MS-E2148 Dynamic optimization

Lecture 1

Topics

1. Continuous-time problem, Calculus of Variations [K4]
2. Transversality conditions, solutions with corners [K4]
3. Solving optimal control problem using CoV [K5]
4. Minimum principle [K 5], ∞-horizon problem [KS15-16]
5. Minimum time and minimum control-effort problems [K5]
6. Discrete-time problem, DP algorithm [B1.2-1.3]
7. DP and finite states, applications [B2.1-2.2, 4.4]
8. DP-applications, imperfect state info [B4.2, 4.3, 5.1]
9. Continuous-time problem revisited, HJB equation [B3.1-3.2, K3.11]
10. Stationary, discounted problems, numerical methods
11. Summary

(B = Bertsekas vol 1, K = Kirk, KS = Kamien/Schwartz)

Dynamic optimization

- Dynamic: something that changes in ”time”, sequential
- Optimization: finding the best

- We will examine two types of problems:
- Continuous-time problems (Lectures 1-5,9)
  - Calculus of Variations (CoV), Euler equation (1750)
  - Differential calculus for functionals
  - Optimal control problems (1950)
- Discrete-time problems (Lectures 6-10)
  - Dynamic Programming (DP) algorithm
  - Bellman equation (1950)

Basic problem

Bertsekas, Ch 1.1.-1.2.

- Discrete time, dynamic (stochastic) system

$$x_{k+1} = f_k(x_k, u_k, w_k) \qquad k = 0, 1, ..., N-1$$

- k is *step*, *time* or some index representing recursion
- State $x_k \in S_k$ is some relevant factor of the system
  - Initial state x_0, final state x_N
- Control $u_k \in U_k$
- Random parameter $w_k \in D_k$ can be noise, *disturbance* or *error*
- Observation $z_k = h_k(x_k, u_{k-1}, v_k)$, where v_k *observation disturbance*

Inventory control

Example

$$x_{k+1} = x_k + u_k - w_k$$

- State x_k is *stock available* at stage k
- Control u_k is *stock ordered* at stage k
- Random parameter w_k is *demand* during stage k
- Purchase cost cu_k, c unit cost
- Holding/shortage cost $r(x_k + u_k - w_k)$, r unit cost

Classification of problems

- Discrete vs. continuous time
- Discrete/finite state vs continuous/infinite state
- Finite vs. infinite time horizon

Examples

- Machine repair problem (DP)
- Brachistochrone problem (CoV)
- Goddard rocket problem (optimal control)
- Flight path/trajectory optimization
- Path from A to B in minimal time/fuel consumption
- Forestry management

Continuous-time problem

Dynamic system

$$\dot{x}(t) = f(x(t), u(t), t), \qquad 0 \le t \le T, \tag{1}$$

where initial state $x(0)$ and final time T are known

- $x(t), \dot{x}(t) \in R^n$, $u(t) \in U \subseteq R^m$, $f : R^{n+m+1} \mapsto R^n$
- The state variable x, its time derivative $\dot{x}$, and control u are vectors; the **system equations** (1) are first-order differential equations (n equations)
- Assume that f_i are continuously differentiable in x and continuous in u

Continuous-time problem

Controls

- The admissible controls are piecewise continuous functions $\{u(t) | t \in [0, T]\}$ s.t. $u(t) \in U$ for $t \in [0, T]$
- The controls are also called *control trajectories*
- Assume that the differential equations (1) have solution for any admissible control; these give the *state trajectories* (see e.g. Picard-Lindelof)

Continuous-time problem

Cost

- The objective is to find the admissible control that minimizes the cost

$$h(x(T), T) + \int_0^T g(x(t), u(t), t)dt \tag{2}$$

- g is the (instant) cost function, h is the final/terminal cost
- Functions h and g are continuously differentiable in x and g is continuous in u

Continuous-time problem

Example

- Mass $m = 1$ moves on a line under force u
- $x_1(t)$ is the location of the mass, $x_2(t)$ is its speed at time t
- The problem is to move the mass from given $(x_1(0), x_2(0))$ near $(\bar{x}_1, \bar{x}_2)$ at time T, and minimize

$$|x_1(T) - \bar{x}_1|^2 + |x_2(T) - \bar{x}_2|^2 \quad (3)$$

- so that the control is restricted to $|u(t)| \leq 1, \ \forall t \in [0, T]$

Continuous-time problem

Example

- Dynamic system (1) is

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= u(t)\end{aligned}$$

and cost (2)

$$\begin{aligned}h(x(T), T) &= |x_1(T) - \bar{x}_1|^2 + |x_2(T) - \bar{x}_2|^2\\ g(x(t), u(t), t) &= 0, \qquad \forall t \in [0, T]\end{aligned}$$

Calculus of variations

- Local analysis of functionals

  $\Rightarrow$ Try to find functions that minimize or maximize some functional locally

- We do not consider control problems yet; they will be handled as an extension later (constrained problem)
- Functionals for the basic problem in calculus of variations are of the form

$$J(x) = \int_{t_0}^{t_f} g(x, \dot{x}, t) dt$$

Calculus of variations

Normed vector space

- X is an arbitrary normed function space, i.e., vector space whose elements are functions $x \in X$
- Norm $\|x\|$ satisfies:
  1. $\|x\| \geq 0$ and $\|x\| = 0$ iff $x(t) = 0, \quad \forall t$
  2. $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all scalars α
  3. $\|x + y\| \leq \|x\| + \|y\|$
- Distance between two functions: $\|x(t) - y(t)\|$
- For example, $\|x\| = \max_{t_0 \leq t \leq t_f} |x(t)|$

Calculus of variations

Functional

- Cost $J : X \mapsto R$ is a functional that maps (vector-valued) function $x(t) \in X$ into a real number
- evaluates how good $x(t)$ is

- Linear functional is
  - Homogenous: $J(\alpha x) = \alpha J(x)$
  - Additive: $J(x + y) = J(x) + J(y)$

- For example:
  - $\int_{t_0}^{t_f} x dt$ is linear
  - $\int_{t_0}^{t_f} x^2 dt$ is not linear

Calculus of variations

Increment of a functional

- The **increment** of a functional is

$$\Delta J(x, \delta x) \equiv J(x + \delta x) - J(x) \tag{4}$$

  where δx is a variation of x

- E.g. the increment of functional $J(x) = \int_{t_0}^{t_f} x^2 dt$ is

$$\begin{aligned} \Delta J &= J(x + \delta x) - J(x) \\ &= \int_{t_0}^{t_f} (x + \delta x)^2 dt - \int_{t_0}^{t_f} x^2 dt \\ &= \int_{t_0}^{t_f} \left(2x\delta x + (\delta x)^2\right) dt \end{aligned}$$

Calculus of variations

Variation of a functional

- The variation of a functional has the same role as differential of a function in the study of extremes
- If the increment (4) of a functional as a function of δx can be written with the linear functional $\delta J(x, \delta x)$:

$$\Delta J(x, \delta x) = \delta J(x, \delta x) + z(x, \delta x)\|\delta x\| \tag{5}$$

where $\lim_{\|\delta x\| \to 0}\{z(x, \delta x)\} = 0$, then J is differentiable in x and δJ is called the (first) **variation** of J with function x

Calculus of variations

Variation of a functional: example

- The increment of functional $J = \int_0^1 (x^2 + 2x)dt$ is

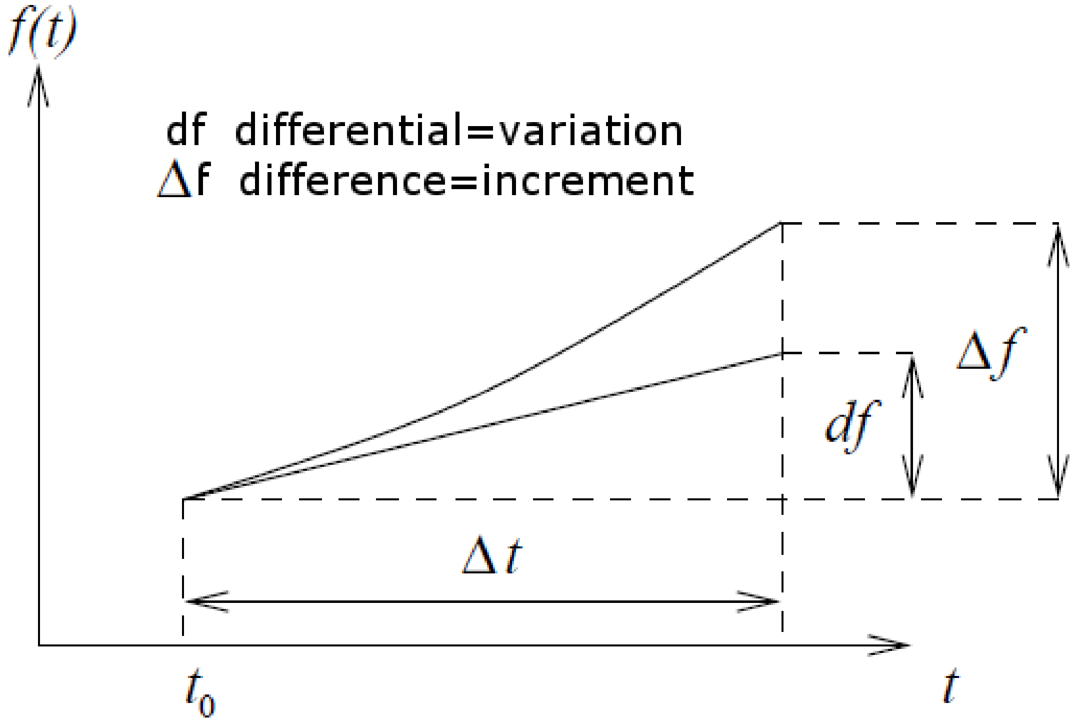
$$\begin{aligned} \Delta J(x, \delta x) &= J(x + \delta x) - J(x) \\ &= \int_0^1 \left((x + \delta x)^2 + 2(x + \delta x)\right) dt - \int_0^1 (x^2 + 2x)dt \\ &= \int_0^1 \left((2x + 2)\delta x\right) dt + \int_0^1 (\delta x)^2 dt \end{aligned}$$

- The second term can be written in form $g(x, \delta x)\|\delta x\|$ and $\lim_{\|\delta x\| \to 0} \left\{g(x, \delta x)\|\delta x\|\right\} = 0$, the variation of a functional J is then

$$\delta J(x, \delta x) = \int_0^1 \left((2x + 2)\delta x\right) dt$$

Calculus of variations

Analogy between variation and differential



- Variation δJ is a *linear approximation* for the difference of the functional values of two functions x and $x + \delta x$
- If $\|\delta x\|$ is small (the functions are close to each other), variation is good approximation for the increment

- Functional J has a local extremum x^* if there is $\epsilon > 0$ so that for all x with $\|x - x^*\| < \epsilon$, the increment of the functional has the same sign

- If $\Delta J = J(x) - J(x^*) \geq 0$, it is a local minimum
- If $\Delta J = J(x) - J(x^*) \leq 0$, it is a local maximum

- If x^* is an extremum, $J(x^*)$ is the extremal value

Calculus of variations

Fundamental theorem

- If x^* is extremum, the variation of J vanishes, i.e.,

$$\delta J(x^*, \delta x) = 0, \qquad \text{for all} \quad \delta x \in X \tag{6}$$

- Proof: Kirk Section 4.1

Calculus of variations

Basic problem

- Let us derive the necessary condition for the variation to vanish in the basic problem
- Let x be a scalar function, and $g(x, \dot{x}, t)$ is twice continuously differentiable both in x and $\dot{x}$
- We try to find on a closed interval $[t_0, t_f]$ the x that satisfies the boundary conditions $x(t_0) = x_0$ and $x(t_f) = x_f$ and is a local extremum for the functional

$$J(x) = \int_{t_0}^{t_f} g(x, \dot{x}, t) dt. \tag{7}$$

- Let us formulate a condition for the vanishing of variational on the optimal x with the help of the increment:

$$\begin{aligned} \Delta J(x, \delta x) &= J(x + \delta x) - J(x) \\ &= \int_{t_0}^{t_f} g(x + \delta x, \dot{x} + \delta \dot{x}, t)dt - \int_{t_0}^{t_f} g(x, \dot{x}, t)dt \end{aligned} \tag{8}$$

- What is the first term on the right side? Let us use the Taylor expansion:

$$\begin{aligned} g(x + \delta x, \dot{x} + \delta \dot{x}, t) &= g(x, \dot{x}, t) + g_x(x, \dot{x}, t)\delta x \\ &\quad + g_{\dot{x}}(x, \dot{x}, t)\delta \dot{x} + O(\delta x, \delta \dot{x}) \end{aligned} \tag{9}$$

- In (9) it holds for the higher order terms $O(\delta x, \delta \dot{x}) \to 0$ when $\|\delta x\|, \|\delta \dot{x}\| \to 0$
- Let us substitute (9) into the increment (8) and take the terms that are linear in δx and $\delta \dot{x}$:

$$\delta J(x, \delta x) = \int_{t_0}^{t_f} \left[g_x(x, \dot{x}, t)\delta x + g_{\dot{x}}(x, \dot{x}, t)\delta \dot{x} \right] dt \qquad (10)$$

- Note that

$$\delta x = \int_{t_0}^{t} \delta \dot{x} dt + \delta x(t_0)$$

- The variation can be expressed in the following form with partial integration:

$$\begin{aligned} \delta J(x, \delta x) &= g_{\dot{x}}(x, \dot{x}, t)\delta x \Big|_{t_0}^{t_f} \\ &+ \int_{t_0}^{t_f} \left[g_x(x, \dot{x}, t) - \frac{d}{dt} g_{\dot{x}}(x, \dot{x}, t) \right] \delta x dt \end{aligned} \qquad (11)$$

- The first term of the right-hand side of equation (11) vanishes, since all x must satisfy the boundary conditions: $\delta x(t_0) = \delta x(t_f) = 0$

Calculus of variations

Basic problem

- According to the fundamental theorem, we need $\delta J(x, \delta x) = 0$ for all δx; what does this mean for the integral term in equation (11)?
- **Fundamental lemma**: if it holds for a continuous function $h(t)$ that $\int_0^T h(t)\delta x(t)dt = 0$ for arbitrary $\delta x(t)$ and $\delta x(0) = \delta x(T) = 0$, then $h(t) = 0$ for all $t \in [0, T]$
- Thus, we get the necessary condition for the extremal x^* based on the fundamental lemma

$$g_x(x^*, \dot{x}^*, t) - \frac{d}{dt} g_{\dot{x}}(x^*, \dot{x}^*, t) = 0 \qquad \forall t \in [t_0, t_f] \tag{12}$$

Calculus of variations

Basic problem

- Equation (12) is the **Euler-Lagrange** or Euler equation
- The total derivative(*) $\frac{d}{dt}$ means that there can be terms of $\ddot{x}$
- Euler equation is a second-order differential equation and usually nonlinear and time dependent
- Caratheodory developed sufficient condition for CoV in 1920/30s

- (*) $\frac{d}{dt}f(t, x(t)) = f_t + f_x\dot{x}$

Example

- The inventory size is $x(t)$, production speed $\dot{x}(t) \geq 0$
- The cost consists of production and inventory costs:

$$J(x, \dot{x}) = \int_0^T \left[C_1 \dot{x}^2 + C_2 x \right] dt \tag{13}$$

- Boundary conditions: $x(0) = 0$, $x(T) = B$
- What is the optimal inventory size $x^*(t)$?

Example

- Now $g(x, \dot{x}, t) = C_1\dot{x}^2 + C_2 x$, and

$$g_x = C_2, \qquad g_{\dot{x}} = 2C_1\dot{x}, \qquad \frac{d}{dt}g_{\dot{x}} = 2C_1\ddot{x}$$

- Euler:

$$\begin{aligned} g_x - \frac{d}{dt}g_{\dot{x}} = C_2 - 2C_1\ddot{x} = 0 \\ \Rightarrow \ddot{x} = \frac{C_2}{2C_1} \end{aligned}$$

- This is non-homogeneous (right-hand side not zero), linear second-order differential equation with constant coefficients (not time dependent), but a special case since the right-hand side is a constant

Example

- We get a candidate for the extremum by integrating twice:

$$x^*(t) = \frac{C_2}{4C_1}t^2 + K_1 t + K_2$$

where K_1 and K_2 are integration constants that are solved from the boundary conditions:

$$\begin{aligned} x(0) = 0 &\quad \Rightarrow \quad K_2 = 0 \\ x(T) = B &\quad \Rightarrow \quad \tfrac{C_2}{4C_1}T^2 + K_1 T = B \quad \Rightarrow \quad K_1 = \tfrac{B}{T} - \tfrac{C_2}{4C_1}T \end{aligned}$$

- Thus, the optimal invetory is

$$x^*(t) = \frac{C_2}{4C_1}t(t - T) + \frac{Bt}{T}$$

that satisfies the constraint $\dot{x}^* \geq 0$ if $B \geq \frac{C_2 T^2}{4C_1}$

Example

- We found a solution to the problem whose optimality depends on if B is big enough compared to T and if the inventory cost C_2 is small enough to the production cost C_1

- The necessary condition $2C_1\ddot{x} = C_2$ can be interpreted:
  - $C_1\dot{x}^2$ is the production cost at time t, i.e., $2C_1\dot{x}$ is the marginal production cost, and $2C_1\ddot{x}$ is its rate of change
  - ... which needs to be in balance with the marginal inventory cost C_2

Euler equation

Functional with many independent functions

- The Euler equation for a functional of a form

$$J(x_1, ..., x_n) = \int_{t_0}^{t_f} g(x_1, ..., x_n, \dot{x}_1, ..., \dot{x}_n, t)dt$$

with boundary conditions $x_i(t_0) = x_{i0}$, $x_i(t_f) = x_{if}$, $\forall i = 1, ..., n$, generalizes to the system of n Euler equations:

$$g_{x_i}(x_1^*, ..., x_n^*, \dot{x}_1^*, ..., \dot{x}_n^*, t) - \frac{d}{dt} g_{\dot{x}_i}(x_1^*, ..., x_n^*, \dot{x}_1^*, ..., \dot{x}_n^*, t) = 0$$

for all $i = 1, ..., n$

Euler equation

Special cases

1) If $g = g(\dot{x})$, i.e., it depends only on $\dot{x}$, the Euler equation is:

$$g_{\dot{x}\dot{x}}\ddot{x}^* = 0$$

either $g_{\dot{x}\dot{x}} = 0$ or $\dot{x}^* = C$. If g is linear in $\dot{x}$, Euler is an identity that is satisfied for all x^*

2) If $g = g(\dot{x}, t)$, i.e., there is no dependency on x, we get

$$g_{\dot{x}} = C$$

- E.g.: $g = 3\dot{x} - t\dot{x}^2$, Euler: $3 - 2t\dot{x} = C \Rightarrow \dot{x} = \frac{3-C}{2t}$ which is solved with one integration

Euler equation

Special cases

3) If $g = g(x, \dot{x})$, i.e., there is no dependency on t, we get:

$$g - \dot{x}g_{\dot{x}} = C$$

which can be solved with one integration (e.g. Brachistochrone problem in exercises)

4) If $g = g(x, t)$, i.e., no dependency on $\dot{x}$, we get:

$$g_x = 0$$

which is not a differential equation and the solution does not contain integration constant. Then x^* is a solution only if it happens to satisfy the boundary conditions.

Euler equation

Special cases

5) If g is linear in $\dot{x}$, i.e., $g = a(x,t) + b(x,t)\dot{x}$, Euler is:

$$a_x(x,t) = b_t(x,t)$$

which is not a differential equation. Then x^* may satisfy the boundary conditions and Euler. Usually this is not the case, and Euler is identity that is satisfied by any x^*.

- E.g.: $g = \dot{x}$, Euler: $0 = 0$ that is satisfied by all x^*

Summary

- Refresh: differential equations, partial derivatives, Taylor expansion, partial integration etc.
- Functional
- Increment
- Variation
- Fundamental theorem in calculus of variations
- Euler equation