translation(3), orientation (3), scale(1)	7

What kind of datum information we get from observations?

- Height difference: scale
- Height observation: translation
- Distance: scale
- Coordinate differences: orientation, scale
- Coordinates: translation
- Azimuth: orientation (1)
- Horizontal angle: direction of the vertical axis
- Zenith angle: direction of the vertical axis

Constraint equations

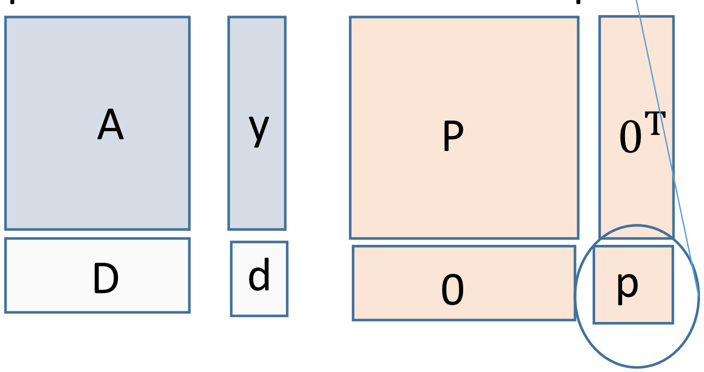
Additional rows and columns to Normal matrix



$$N = A^T P A$$

$$u = A^T P y$$

Additional observations, weighted parameters or conditions between parameters to observation equations



Example: levelling network

A =

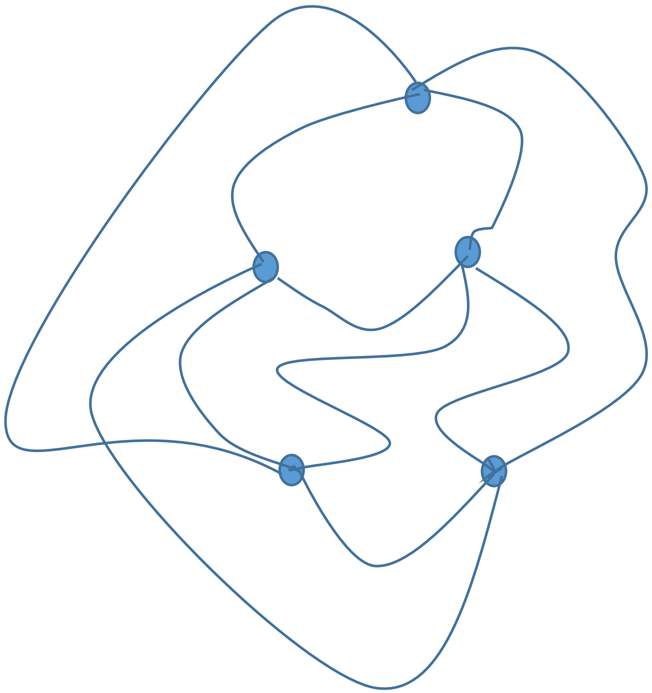
-1	1	0	0	0
-1	0	1	0	0
-1	0	0	1	0
-1	0	0	0	1
0	-1	1	0	0
0	-1	0	1	0
0	-1	0	0	1
0	0	-1	1	0
0	0	-1	0	1
0	0	0	-1	1

Diag(P)=

1.0829
1.3815
1.0191
1.1445
1.2726
1.5118
5.8017
1.3329
5.5142
1.7395

y =

-4.39640
1.87664
-5.03596
-0.31572
6.26616
-0.64889
4.07619
-6.91318
-2.19066
4.71801



Normal equation matrix N and vector u

N =

4.6281	-1.0829	-1.3815	-1.0191	-1.1445
-1.0829	9.6691	-1.2726	-1.5118	-5.8017
-1.3815	-1.2726	9.5012	-1.3329	-5.5142
-1.0191	-1.5118	-1.3329	5.6033	-1.7395
-1.1445	-5.8017	-5.5142	-1.7395	14.2000

u =

7.6619
-35.4034
31.8613
-23.5348
19.4150

Eigenvalues of N:(eig(N)

-4.4756e-016
5.7499e+000
7.1932e+000
1.0866e+001
1.9793e+001

The rank of A- matrix and N-matrix is 4, N is singular matrix due to the datum defect. The translation (height level of the network) information is missing.
We need to bring the height level some how to the adjustment

Minimum constraints using inner constraints

N=

						u=
4.62812	-1.08294	-1.38154	-1.01911	-1.14453	1.00000	7.6619
-1.08294	9.66905	-1.27261	-1.51177	-5.80173	1.00000	-35.4034
-1.38154	-1.27261	9.50123	-1.33289	-5.51418	1.00000	31.8613
-1.01911	-1.51177	-1.33289	5.60330	-1.73953	1.00000	-23.5348
-1.14453	-5.80173	-5.51418	-1.73953	14.19997	1.00000	19.4150
1.00000	1.00000	1.00000	1.00000	1.00000	0.00000	0

$$\sum H = H_1 + H_2 + H_3 + H_4 + H_5 = 0$$

```
x=inv(N)*u
x =

   1.5742
  -2.8180
   3.4490
  -3.4632
   1.2579
   0
```

Or fixing one height

N =

4.6281	-1.0829	-1.3815	-1.0191	-1.1445
-1.0829	9.6691	-1.2726	-1.5118	-5.8017
-1.3815	-1.2726	9.5012	-1.3329	-5.5142
-1.0191	-1.5118	-1.3329	5.6033	-1.7395
-1.1445	-5.8017	-5.5142	-1.7395	14.2000

u =

7.6619
-35.4034
31.8613
-23.5348
19.4150

By remowing the row and the column from N

or

N1 =

4.62812	-1.08294	-1.38154	-1.01911	-1.14453	1.00000
-1.08294	9.66905	-1.27261	-1.51177	-5.80173	0.00000
-1.38154	-1.27261	9.50123	-1.33289	-5.51418	0.00000
-1.01911	-1.51177	-1.33289	5.60330	-1.73953	0.00000
-1.14453	-5.80173	-5.51418	-1.73953	14.19997	0.00000
1.00000	0.00000	0.00000	0.00000	0.00000	0.00000

By adding constraint equation

Or by adding Height observation to observation equations as a weighted parameter

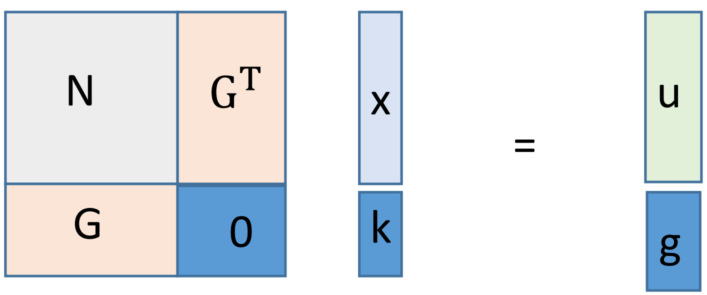
Connecting the levelling network to reference frame

- Height differences don’t determine height level of the network
- We need at least one known height
  - In order to save the shape of the network we like to use minimum constraints
    - Adding one height observation to observation equations
    - Or using inner constraints
- Fixing more than one point we affect the shape of the network
    - If we like to study time series of networks, we use minimum constraints
    - In hierarchical networks (densification of the network) it is quite usual to fix the known points

Connecting the plain network to the reference frame

- Distances (measured with calibrated instrument) bring the scale to the network
- If we have two fixed points we have brought the orientation (and scale) to the network
- With an atzimuth observation and one fixed point we get the orientation and position
- If we have more constraints (more fixed points in network) than what is necessary, we have over constraint network.
    - Fixed points or extra constraints affect the shape and size of the network
    - Over constraint network is quite usual in In hierarcical measurements (densification of the network)

Minimum constraints with inner constraint equations



$$G^T = \begin{pmatrix} x_1 & -y_1 & 1 & 0 \\ y_1 & x_1 & 0 & 1 \\ x_2 & -y_2 & 1 & 0 \\ y_2 & x_2 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_m & -y_m & 1 & 0 \\ y_m & x_m & 0 & 1 \end{pmatrix}$$

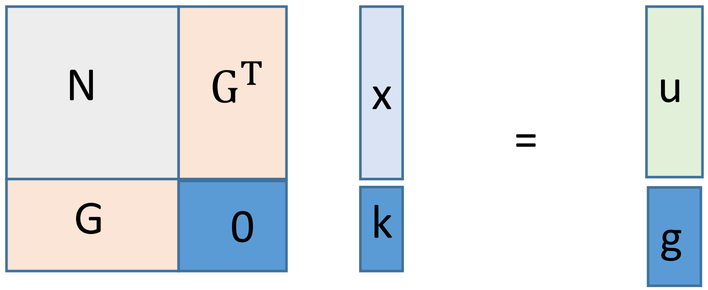
Inner constraints in 3D network

$$G^T = \begin{pmatrix} x_1 & 0 & -z_1 & y_1 & 1 & 0 & 0 \\ y_1 & z_1 & 0 & -x_1 & 0 & 1 & 0 \\ z_1 & -y_1 & x_1 & 0 & 0 & 0 & 1 \\ x_2 & 0 & -z_2 & y_2 & 1 & 0 & 0 \\ y_2 & z_2 & 0 & -x_2 & 0 & 1 & 0 \\ z_2 & -y_2 & x_2 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_m & 0 & -z_m & y_m & 1 & 0 & 0 \\ y_m & z_m & 0 & -x_m & 0 & 1 & 0 \\ z_m & -y_m & x_m & 0 & 0 & 0 & 1 \end{pmatrix}$$

scale

orientation

translation

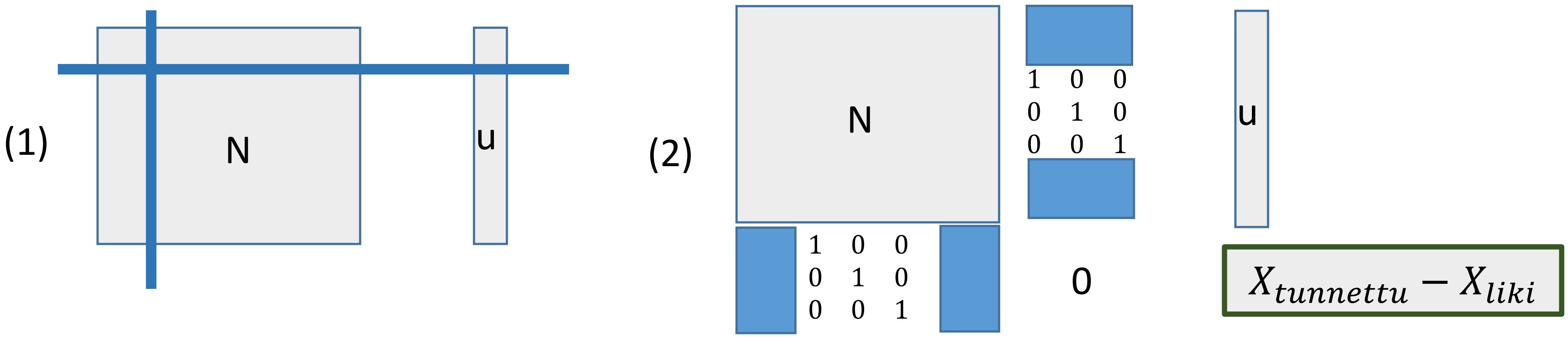


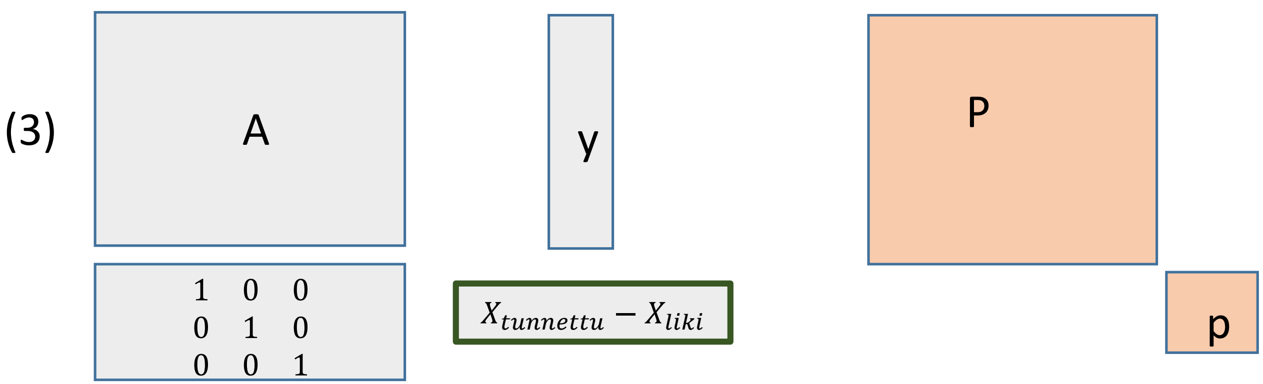
If observations already determine some of the datum elements, then the respective column in G should be removed

It is usual to apply inner constraints over part of the points (datum points)

Fixed positions

- By removing the respective rows and columns from normal matrix (1)
- By adding constraint equations in normal equationst (2)
- By using weighted parameter equations in observation equations painotettuna parametrina havaintoyhtälöihin (3)





Weighted parameters is better than removing rows and columns from N, because the constraints are possible to be remove.

Principle of stacking normal equations

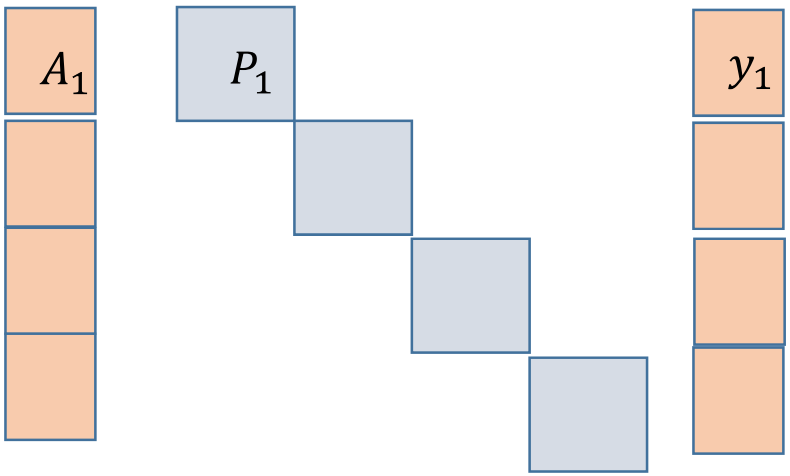
$$A^T = (A_1^T \quad A_2^T \quad \cdots \quad A_n^T)$$

$$A^T P = (A_1^T P_1 + A_2^T 0 + \cdots + A_n^T 0 \quad A_2^T P_2 \quad \cdots \quad A_n^T P_n)$$

$$A^T PA = A_1^T P_1 A_1 + A_2^T P_2 A_2 + \cdots + A_n^T P_n A_n$$

$$A^T Py = A_1^T P_1 y_1 + A_2^T P_2 y_2 + \cdots + A_n^T P_n y_n$$

A, P and y are partitioned. The Normal equation matrix and vector is a sum of the normal equations of the uncorrelated parts of observations. It is possible to update normal equation with new observations one observation in time (adding or removing)



Benifits of the stacking:

- Combination of different epochs (sub networks) in large networks is easy
- It is not necessary to have the full A-matrix, save memory space

Least squares process