

MS-E2148 Dynamic optimization

Lecture 5

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Recap

- ▶ Infinite horizon problems
- ▶ Minimum principle in solving control problems

Minimum-time problem

- ▶ The task is to get the state into a set $G(t)$ in minimum time:

$$\min J = \int_{t_0}^{t_f} dt, \quad \dot{x} = f(x, u, t), \quad x(t_0) = x_0, \quad x(t_f) \in G(t_f)$$

when the controls are bounded: $|u| \leq 1$

- ▶ Due to bounded controls, it may be that there is no minimum time for all initial states (the goal set runs away)
- ▶ *Reachable states* $R(t)$ are the states that can be reached at time t from the initial state x_0 using admissible controls
- ▶ See controllability and observability of (linear) systems

Minimum-time problem

Example of reachable states

- ▶ The system $\dot{x} = u$, with constraint $|u| \leq 1$
- ▶ The solution candidates are of the form $x = x_0 + \int_{t_0}^t u(\tau) d\tau$
- ▶ With admissible controls

$$x_0 - (t - t_0) \leq x \leq x_0 + (t - t_0)$$

so the set of reachable states increases as a function of t

- ▶ If the goal set and the reachable states do not have a common state, there is no optimal control for the minimum-time problem

Minimum-time problem

- ▶ Let us examine the problem

$$\min \int_{t_0}^{t_f} dt, \quad \dot{x} = a(x, t) + B(x, t)u \quad (1)$$

where a is a vector and B is a suitable size matrix that depends explicitly on state and time

- ▶ The controls are bounded: $M^- \leq u \leq M^+$
- ▶ The Hamiltonian is

$$H(x, u, p, t) = 1 + p^T[a(x, t) + B(x, t)u] \quad (2)$$

Minimum-time problem

- ▶ Minimum principle:

$$\begin{aligned} 1 + p^{*T}[a(x^*, t) + B(x^*, t)u^*] &\leq 1 + p^{*T}[a(x^*, t) + B(x^*, t)u] \\ \Rightarrow p^{*T}B(x^*, t)u^* &\leq p^{*T}B(x^*, t)u \end{aligned}$$

- ▶ Since $B = [b_1, b_2, \dots, b_m]$ is a matrix and u is a vector, the term $p^*B(x^*, t)u$ can be written

$$p^{*T}B(x^*, t)u = \sum_{i=1}^m p^{*T}b_i(x^*, t)u_i \quad (3)$$

Minimum-time problem

- ▶ We can solve the optimal control from the minimum principle:

$$u_i^* = \begin{cases} M_i^+, & s_i(t) < 0 \\ \text{undefined}, & s_i(t) = 0 \\ M_i^-, & s_i(t) > 0 \end{cases}$$

- ▶ This is called the *bang-bang* control

Minimum-time problem

Switching function and singular interval

- ▶ We denote the multiplier of the linear term of the control, $s(t) = p^{*T} B(x^*, t)$, in the Hamiltonian and it is called the *switching function* (multiplier of u in Hamiltonian)
- ▶ The sign of the switching function tells which end of the bang-bang control is used
- ▶ Time interval $t \in [t_1, t_2]$, $t_2 > t_1$, where $s(t) \equiv 0$ is a *singular interval* – this is an interval where the necessary condition does not give a unique solution to u^*

Minimum-time problem

Example

- ▶ Let us examine a rocket landing on the surface of Mars:

$$m(t)\ddot{x} = -gm(t) + T(t) = -gm(t) - k\dot{m}(t)$$

- ▶ The thrust $T(t)$ is proportional to the usage of fuel ("mass flow")
- ▶ x = distance from the surface; $x(t_0)$ = distance initially
- ▶ We can write this as a first-order system, where $x_1 = x$ and $u = \dot{m}$:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -g - ku/x_3$$

$$\dot{x}_3 = u$$

Minimum-time problem

Example

- ▶ Objective: land in minimum time $\int_{t_0}^{t_f} dt$, where t_f is free
- ▶ Hamiltonian:

$$H(x, u, p) = 1 + p_1 x_2 - g p_2 - p_2 \frac{ku}{x_3} + p_3 u$$

- ▶ Minimum principle:

$$\left(-p_2^* \frac{k}{x_3^*} + p_3^* \right) u^* \leq \left(-p_2^* \frac{k}{x_3^*} + p_3^* \right) u$$

- ▶ Switching function: $s(t) = -p_2^* \frac{k}{x_3^*} + p_3^*$

Minimum-time problem

Example

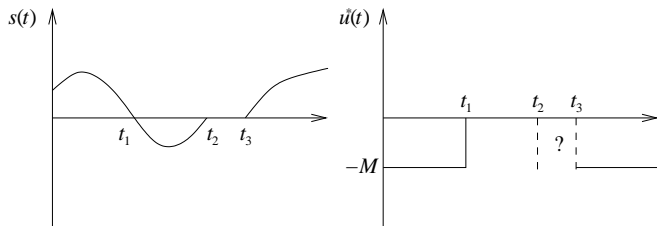
- ▶ Control $u = \dot{m}$ is bounded between $[-M, 0]$ and by the bang-bang principle

$$u^* = \begin{cases} 0, & s(t) < 0 \\ \text{undefined}, & s(t) = 0 \\ -M, & s(t) > 0 \end{cases}$$

- ▶ The roots of the switching function $s(t)$ give when the control jumps between the extreme controls

Minimum-time problem

Example



- ▶ If $s(t) = 0$ for some time interval $[t_2, t_3]$, the control is undefined and we say it is a singular interval; we should always check the possibility of singular solutions

Minimum control-effort problems

- ▶ Let us examine the problem

$$\min \int_{t_0}^{t_f} \left[\sum_{i=1}^m |u_i| \right] dt, \quad \dot{x} = a(x, t) + B(x, t)u$$

where the system is the same as in the minimum-time problem

- ▶ The controls are bounded $|u| \leq 1$
- ▶ The components of the control vector are also independent; i.e., the choice of u_i does not influence u_j , $i \neq j$

Minimum control-effort problems

- ▶ Hamiltonian:

$$H(x, u, p, t) = \sum_{i=1}^m |u_i| + p^T [a(x, t) + B(x, t)u] \quad (4)$$

- ▶ Minimum principle:

$$\sum_{i=1}^m |u_i^*| + p^{*T} B(x^*, t)u^* \leq \sum_{i=1}^m |u_i| + p^{*T} B(x^*, t)u$$

which can be written in form (see equation (3))

$$|u_i^*| + p^{*T} b_i(x^*, t)u_i^* \leq |u_i| + p^{*T} b_i(x^*, t)u_i \quad i = 1, 2, \dots, m$$

Minimum control-effort problems

- ▶ It is not yet possible to derive the optimal control from the minimum principle because of the nonlinear absolute value
- ▶ By the properties of absolute value, we can write

$$|u_i| + p^{*T} b_i(x^*, t) u_i = \begin{cases} (1 + s_i(t)) u_i, & u_i \geq 0 \\ (-1 + s_i(t)) u_i, & u_i \leq 0 \end{cases}$$

where the components of the switching function are
 $s_i(t) = p^{*T} b_i(x^*, t)$

Minimum control-effort problems

- ▶ The optimal control is

$$u_i^* = \begin{cases} 1, & s_i(t) < -1 \\ 0, & -1 < s_i(t) < 1 \\ -1, & s_i(t) > 1 \\ \text{undefined but } \geq 0, & s_i(t) = -1 \\ \text{undefined but } \leq 0, & s_i(t) = 1 \end{cases}$$

- ▶ This is called *bang-off-bang* control

Minimum control-effort problems

Example

- ▶ System is $\dot{x} = -ax + u$, where $a > 0$
- ▶ Controls are bounded $|u| \leq 1$
- ▶ The objective is

$$\min J(u) = \int_{t_0}^{t_f} |u| dt,$$

where t_f is free and the initial and final states are fixed:
 $x(t_0) = x_0$ and $x(t_f) = 0$

Minimum control-effort problems

Example

- ▶ Hamiltonian: $H(x, u, p) = |u| - pax + pu$
- ▶ In (4) we have $b_i = B = 1$, and the optimal control is of the form

$$u^* = \begin{cases} 1, & p^* < -1 \\ \text{undefined but } \geq 0, & p^* = -1 \\ 0, & -1 < p^* < 1 \\ \text{undefined but } \leq 0, & p^* = 1 \\ -1, & p^* > 1 \end{cases}$$

Minimum control-effort problems

Example

- ▶ Can there be singular intervals, i.e., $[t_1, t_2]$ where $p^* = -1$ or $p^* = 1$?
- ▶ By the costate equations $\dot{p}^* = -H_x$ the costate is of the form $p^* = c_1 e^{at}$, where c_1 is an integration constant
- ▶ The exponent function is strictly monotone and $a > 0$, so there cannot be any singular intervals \Rightarrow the control can be undefined only momentarily (pointwise)

Singular solutions

- ▶ In the previous, the singularity in the minimum-time and minimum control-effort problems was due to the fact that the Hamiltonian was linear in control
- ▶ Then we define the switching function that gives the multiplier for the control in the Hamiltonian
- ▶ If the switching function is zero *more than in a single point*, we call it singular interval and the corresponding solution is the singular solution
- ▶ Note! The possibility of singular solutions is not, of course, restricted to minimum-time or minimum control-effort problems!

Singular solutions

Example

- ▶ Let us examine the problem

$$\min \int_0^{t_f} \frac{1}{2} [x_1^2 + x_2^2] dt$$

when t_f and $x(t_f)$ are free, and

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

and the control is bounded: $|u| \leq 1$

Singular solutions

Example

- ▶ Hamiltonian: $H = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + p_1x_2 + p_2u$ is linear in u and the switching function is $s(t) = p_2$
- ▶ The control that minimizes Hamiltonian is

$$u^* = \begin{cases} -1, & \text{when } p_2 > 0 \\ 1, & \text{when } p_2 < 0 \end{cases}$$

- ▶ The costate equations are

$$\dot{p}_1^* = -H_{x_1} = -x_1^*$$

$$\dot{p}_2^* = -H_{x_2} = -x_2^* - p_1^*$$

Singular solutions

Example

- ▶ On a singular interval we have $s(t) = p_2^* = 0$, and also $\dot{p}_2 = 0$; from the second costate equation we get

$$\dot{p}_2^* = -x_2^* - p_1^* = 0$$

- ▶ We use the condition $H = 0$, and on the singular interval it holds

$$\begin{aligned}\frac{1}{2}x_1^{*2} + \frac{1}{2}x_2^{*2} + p_1^*x_2^* + p_2^*u^* &= 0 \\ \Rightarrow x_1^{*2} + x_2^{*2} - 2x_2^{*2} &= 0 \\ \Rightarrow x_1^{*2} - x_2^{*2} &= 0 \\ \Rightarrow (x_1^* - x_2^*)(x_1^* + x_2^*) &= 0\end{aligned}$$

Singular solutions

Example

- ▶ The singularity condition holds on $t \in [t_1, t_2]$

$$x_1^* - x_2^* = 0 \quad \text{if} \quad x_1^* + x_2^* = 0$$

- ▶ (Solve the problem and draw the figures of x and u)

Summary

- ▶ Minimum-time problems
- ▶ Minimum control-effort problems
- ▶ Switching function and singular intervals