MS-E2148 Dynamic optimization

Lecture 3

Contents

- We derive the necessary conditions for the optimal control problem from the basic problem of calculus of variations
- Material Kirk 5

Recap

- We derived the transversality and corner point conditions for the basic problem in calculus of variations

Calculus of variations

Problem with differential equation constraints

$$J(x) = \int_{t_0}^{t_f} g(x, \dot{x}, t)dt \tag{1}$$

where the variables are constrained to $f(x, \dot{x}, t) = 0$

- Elimination of variables is not possible in general, so we form an extended functional

$$\tilde{J}(x, p) = \int_{t_0}^{t_f} \left[g(x, \dot{x}, t) + p^T [f(x, \dot{x}, t)] \right] dt \tag{2}$$

where $p \equiv p(t)$ are the *Lagrange multipliers* or *costate/adjoint variables*

Calculus of variations

Differential equation constraints

- The variation of the extended functional (2) is

$$\delta\tilde{J}(x,\delta x,p,\delta p) = \int_{t_0}^{t_f}\Big\{\Big[g_x(x,\dot{x},t)+p^T[f_x(x,\dot{x},t)] - \frac{d}{dt}\Big(g_{\dot{x}}(x,\dot{x},t)+p^T[f_{\dot{x}}(x,\dot{x},t)]\Big)\Big]\delta x + f(x,\dot{x},t)\delta p\Big\}dt$$

- It must hold on the extremal that $\delta\tilde{J}(x^*,p)=0$ and the differential equations must be satisfied, i.e., $f(x^*,\dot{x}^*,t)=0$, which means that the multiplier of δp in the variation must be zero

Calculus of variations

Differential equation constraints

- In general, x is a vector of length $(n+m)$, but there are n differential equation constraints, so p is vector of length n
- Thus, with suitable choice of p, we can only make n of the multipliers of δx as zero in $\delta \tilde{J}$, and this leaves m multipliers of δx
- We require all the multipliers of δx as zero on the interval $[t_0, t_f]$:

$$\begin{aligned} 0 &= g_x(x^*, \dot{x}^*, t) + p^T[f_x(x^*, \dot{x}^*, t)] \\ &\quad - \frac{d}{dt}\left(g_{\dot{x}}(x^*, \dot{x}^*, t) + p^T[f_{\dot{x}}(x^*, \dot{x}^*, t)]\right) \end{aligned}$$

These are Euler equations for the extended integrand $\tilde{g} \equiv g + pf$

Calculus of variations

Differential equation constraints

- **E.g.**: $J(x) = \int_{t_0}^{t_f} \frac{1}{2}[x_1^2 + x_2^2]dt$, where $\dot{x}_1 = x_2$
- It has $n = 1$ differential equations, p is of length $n = 1$, x is of length $n + m = 1 + 1 = 2$
- Extended integrand: $\tilde{g} = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + px_2 - p\dot{x}_1$
- Euler for the extended integrand:

$$x_1^* + \dot{p}^* = 0$$
$$x_2^* + p^* = 0$$

and it should also satisfy $\dot{x}_1^* = x_2^*$

Optimal control problem

- Let us find the admissible control u^* that makes the following system
$$\dot{x} = f(x, u, t) \tag{3}$$
to follow the feasible state trajectory x^* that minimizes the cost
$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, u, t)dt \tag{4}$$
- In general, x is a vector of length n and u is vector of length m
- Assume that t_0 and $x(t_0) = x_0$ are fixed
- Here, u is assumed smooth but this is later extended using maximum principle

Optimal control problem

Difference to basic problem of calculus of variations

- In (4) there is an extra term to the basic problem of calculus of variations

$$h(x(t_f), t_f) = \int_{t_0}^{t_f} \frac{d}{dt} h(x, t) dt + h(x(t_0), t_0)$$

which can be included in the integrand (4):

$$J(u) = \int_{t_0}^{t_f} \left[g(x, u, t) + \frac{d}{dt} h(x, t) \right] dt + h(x(t_0), t_0)$$

- Since $x(t_0)$ and t_0 are fixed, it is enough to minimize the functional

$$J(u) = \int_{t_0}^{t_f} \left[g(x, u, t) + \frac{d}{dt} h(x, t) \right] dt \tag{5}$$

Optimal control problem

Extended functional

- The total derivative can be simplified in (5):

$$J(u) = \int_{t_0}^{t_f} \Big[g(x,u,t) + h_x(x,t)^T \dot{x} + h_t(x,t) \Big] dt \qquad (6)$$

- The optimal control problem has also a differential equation constraint (3); it can be included into the objective using the extended functional:

$$\begin{aligned} \tilde{J}(u) \quad = \quad & \textstyle\int_{t_0}^{t_f} \Big[g(x,u,t) + h_x(x,t)^T \dot{x} + h_t(x,t) \\ & + p^T [f(x,u,t) - \dot{x}] \Big] dt \end{aligned} \qquad (7)$$

  where $p \equiv p(t)$ is n vector of Lagrange multipliers or costate variables

Optimal control problem
Extended integrand

- Let us write the extended integrand:

$$\tilde{g}(x, \dot{x}, u, p) \equiv g(x, u, t) + p^T[f(x, u, t) - \dot{x}] + h_x(x, t)^T \dot{x} + h_t(x, t) \tag{8}$$

so we get that (7) is

$$\tilde{J}(u) = \int_{t_0}^{t_f} \tilde{g}(x, \dot{x}, u, p, t) dt \tag{9}$$

- To derive the necessary conditions, let us form the variation of $\delta \tilde{J}(u)$ that depends linearly on variations δx, $\delta \dot{x}$, δu, δp, and δt_f
- Assume the boundary values as fixed or free

Optimal control problem

Variation

$$
\begin{aligned}
\delta\tilde{J}(u^*) \quad = \quad & \tilde{g}_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f)^T \delta x_f \\
+ \quad & \Big[\tilde{g}(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f) \\
- \quad & \tilde{g}_{\dot{x}}(x^*(t_f), \dot{x}^*(t_f), u^*(t_f), p^*(t_f), t_f)^T \dot{x}^*(t_f)\Big] \delta t_f \\
\\
+ \quad & \int_{t_0}^{t_f} \Big[\Big(\tilde{g}_x(x^*, \dot{x}^*, u^*, p^*, t)^T \\
& \quad - \quad \frac{d}{dt}\tilde{g}_{\dot{x}}(x^*, \dot{x}^*, u^*, p^*, t)^T\Big)\delta x \\
& \quad + \quad \tilde{g}_u(x^*, \dot{x}^*, u^*, p^*, t)^T \delta u \\
& \quad + \quad \tilde{g}_p(x^*, \dot{x}^*, u^*, p^*, t)^T \delta p\Big] dt
\end{aligned}
$$

Optimal control problem

Integral term of variation

- The equation corresponding to Euler would be to require that the integrand $\delta\tilde{J}(u^*)$ is zero
- Based on (8) it has function h whose effect vanishes in the integrand on the extremal (as long as h is twice continuously differentiable):

$$\partial_x[h_x(x^*,t)^T\dot{x}^* + h_t(x^*,t)] - \tfrac{d}{dt}\partial_{\dot{x}}[h_x(x^*,t)^T\dot{x}^*]$$
$$= h_{xx}(x^*,t)\dot{x}^* + h_{tx}(x^*,t) - h_{xx}(x^*,t)\dot{x} - h_{xt}(x^*,t) = 0$$

- The integrand for the variation is then

$$\int_{t_0}^{t_f}\Big[\Big(g_x(x^*,u^*,t) + p^*[f_x(x^*,u^*,t)] - \tfrac{d}{dt}(-p^*)\Big)^T\delta x$$
$$+\Big(g_u(x^*,u^*,t) + p^*[f_u(x^*,u^*,t)]\Big)^T\delta u + \Big(f(x^*,u^*,t) - \dot{x}^*\Big)^T\delta p\Big]d$$

Optimal control problem

Necessary conditions

- Integrand vanishes on the extremal if the multipliers of δx, δu and δp are zero, i.e.,

$$
\begin{aligned}
\dot{p}^* &= -p^*[f_x(x^*, u^*, t)] - g_x(x^*, u^*, t) \\
0 &= g_u(x^*, u^*, t) + p^*[f_u(x^*, u^*, t)] \\
\dot{x}^* &= f(x^*, u^*, t)
\end{aligned}
\tag{10}
$$

- On the extremal, the other terms need to vanish in the variation $\delta \tilde{J}$

$$
\begin{aligned}
&\Big[h_x(x^*(t_f), t_f) - p^*(t_f)\Big]^T \delta x_f + \Big[g(x^*(t_f), u^*(t_f), t_f) \\
&+ h_t(x^*(t_f), t_f) + p^*(t_f)(f(x^*(t_f), u^*(t_f), t_f)\Big]\delta t_f = 0
\end{aligned}
\tag{11}
$$

Optimal control problem

Necessary conditions and Hamiltonian

- From equation (11) we can derive the transversality conditions for the control problem as before
- We can simplify the equations (10) and (11) using the *Hamiltonian*:

$$H(x, u, p, t) = g(x, u, t) + p \cdot f(x, u, t) \tag{12}$$

Optimal control problem

Necessary conditions and Hamiltonian

- The equations (10) are then: for all $t \in [t_0, t_f]$

$$\begin{aligned} \dot{p}^* &= -H_x(x^*, u^*, p^*, t) \\ 0 &= H_u(x^*, u^*, p^*, t) \\ \dot{x}^* &= H_p(x^*, u^*, p^*, t) \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\Big[h_x(x^*(t_f), t_f) - p^*(t_f)\Big]^T \delta x_f \\ &+\Big[H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + h_t(x^*(t_f), t_f)\Big]\delta t_f \quad = \quad 0 \end{aligned} \tag{14}$$

Optimal control problem

Example

- Let us optimize the inventory where x is the amount of product at stage t and u is the production speed, $\dot{x} = u$
- The costs are

$$J(u) = \int_0^T \left[C_1 u^2 + C_2 x \right] dt$$

and boundary conditions $x(0) = 0$, $x(T) = B$

- Hamiltonian: $H(x, u, p, t) = C_1 u^2 + C_2 x + pu$
- Necessary conditions:

$$\begin{array}{lllll} H_u = 0 & \Rightarrow & 2C_1 u^* + p^* = 0 & \Rightarrow & u^* = -\frac{p^*}{2C_1} \\ H_x = -\dot{p}^* & \Rightarrow & C_2 = -\dot{p}^* & \Rightarrow & p^* = -C_2 t + K_1 \end{array}$$

Optimal control problem

Example

- We get

$$x^* = -\frac{1}{2C_1}\Big[-\frac{C_2}{2}t^2 + K_1 t + K_2\Big]$$

and the integration constants are solved using the boundary conditions:

$$\begin{array}{lll} x(0) = 0 & \Rightarrow & K_2 = 0 \\ x(T) = B & \Rightarrow & 2C_1 B = \frac{C_2}{2}T^2 - K_1 T \quad \Rightarrow \quad K_1 = \frac{C_2}{2}T - \frac{2C_1 B}{T} \end{array}$$

Optimal control problem

- The two first equations in (13) are called *costate equations* and *stationary condition*
- Conditions can be generalized to vector valued x, u, and p
- Transversality conditions can be derived for each case from (14)
- These are necessary conditions
- Constrained controls are not examined yet; we will derive *minimum principle* for them
- See also viscosity solutions examined in 1980s

Optimal control problem

Sufficient conditions

- The above conditions are also sufficient for minimality (maximality) if the function f and the integrand g are convex (concave) in x and u and $p \geq 0$
- Usually the controls are constrained, and then the sufficiency should be examined case by case

Optimal control problem

Second-order conditions

- We can derive the second-order condition for the minimum/maximum based on convexity/concavity

$$\begin{aligned} H_{uu} &\geq 0 \qquad \text{minimum} \\ H_{uu} &\leq 0 \qquad \text{maximum} \end{aligned}$$

- This condition cannot be used in all cases (e.g., constrained controls)
- Legendre-Clebsch condition

Transversality conditions in optimal control

Free final time

- In equation (14) $\delta x_f = 0$, but δt_f is arbitrary, so we must have

$$H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + h_t(x^*(t_f), t_f) = 0 \tag{15}$$

Transversality conditions in optimal control

Free final state

- In equation (14) $\delta t_f = 0$, but δx_f is arbitrary, so we must have
$$h_x(x^*(t_f), t_f) - p^*(t_f) = 0 \tag{16}$$

Transversality conditions in optimal control

Free final state and time

- In equation (14) both δt_f and δx_f are arbitrary, so we must have

$$\begin{aligned} &h_x(x^*(t_f), t_f) - p^*(t_f) = 0 \\ &H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + h_t(x^*(t_f), t_f) = 0 \end{aligned} \tag{17}$$

Transversality conditions in optimal control

Final state and time are dependent

- Now, $x(t_f) = \theta(t_f)$ and δx_f depends on δt_f:

$$\delta x_f = \dot{\theta}(t_f)\delta t_f$$

- In condition (14), we require that the multiplier of δt_f is zero

$$\begin{aligned} H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + h_t(x^*(t_f), t_f) & \\ + \Big[h_x(x^*(t_f), t_f) - p^*(t_f)\Big]\dot{\theta}(t_f) &= 0 \end{aligned} \tag{18}$$

Transversality conditions in optimal control
Example

- Let us maximize $J(u) = \int_0^1 (x+u)dt$ with condition $\dot{x} = 1 - u^2$ and $x(0) = 1$, free final state
- Hamiltonian: $H(x, u, p, t) = x + u + p(1 - u^2)$
- Stationary condition: $H_u = 1 - 2pu = 0 \Rightarrow u^* = 1/(2p^*)$
- Costate equation: $\dot{p} = -H_x = -1$
- Transversality condition: $p^*(1) = 0$
- Let us integrate the costate equation and use the transversality condition: $p^* = 1 - t$, and thus $u^* = 1/(2 - 2t)$
- 2nd order condition: $H_{uu} = -2(1-t) \leq 0$ for all $t \in [0, 1]$, i.e., u^* gives a maximum

Transversality conditions in optimal control

Example of vector-valued x and p

- Let us minimize $J(u) = \int_0^2 u^2/2dt$ with constraint $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_2 + u$
- Hamiltonian: $H = u^2/2 + p_1 x_2 - p_2 x_2 + p_2 u$
- Necessary conditions: stationary condition

$$0 = H_u = u^* + p_2^* \Rightarrow u^* = -p_2^*$$

and costate equations:

$$\begin{aligned} \dot{p}_1^* &= -H_{x_1} = 0 \\ \dot{p}_2^* &= -H_{x_2} = -p_1^* + p_2^* \end{aligned}$$

Transversality conditions in optimal control

Example of vector-valued x and p

- Let us substitute the optimal control u^* from the stationary condition into the costate equations, we get total of four first-order differential equations, so we need four boundary conditions for solving the integration constants

1) If the boundary points are fixed, the boundary conditions are $x(t_0) = x_0$ and $x(t_f) = x_f$
2) What if the boundary points are free? We require that

$$0 = h_{x_1}(x_1^*(t_f), t_f) - p_1^*(t_f) = -p_1^*(t_f)$$
$$0 = h_{x_2}(x_2^*(t_f), t_f) - p_2^*(t_f) = -p_2^*(t_f)$$

that replace the two equations

Summary

- Differential equation constraint
- Optimal control problem
- Necessary conditions for optimal control
- Hamiltonian