MS-E2148 Dynamic optimization

Recap

Summary

- The course considered two kinds of problems:
  1. Discrete-time problems
  2. Continuous-time problems
- These were solved with
  1. Dynamic programming (DP)
  2. Calculus of variations
  3. Pontryagin's minimum principle
- HJB equation is DP algorithm in continuous-time problems
- Bellman equation is DP algorithm for ∞ horizon discounted, stationary problem
- Calculus of variations was also used in deriving the necessary conditions for the control problem
- Stochastic was only in the discrete-time problems

Calculus of variations

Increment and variation

- The increment of a functional is

$$\Delta J(x, \delta x) \equiv J(x + \delta x) - J(x) \tag{1}$$

- If the increment can be expressed with linear functional $\delta J(x, \delta x)$ in δx:

$$\Delta J(x, \delta x) = \delta J(x, \delta x) + g(x, \delta x)\|\delta x\| \tag{2}$$

  where $\lim_{\|\delta x\| \to 0}\{g(x, \delta x)\} = 0$, then J is differentiable in x and δJ is the variation of J with function/variation x

- On the course, we assume that J is differentiable in x, so the variation is linear approximation to the increment

Calculus of variations

Necessary conditions

- On the extremal x^* the variation vanishes; the necessary condition is the Euler equation

$$g_x(x^*, \dot{x}^*, t) - \frac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) = 0 \qquad \forall t \in [t_0, t_f] \tag{3}$$

- If the boundary points are fixed, i.e.,

$$x(t_0) = x_0 \qquad \text{and} \qquad x(t_f) = x_f \tag{4}$$

there are no additional conditions to Euler

Calculus of variations

Transversality conditions to complement Euler

- $x(t_f)$ free, t_f fixed:

$$g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0 \tag{5}$$

- t_f free, $x(t_f)$ fixed:

$$g(x^*(t_f), \dot{x}^*(t_f), t_f) - g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\dot{x}^*(t_f) = 0 \tag{6}$$

- $x(t_f)$ and t_f free but independent:

  (5) and (6)

- $x(t_f)$ and t_f free and $x(t_f) = \theta(t_f)$:

$$g_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\left[\dot{\theta}(t_f) - \dot{x}^*(t_f)\right] + g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0 \tag{7}$$

Calculus of variations

Weierstrass-Erdmann corner point conditions

$$g_{\dot{x}}(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-) = g_{\dot{x}}(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+)$$

and

$$\begin{aligned} & g(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-) - \left[g_{\dot{x}}(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-)\right]\dot{x}^*(t_1^-) \\ &= g(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+) - \left[g_{\dot{x}}(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+)\right]\dot{x}^*(t_1^+) \end{aligned}$$

but

$$\dot{x}^*(t_1^-) \neq \dot{x}^*(t_1^+)$$

where the corner is at $t = t_1$

Control problem

Unbounded controls

- Hamiltonian: $H(x, u, p, t) = g(x, u, t) + p^T[f(x, u, t)]$
- Necessary conditions:

$$\begin{aligned} \dot{p}^* &= -H_x(x^*, u^*, p^*, t) \\ 0 &= H_u(x^*, u^*, p^*, t) \\ \dot{x}^* &= H_p(x^*, u^*, p^*, t) \end{aligned} \quad (8)$$

and

$$\begin{aligned} &\Big[h_x(x^*(t_f), t_f) - p^*(t_f)\Big]^T \delta x_f \\ &+ \Big[H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + h_t(x^*(t_f), t_f)\Big]\delta t_f = 0 \end{aligned} \quad (9)$$

- The boundary conditions have to be derived from (9) case by case

Control problem

Bounded controls

- Necessary conditions:

$$\dot{p}^* = -H_x(x^*, u^*, p^*, t)$$
$$H(x^*, u^*, p^*, t) \leq H(x^*, u, p^*, t)$$
$$\dot{x}^* = H_p(x^*, u^*, p^*, t)$$

and complemented with (9)

Special cases

- Control problems where the final time is free and the Hamiltonian has no explicit time depend.: $H = 0$ for all t
- ∞ horizon calculus of variations: use stationary condition $\dot{x} = \ddot{x} = 0$
- Minimum time problem: use bang-bang control
- Minimum control-effort problem: use bang-off-bang control
- Singular solutions: examine if the switching function can have root for finite length time interval
- ∞ horizon, discrete-time, stationary discounted problem: Bellman equation gives the optimal J^*:

$$J^*(x) = \min_u E_w \Big\{ g(x, u, w) + \alpha J^*(f(x, u, w)) \Big\} \qquad (10)$$

Hamilton-Jacobi-Bellman equation

Cost-to-go for the continuous-time problem

$$0 = \min_{u \in U}[g(x,u) + \nabla_t J^*(t,x) + \nabla_x J^*(t,x)^T f(x,u)], \quad \forall t, x, \quad (11)$$

with boundary condition $J^*(T, x) = h(x)$

DP algorithm

The cost-to-go for discrete-time problem

- For each initial state x_0 the optimal cost $J^*(x_0)$ follows from the next state:

$$J_N(x_N) = g_N(x_N), \tag{12}$$

$$J_k(x_k) = \min_{u_k \in U_k(x_k)} E_{w_k}\left\{g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k))\right\}, \tag{13}$$

$k = 0, 1, ..., N-1$

- If $u_k^* = \mu_k^*(x_k)$ minimizes equation (13) right-hand side for each x_k and k, then the control law $\pi^* = \{\mu_0^*, ..., \mu_{N-1}^*\}$ is optimal

DP algorithm

Shortest path problem

- Deterministic problem, finite states
- Optimal cost from i to t in $N-k$ steps is

$$J_k(i) = \min_{j=1,...,N} [a_{ij} + J_{k+1}(j)], \quad k = 0, 1, ..., N-2,$$

  where $J_{N-1}(i) = a_{it}$, $i = 1, 2, ..., N$.
- Note that in above $J_{k+1}(j) =$ optimal cost from j to t in $N-k-1$ steps