Chapter 4
Brownian Motion and Stochastic Calculus

The modeling of random assets in finance is based on stochastic processes, which are families $(X_t)_{t\in I}$ of random variables indexed by a time interval I. In this chapter we present a description of Brownian motion and a construction of the associated Itô stochastic integral.

4.1 Brownian Motion

We start by recalling the definition of Brownian motion, which is a fundamental example of a stochastic process. The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of Brownian motion can be constructed on the space $\Omega = \mathcal{C}_0(\mathbb{R}_+)$ of continuous real-valued functions on $\mathbb{R}_+$ started at 0.

Definition 4.1. *The standard Brownian motion is a stochastic process $(B_t)_{t\in\mathbb{R}_+}$ such that*

(i) $B_0 = 0$ almost surely,

(ii) The sample trajectories $t \mapsto B_t$ are continuous, with probability 1.

(iii) For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the increments

$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$$

are independent.

(iv) For any given times $0 \le s < t$, $B_t - B_s$ has the Gaussian distribution $\mathcal{N}(0, t-s)$ with mean zero and variance $t - s$.

We refer to Theorem 10.28 of [28] and to Chapter 1 of [72] for the proof of the existence of Brownian motion as a stochastic process $(B_t)_{t\in\mathbb{R}_+}$ satisfying the above properties (i)-(iv).

In particular, Condition (iv) above implies

$$\mathbb{E}[B_t - B_s] = 0 \quad \text{and} \quad \text{Var}[B_t - B_s] = t - s, \qquad 0 \le s \le t.$$

In the sequel the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ will be generated by the Brownian paths up to time t, in other words we write

$$\mathcal{F}_t = \sigma(B_s \ : \ 0 \le s \le t), \qquad t \ge 0. \tag{4.1}$$

A random variable F is said to be $\mathcal{F}_t$*-measurable* if the knowledge of F depends only on the information known up to time t. As an example, if t =today,

- the date of the past course exam is $\mathcal{F}_t$-measurable, because it belongs to the past.

- the date of the next Chinese new year, although it refers to a future event, is also $\mathcal{F}_t$-measurable because it is known at time t.

- the date of the next typhoon is not $\mathcal{F}_t$-measurable since it is not known at time t.

- the maturity date T of a European option is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$, because it has been determined at time 0.

- the exercise date τ of an American option after time t (see Section 9.4) is not $\mathcal{F}_t$-measurable because it refers to a future random event.

Property (iii) above shows that $B_t - B_s$ is independent of all Brownian increments taken before time s, *i.e.*

$$(B_t - B_s) \perp\!\!\!\perp (B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}),$$

$0 \le t_0 \le t_1 \le \cdots \le t_n \le s \le t$, hence $B_t - B_s$ is also independent of the whole Brownian history up to time s, hence $B_t - B_s$ is in fact independent of $\mathcal{F}_s$, $s \ge 0$.

For convenience we will informally regard Brownian motion as a random walk over infinitesimal time intervals of length Δt, with increments

$$\Delta B_t := B_{t+\Delta t} - B_t$$

over the time interval $[t, t + \Delta t]$ given by

$$\Delta B_t = \pm\sqrt{\Delta t} \tag{4.2}$$

with equal probabilities $(1/2, 1/2)$.

The choice of the square root in (4.2) is in fact not fortuitous. Indeed, any choice of $\pm(\Delta t)^\alpha$ with a power $\alpha > 1/2$ would lead to explosion of the process as dt tends to zero, whereas a power $\alpha \in (0, 1/2)$ would lead to a vanishing process.

Note that we have

$$\mathbb{E}[\Delta B_t] = \frac{1}{2}\sqrt{\Delta t} - \frac{1}{2}\sqrt{\Delta t} = 0,$$

and

$$\mathrm{Var}[\Delta B_t] = \mathbb{E}[(\Delta B_t)^2] = \frac{1}{2}\Delta t + \frac{1}{2}\Delta t = \Delta t.$$

According to this representation, the paths of Brownian motion are not differentiable, although they are continuous by Property (ii), as we have

$$\frac{dB_t}{dt} \simeq \frac{\pm\sqrt{dt}}{dt} = \pm\frac{1}{\sqrt{dt}} \simeq \pm\infty. \tag{4.3}$$

After splitting the interval $[0, T]$ into N intervals

$$\left(\frac{k-1}{N}T, \frac{k}{N}T\right], \qquad k = 1, \ldots, N,$$

of length $\Delta t = T/N$ with N "large", and letting

$$X_k = \pm\sqrt{T} = \pm\sqrt{N}\sqrt{\Delta t} = \sqrt{N}\Delta B_t$$

with probabilities $(1/2, 1/2)$ we have $Var(X_k) = T$ and

$$\Delta B_t = \frac{X_k}{\sqrt{N}} = \pm\sqrt{\Delta t}$$

is the increment of B_t over $((k-1)\Delta t, k\Delta t]$, and we get

$$B_T \simeq \sum_{0<t<T} \Delta B_t \simeq \frac{X_1 + \cdots + X_N}{\sqrt{N}}.$$

Hence by the central limit theorem we recover the fact that B_T has a centered Gaussian distribution with variance T, cf. point (iv) of the above definition of Brownian motion. Indeed, the central limit theorem states that given any sequence $(X_k)_{k\geq 1}$ of independent identically distributed centered random variables with variance $\sigma^2 = Var(X_k) = T$, the normalized sum

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

converges (in distribution) to a centered Gaussian random variable $\mathcal{N}(0, \sigma^2)$ with variance σ^2 as n goes to infinity. As a consequence, ΔB_t could in fact be replaced by any centered random variable with variance Δt in the above description.

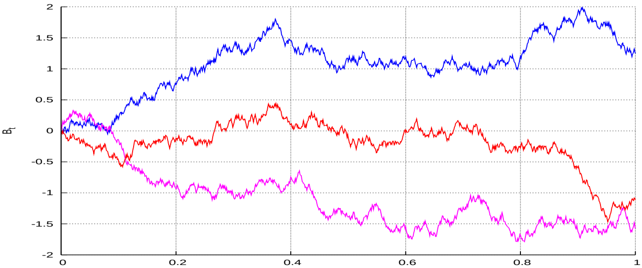

Fig. 4.1: Sample paths of a one-dimensional Brownian motion.

In Figure 4.1 we draw three sample paths of a standard Brownian motion obtained by computer simulation using (4.2). Note that there is no point in "computing" the value of B_t as it is a *random variable* for all $t > 0$, however we can generate samples of B_t, which are distributed according to the centered Gaussian distribution with variance t.

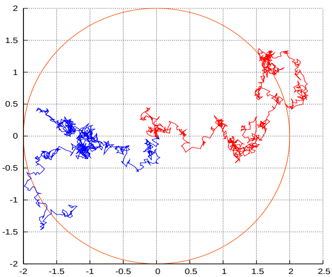

Fig. 4.2: Two sample paths of a two-dimensional Brownian motion.

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The n-dimensional Brownian motion can be constructed as $(B_t^1, \ldots, B_t^n)_{t\in\mathbb{R}_+}$ where $(B_t^1)_{t\in\mathbb{R}_+}$, $\ldots$,$(B_t^n)_{t\in\mathbb{R}_+}$ are independent copies of $(B_t)_{t\in\mathbb{R}_+}$. Next, we turn to simulations of 2 dimensional and 3 dimensional Brownian motions in Figures 4.2 and 4.3. Recall that the movement of pollen particles originally observed by R. Brown in 1827 was indeed 2-dimensional.

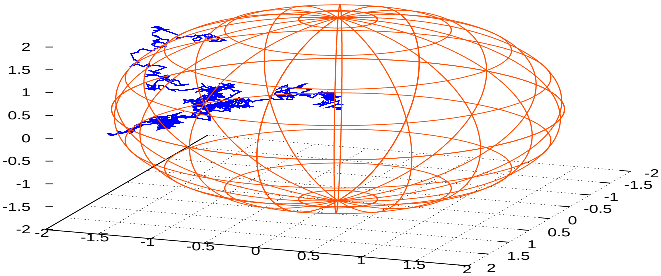

Fig. 4.3: Sample paths of a three-dimensional Brownian motion.

4.2 Wiener Stochastic Integral

In this section we construct the Itô stochastic integral of square-integrable deterministic function with respect to Brownian motion.

Recall that Bachelier originally modeled the price S_t of a risky asset by $S_t = \sigma B_t$ where σ is a volatility parameter. The stochastic integral

$$\int_0^T f(t)dS_t = \sigma \int_0^T f(t)dB_t$$

can be used to represent the value of a portfolio as a sum of profits and losses $f(t)dS_t$ where dS_t represents the stock price variation and $f(t)$ is the quantity invested in the asset S_t over the short time interval $[t, t + dt]$.

A naive definition of the stochastic integral with respect to Brownian motion would consist in writing

$$\int_0^\infty f(t)dB_t = \int_0^\infty f(t)\frac{dB_t}{dt}dt,$$

and evaluating the above integral with respect to dt. However this definition fails because the paths of Brownian motion are not differentiable, cf. (4.3).

Next we present Itô's construction of the stochastic integral with respect to Brownian motion. Stochastic integrals will be first constructed as integrals of simple step functions of the form

$$f(t) = \sum_{i=1}^{n} a_i \mathbf{1}_{(t_{i-1},t_i]}(t), \qquad t \in \mathbb{R}_+, \tag{4.4}$$

i.e. the function f takes the value a_i on the interval $(t_{i-1}, t_i]$, $i = 1, \ldots, n$, with $0 \le t_0 < \cdots < t_n$, as illustrated in Figure 4.4.

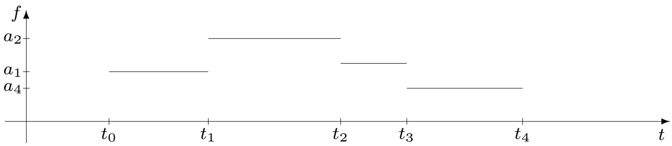

Fig. 4.4: Step function.

Note that the set of simple step functions f of the form (4.4) is a linear space which is dense in $L^2(\mathbb{R}_+)$ for the norm

$$\|f\|_{L^2(\mathbb{R}_+)} := \sqrt{\int_0^\infty |f(t)|^2 dt}.$$

Recall also that the classical integral of f given in (4.4) is interpreted as the area under the curve f and computed as

$$\int_0^\infty f(t)dt = \sum_{i=1}^{n} a_i (t_i - t_{i-1}).$$

In the next definition we adapt this construction to the setting of integration with respect to Brownian motion.

Definition 4.2. *The stochastic integral with respect to Brownian motion* $(B_t)_{t\in\mathbb{R}_+}$ *of the simple step functions* f *of the form* (4.4) *is defined by*

$$\int_0^\infty f(t)dB_t := \sum_{i=1}^{n} a_i (B_{t_i} - B_{t_{i-1}}). \tag{4.5}$$

In the next Proposition 4.1 we determine the probability distribution of $\int_0^\infty f(t)dB_t$ and we show that it is independent of the particular representation (4.4) chosen for $f(t)$. In the sequel we will make a repeated use of the space $L^2(\mathbb{R}_+)$ of functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that

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$$\|f\|_{L^2(\mathbb{R}_+)}^2 := \int_0^\infty |f(t)|^2 dt < \infty,$$

called square-integrable functions.

Proposition 4.1. *The definition of the stochastic integral $\int_0^\infty f(t)dB_t$ can be extended to any measurable function $f \in L^2(\mathbb{R}_+)$, i.e. to f such that*

$$\int_0^\infty |f(t)|^2 dt < \infty. \tag{4.6}$$

In this case, $\int_0^\infty f(t)dB_t$ has a centered Gaussian distribution

$$\int_0^\infty f(t)dB_t \simeq \mathcal{N}\left(0, \int_0^\infty |f(t)|^2 dt\right)$$

with variance $\int_0^\infty |f(t)|^2 dt$ and we have the Itô isometry

$$\mathbb{E}\left[\left(\int_0^\infty f(t)dB_t\right)^2\right] = \int_0^\infty |f(t)|^2 dt. \tag{4.7}$$

Proof. Recall that if $X_1, \ldots, X_n$ are independent Gaussian random variables with probability laws $\mathcal{N}(m_1, \sigma_1^2), \ldots, \mathcal{N}(m_n, \sigma_n^2)$ then then sum $X_1 + \cdots + X_n$ is a Gaussian random variable with probability law $\mathcal{N}(m_1 + \cdots + m_n, \sigma_1^2 + \cdots + \sigma_n^2)$.

As a consequence, when f is the simple function

$$f(t) = \sum_{i=1}^n a_i \mathbf{1}_{(t_{i-1}, t_i]}(t), \qquad t \in \mathbb{R}_+,$$

the sum

$$\int_0^\infty f(t)dB_t = \sum_{k=1}^n a_k (B_{t_k} - B_{t_{k-1}})$$

has a centered Gaussian distribution with variance

$$\sum_{k=1}^n |a_k|^2 (t_k - t_{k-1}),$$

since

$$\mathrm{Var}\,[a_k(B_{t_k} - B_{t_{k-1}})] = a_k^2 \mathrm{Var}\,[B_{t_k} - B_{t_{k-1}}] = a_k^2 (t_k - t_{k-1}),$$

hence the stochastic integral

$$\int_0^\infty f(t)dB_t = \sum_{k=1}^n a_k (B_{t_k} - B_{t_{k-1}})$$

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of the step function

$$f(t) = \sum_{k=1}^{n} a_k \mathbf{1}_{(t_{k-1},t_k]}(t)$$

has a centered Gaussian distribution with variance

$$\begin{aligned}
\mathrm{Var}\left[\int_0^\infty f(t)dB_t\right] &= \sum_{k=1}^{n} |a_k|^2 (t_k - t_{k-1}) \\
&= \sum_{k=1}^{n} |a_k|^2 \int_{t_{k-1}}^{t_k} dt \\
&= \int_0^\infty \sum_{k=1}^{n} |a_k|^2 1_{(t_{k-1},t_k]}(t) dt \\
&= \int_0^\infty |f(t)|^2 dt.
\end{aligned}$$

Finally we note that

$$\begin{aligned}
\mathrm{Var}\left[\int_0^\infty f(t)dB_t\right] &= \mathbb{E}\left[\left(\int_0^\infty f(t)dB_t\right)^2\right] - \left(\mathbb{E}\left[\int_0^\infty f(t)dB_t\right]\right)^2 \\
&= \mathbb{E}\left[\left(\int_0^\infty f(t)dB_t\right)^2\right].
\end{aligned}$$

The extension of the stochastic integral to all functions satisfying (4.6) is obtained by density and a Cauchy sequence argument, based on the isometry relation (4.7). Namely, given f a function satisfying (4.6) and $(f_n)_{n\in\mathbb{N}}$ a sequence of simple functions converging to f for the norm

$$\|f - f_n\|_{L^2(\mathbb{R}_+)} := \left(\int_0^\infty |f(t) - f_n(t)|^2 dt\right)^{1/2}$$

i.e. in $L^2(\mathbb{R}_+)$, the isometry (4.7) shows that $\left(\int_0^\infty f_n(t)dB_t\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $L^2(\Omega)$ of square-integrable random variables $F : \Omega \to \mathbb{R}$ such that

$$\|F\|^2_{L^2(\Omega\times\mathbb{R}_+)} := \mathbb{E}[F^2] < \infty.$$

Indeed, we have

$$\begin{aligned}
&\left\|\int_0^\infty f_k(t)dB_t - \int_0^\infty f_n(t)dB_t\right\|_{L^2(\Omega)} \\
&= \left(\mathbb{E}\left[\left(\int_0^\infty f_k(t)dB_t - \int_0^\infty f_n(t)dB_t\right)^2\right]\right)^{1/2} \\
&= \|f_k - f_n\|_{L^2(\mathbb{R}_+)} \\
&\leq \|f - f_k\|_{L^2(\mathbb{R}_+)} + \|f - f_n\|_{L^2(\mathbb{R}_+)},
\end{aligned}$$

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which tends to 0 as k, n tend to infinity, hence $\left(\int_0^\infty f_n(t)dB_t\right)_{n\in\mathbb{N}}$ is it converges for the L^2-norm as $L^2(\Omega)$ is a complete space, cf. *e.g.* Chapter 4 of [21]. In this case we let

$$\int_0^\infty f(t)dB_t := \lim_{n\to\infty} \int_0^\infty f_n(t)dB_t$$

and the limit is unique from (4.7). □

For example, $\displaystyle\int_0^\infty e^{-t}dB_t$ has a centered Gaussian distribution with variance

$$\int_0^\infty e^{-2t}dt = \left[-\frac{1}{2}e^{-2t}\right]_0^\infty = \frac{1}{2}.$$

Again, the Wiener stochastic integral $\displaystyle\int_0^\infty f(s)dB_s$ is nothing but a Gaussian random variable and it cannot be "computed" in the way standard integral are computed via the use of primitives. However, when $f \in L^2(\mathbb{R}_+)$ is $\mathcal{C}^1$ on $\mathbb{R}_+$, we have the following formula

$$\int_0^\infty f(t)dB_t = -\int_0^\infty f'(t)B_t dt,$$

provided $\lim_{t\to\infty} t|f(t)|^2 = 0$ and $f \in L^2(\mathbb{R}_+)$, cf. *e.g.* Remark 2.5.9 in [66].

4.3 Itô Stochastic Integral

In this section we extend the Wiener stochastic integral to square-integrable *adapted* processes. Recall that a process $(X_t)_{t\in\mathbb{R}_+}$ is said to be $\mathcal{F}_t$-adapted if X_t is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$, where the information flow $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ has been defined in (4.1).

In other words, a process $(X_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t$-adapted if the value of X_t at time t depends only on information known up to time t. Note that the value of X_t may still depend on "known" future data, for example a fixed future date in the calendar, such as a maturity time $T > t$, as long as its value is known at time t.

The extension of the stochastic integral to adapted random processes is actually necessary in order to compute a portfolio value when the portfolio process is no longer deterministic. This happens in particular when one needs to update the portfolio allocation based on random events occurring on the market.

Stochastic integrals of adapted processes will be first constructed as integrals of simple predictable processes $(u_t)_{t\in\mathbb{R}_+}$ of the form

$$u_t = \sum_{i=1}^{n} F_i \mathbf{1}_{(t_{i-1},t_i]}(t), \qquad t \in \mathbb{R}_+, \tag{4.8}$$

where F_i is an $\mathcal{F}_{t_{i-1}}$-measurable random variable for $i = 1, \ldots, n$. The notion of simple predictable process is natural in the context of portfolio investment, in which F_i will represent an investment allocation decided at time t_{i-1} and to remain unchanged over the time period $(t_{i-1}, t_i]$.

By convention, $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is denoted by $u_t(\omega)$, $t \in \mathbb{R}_+$, $\omega \in \Omega$, and the random outcome ω is often dropped for convenience of notation.

Definition 4.3. *The stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ of any simple predictable process $(u_t)_{t\in\mathbb{R}_+}$ of the form (4.8) is defined by*

$$\int_0^\infty u_t dB_t := \sum_{i=1}^{n} F_i (B_{t_i} - B_{t_{i-1}}). \tag{4.9}$$

The next proposition gives the extension of the stochastic integral from simple predictable processes to square-integrable $\mathcal{F}_t$-adapted processes $(X_t)_{t\in\mathbb{R}_+}$ for which the value of X_t at time t only depends on information contained in the Brownian path up to time t. This also means that knowing the future is not permitted in the definition of the Itô integral, for example a portfolio strategy that would allow the trader to "buy at the lowest" and "sell at the highest" is not possible as it would require knowledge of future market data.

Note that the difference between Relation (4.10) below and Relation (4.7) is the expectation on the right hand side.

Proposition 4.2. *The stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ extends to all adapted processes $(u_t)_{t\in\mathbb{R}_+}$ such that*

$$\mathbb{E}\left[\int_0^\infty |u_t|^2 dt\right] < \infty,$$

with the Itô isometry

$$\mathbb{E}\left[\left(\int_0^\infty u_t dB_t\right)^2\right] = \mathbb{E}\left[\int_0^\infty |u_t|^2 dt\right]. \tag{4.10}$$

In addition, the Itô integral of an adapted process $(u_t)_{t\in\mathbb{R}_+}$ is always a centered random variable:

$$\mathbb{E}\left[\int_0^\infty u_s dB_s\right] = 0. \tag{4.11}$$

Proof. We start by showing that the Itô isometry (4.10) holds for the simple predictable process u of the form (4.8). We have

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$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^\infty u_t dB_t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n F_i(B_{t_i} - B_{t_{i-1}})\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i,j=1}^n F_i F_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n |F_i|^2 (B_{t_i} - B_{t_{i-1}})^2\right] \\
&\quad +2\,\mathbb{E}\left[\sum_{1\le i<j\le n} F_i F_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})\right] \\
&= \sum_{i=1}^n \mathbb{E}[\mathbb{E}[|F_i|^2 (B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}]] \\
&\quad +2 \sum_{1\le i<j\le n} \mathbb{E}[\mathbb{E}[F_i F_j (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}}) | \mathcal{F}_{t_{j-1}}]] \\
&= \sum_{i=1}^n \mathbb{E}[|F_i|^2\, \mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2 | \mathcal{F}_{t_i}]] \\
&\quad +2 \sum_{1\le i<j\le n} \mathbb{E}[F_i F_j (B_{t_i} - B_{t_{i-1}})\, \mathbb{E}[(B_{t_j} - B_{t_{j-1}}) | \mathcal{F}_{t_{j-1}}]] \\
&= \sum_{i=1}^n \mathbb{E}[|F_i|^2\, \mathbb{E}[(B_{t_i} - B_{t_{i-1}})^2]] \\
&\quad +2 \sum_{1\le i<j\le n} \mathbb{E}[F_i F_j (B_{t_i} - B_{t_{i-1}})\, \mathbb{E}[(B_{t_j} - B_{t_{j-1}})]] \\
&= \sum_{i=1}^n \mathbb{E}[|F_i|^2 (t_i - t_{i-1})] \\
&= \mathbb{E}\left[\sum_{i=1}^n |F_i|^2 (t_i - t_{i-1})\right] \\
&= \mathbb{E}\left[\int_0^\infty |u_t|^2 dt\right],
\end{aligned}$$

where we used the "tower property" (16.24) of conditional expectations and the facts that $B_{t_i} - B_{t_{i-1}}$ is independent of $\mathcal{F}_{t_{i-1}}$ and

$$\mathbb{E}[B_{t_i} - B_{t_{i-1}}] = 0, \qquad \mathbb{E}\left[(B_{t_i} - B_{t_{i-1}})^2\right] = t_i - t_{i-1}, \qquad i = 1, \ldots, n.$$

The extension of the stochastic integral to square-integrable adapted processes $(u_t)_{t\in\mathbb{R}_+}$ is obtained as in Proposition 4.1 by density and a Cauchy sequence argument using the isometry (4.10), in the same way as in the proof of Proposition 4.1. Let $L^2(\Omega \times \mathbb{R}_+)$ denote the space of square-integrable stochastic processes $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|u\|^2_{L^2(\Omega\times\mathbb{R}_+)} := \mathbb{E}\left[\int_0^\infty |u_t|^2 dt\right] < \infty.$$

By Lemma 1.1 of [41], p. 22 and p. 46, or Proposition 2.5.3 of [66], the set of simple predictable processes forms a linear space which is dense in the subspace $L^2_{ad}(\Omega \times \mathbb{R}_+)$ made of square-integrable adapted processes in $L^2(\Omega \times \mathbb{R}_+)$. In other words, given u a square-integrable adapted process there exists a sequence $(u^n)_{n\in\mathbb{N}}$ of simple predictable processes converging to u in $L^2(\Omega \times \mathbb{R}_+)$, and the isometry (4.10) shows that $\left(\int u^n_t dB_t\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$, hence it converges in the complete space $L^2(\Omega)$. In this case we let

$$\int_0^\infty u_t dB_t := \lim_{n\to\infty} \int_0^\infty u^n_t dB_t$$

and the limit is unique from (4.10). □

Note also that the Itô isometry (4.10) can also be written as

$$\mathbb{E}\left[\int_0^\infty u_t dB_t \int_0^\infty v_t dB_t\right] = \mathbb{E}\left[\int_0^\infty u_t v_t dt\right],$$

for all square-integrable adapted processes u, v.

In addition, when the integrand $(u_t)_{t\in\mathbb{R}_+}$ is not a deterministic function, the random variable $\int_0^\infty u_s dB_s$ no longer has a Gaussian distribution, except in some exceptional cases.

The stochastic integral of u over the interval $[a, b]$ is defined as

$$\int_a^b u_t dB_t := \int_0^\infty \mathbf{1}_{[a,b]}(t) u_t dB_t.$$

In particular we have

$$\int_0^\infty \mathbf{1}_{[a,b]}(t) dB_t = B_b - B_a, \qquad 0 \le a \le b,$$

and

$$\int_a^b dB_t = B_b - B_a, \qquad 0 \le a \le b.$$

We also have the Chasles relation

$$\int_a^c u_t dB_t = \int_a^b u_t dB_t + \int_b^c u_t dB_t, \qquad 0 \le a \le b \le c,$$

and the stochastic integral has the following linearity property:

$$\int_0^\infty (u_t + v_t) dB_t = \int_0^\infty u_t dB_t + \int_0^\infty v_t dB_t, \qquad u, v \in L^2(\mathbb{R}_+).$$

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In the sequel we will define the return at time $t \in \mathbb{R}_+$ of the risky asset $(S_t)_{t\in\mathbb{R}_+}$ as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$. This equation can be formally rewritten in integral form as

$$S_T = S_0 + \mu \int_0^T S_t dt + \sigma \int_0^T S_t dB_t,$$

hence the need to define an integral with respect to dB_t, in addition to the usual integral with respect to dt.

In Proposition 4.2 we have defined the stochastic integral of square-integrable processes with respect to Brownian motion, thus we have made sense of the equation

$$S_T = S_0 + \mu \int_0^T S_t dt + \sigma \int_0^T S_t dB_t,$$

for $(S_t)_{t\in\mathbb{R}_+}$ an $\mathcal{F}_t$-adapted process, which can be rewritten in differential notation as

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

or

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \tag{4.12}$$

This model will be used to represent the random price S_t of a risky asset at time t. Here the return dS_t/S_t of the asset is made of two components: a constant return μdt and a random return σdB_t parametrized by the coefficient σ, called the volatility.

Our goal is now to solve Equation (4.12) and for this we will need to introduce Itô's calculus in Section 4.5 after reviewing classical deterministic calculus in Section 4.4.

4.4 Deterministic Calculus

The *fundamental theorem of calculus* states that for any continuously differentiable (deterministic) function f we have

$$f(x) = f(0) + \int_0^x f'(y)dy.$$

In differential notation this relation is written as the first order expansion

$$df(x) = f'(x)dx, \tag{4.13}$$

where dx is "small". Higher order expansions can be obtained from *Taylor's formula*, which, letting

$$df(x) = f(x + dx) - f(x),$$

states that

$$df(x) = f'(x)dx + \frac{1}{2}f''(x)(dx)^2 + \frac{1}{3!}f'''(x)(dx)^3 + \frac{1}{4!}f^{(4)}(x)(dx)^4 + \cdots .$$

Note that Relation (4.13) can be obtained by neglecting the terms of order larger than one in Taylor's formula, since $(dx)^n << dx$ when $n \geq 2$ and dx is "small".

4.5 Stochastic Calculus

Let us now apply Taylor's formula to Brownian motion, taking

$$dB_t = B_{t+dt} - B_t,$$

and letting

$$df(B_t) = f(B_{t+dt}) - f(B_t),$$

we have

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2 + \frac{1}{3!}f'''(B_t)(dB_t)^3 + \frac{1}{4!}f^{(4)}(B_t)(dB_t)^4 + \cdots .$$

From the construction of Brownian motion by its small increments $dB_t = \pm\sqrt{dt}$, it turns out that the terms in $(dt)^2$ and $dtdB_t = \pm(dt)^{3/2}$ can be neglected in Taylor's formula at the first order of approximation in dt. However, the term of order two

$$(dB_t)^2 = (\pm\sqrt{dt})^2 = dt$$

can no longer be neglected in front of dt.

Hence Taylor's formula written at the second order for Brownian motion reads

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt, \tag{4.14}$$

for "small" dt. Note that writing this formula as

$$\frac{df(B_t)}{dt} = f'(B_t)\frac{dB_t}{dt} + \frac{1}{2}f''(B_t)$$

does not make sense because the derivative

$$\frac{dB_t}{dt} \simeq \pm \frac{\sqrt{dt}}{dt} \simeq \pm \frac{1}{\sqrt{dt}} \simeq \pm\infty$$

does not exist.

Integrating (4.14) on both sides and using the relation

$$f(B_t) - f(B_0) = \int_0^t df(B_s)$$

we get the integral form of Itô's formula for Brownian motion, *i.e.*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

We now turn to the general expression of Itô's formula which applies to Itô processes of the form

$$X_t = X_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+, \tag{4.15}$$

or in differential notation

$$dX_t = v_t dt + u_t dB_t,$$

where $(u_t)_{t\in\mathbb{R}_+}$ and $(v_t)_{t\in\mathbb{R}_+}$ are square-integrable adapted processes.

Given $f(t,x)$ a smooth function of two variables, from now on we let $\frac{\partial f}{\partial x}$ denote partial differentiation with respect to the *second* variable in $f(t,x)$, while $\frac{\partial f}{\partial s}$ denote partial differentiation with respect to the *first* (time) variable in $f(t,x)$.

Theorem 4.1. *(Itô formula for Itô processes). For any Itô process* $(X_t)_{t\in\mathbb{R}_+}$ *of the form* (4.15) *and any* $f \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ *we have*

$$\begin{aligned} f(t,X_t) = f(0,X_0) &+ \int_0^t v_s \frac{\partial f}{\partial x}(s,X_s)ds + \int_0^t u_s \frac{\partial f}{\partial x}(s,X_s)dB_s \\ &+ \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \frac{1}{2}\int_0^t |u_s|^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)ds. \end{aligned} \tag{4.16}$$

Proof. cf. [71], Theorem II-32. □

Using the relation

$$\int_0^t df(s,X_s) = f(t,X_t) - f(0,X_0),$$

we get

$$\int_0^t df(s,X_s) = \int_0^t v_s \frac{\partial f}{\partial x}(s,X_s)ds + \int_0^t u_s \frac{\partial f}{\partial x}(s,X_s)dB_s$$
$$+ \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \frac{1}{2}\int_0^t |u_s|^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)ds,$$

which allows us to rewrite Itô's formula (4.16) in differential notation as

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)dt + u_t\frac{\partial f}{\partial x}(t,X_t)dB_t + v_t\frac{\partial f}{\partial x}(t,X_t)dt + \frac{1}{2}|u_t|^2\frac{\partial^2 f}{\partial x^2}(t,X_t)dt, \tag{4.17}$$

or

$$df(t,X_t) = \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}|u_t|^2\frac{\partial^2 f}{\partial x^2}(t,X_t)dt.$$

Next, given two processes $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ written as

$$X_t = X_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \qquad t \in \mathbb{R}_+,$$

and

$$Y_t = Y_0 + \int_0^t b_s ds + \int_0^t a_s dB_s, \qquad t \in \mathbb{R}_+,$$

the Itô formula also shows that

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t$$

where the product $dX_t \cdot dY_t$ is computed according to the *Itô rule*

$$(dt)^2 = 0, \qquad dt dB_t = 0, \qquad (dB_t)^2 = dt, \tag{4.18}$$

i.e.

$$\begin{aligned} dX_t \cdot dY_t &= (v_t dt + u_t dB_t)(b_t dt + a_t dB_t) \\ &= b_t v_t (dt)^2 + b_t u_t dt dB_t + a_t v_t dt dB_t + a_t u_t (dB_t)^2 \\ &= u_t a_t dt. \end{aligned}$$

Hence we have

$$\begin{aligned} (dX_t)^2 &= (v_t dt + u_t dB_t)^2 \\ &= (v_t)^2(dt)^2 + (u_t)^2(dB_t)^2 + 2u_t v_t dt \cdot dB_t \\ &= (u_t)^2 dt, \end{aligned}$$

according to the Itô multiplication table

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$$\begin{array}{|c|c|c|} \hline \cdot & dt & dB_t \\ \hline dt & 0 & 0 \\ \hline dB_t & 0 & dt \\ \hline \end{array} \tag{4.19}$$

and (4.17) can also be rewriten as

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2.$$

Taking $u_t = 1$ and $v_t = 0$ in (4.15) yields $X_t = B_t$, in which case the Itô formula (4.16) reads

$$f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial s}(s, B_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)ds,$$

i.e. in differential notation:

$$df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)dt. \tag{4.20}$$

As another example, applying Itô's formula (4.20) to B_t^2 with

$$B_t^2 = f(t, B_t) \quad \text{and} \quad f(t, x) = x^2,$$

we get

$$\begin{aligned} dB_t^2 &= df(B_t) \\ &= \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)dt \\ &= 2B_t dB_t + dt, \end{aligned}$$

since

$$\frac{\partial f}{\partial t}(t, x) = 0, \qquad \frac{\partial f}{\partial x}(t, x) = 2x, \qquad \text{and} \qquad \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, x) = 1,$$

hence by integration we find

$$\begin{aligned} B_T^2 &= B_0 + 2\int_0^T B_s dB_s + \int_0^T dt \\ &= 2\int_0^T B_s dB_s + T, \end{aligned}$$

and

$$\int_0^T B_s dB_s = \frac{B_T^2}{2} - \frac{T}{2}.$$

We close this section with some comments on the practice of Itô's calculus. In some finance textbooks, Itô's formula for *e.g.* geometric Brownian motion can be found written in the notation

$$f(T,S_T) = f(0,X_0) + \sigma \int_0^T S_t \frac{\partial f}{\partial S_t}(t,S_t)dB_t + \mu \int_0^T S_t \frac{\partial f}{\partial S_t}(t,S_t)dt$$
$$+ \int_0^T \frac{\partial f}{\partial t}(t,S_t)dt + \frac{1}{2}\sigma^2 \int_0^T S_t^2 \frac{\partial^2 f}{\partial S_t^2}(t,S_t)dt,$$

or

$$df(S_t) = \sigma S_t \frac{\partial f}{\partial S_t}(S_t)dB_t + \mu S_t \frac{\partial f}{\partial S_t}(S_t)dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2}(S_t)dt.$$

The notation $\frac{\partial f}{\partial S_t}(S_t)$ can in fact be easily misused in combination with the fundamental theorem of classical calculus, and lead to the *wrong* identity

$$df(S_t) = \frac{\partial f}{\partial S_t}(S_t)dS_t.$$

Similarly, writing

$$df(B_t) = \frac{df}{dx}(B_t)dB_t + \frac{1}{2}\frac{d^2 f}{dx^2}(B_t)dt$$

is consistent, while writing

$$df(B_t) = \frac{df(B_t)}{dB_t}dB_t + \frac{1}{2}\frac{d^2 f(B_t)}{dB_t^2}dt$$

is potentially a source of confusion.

4.6 Geometric Brownian Motion

Our aim in this section is to solve the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{4.21}$$

that will defined the price S_t of a risky asset at time t, where $\mu \in \mathbb{R}$ and $\sigma > 0$. This equation is rewritten in *integral form* as

$$S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dB_s, \qquad t \in \mathbb{R}_+. \tag{4.22}$$

It can be solved by applying Itô's formula to $f(S_t) = \log S_t$ with $f(x) = \log x$, which shows that

$$d\log S_t = \mu S_t f'(S_t)dt + \sigma S_t f'(S_t)dB_t + \frac{1}{2}\sigma^2 S_t^2 f''(S_t)dt$$
$$= \mu dt + \sigma dB_t - \frac{1}{2}\sigma^2 dt,$$

hence

$$\log S_t - \log S_0 = \int_0^t d\log S_r$$
$$= \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) dr + \int_0^t \sigma dB_r$$
$$= \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t, \qquad t \in \mathbb{R}_+,$$

and

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right), \qquad t \in \mathbb{R}_+.$$

The above provides a proof of the next proposition.

Proposition 4.3. *The solution of* (4.21) *is given by*

$$S_t = S_0 e^{\mu t + \sigma B_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+.$$

Proof. Let us provide an alternative proof by searching for a solution of the form

$$S_t = f(t, B_t)$$

where $f(t, x)$ is a function to be determined. By Itô's formula (4.20) we have

$$dS_t = df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)dt.$$

Comparing this expression to (4.21) and identifying the terms in dB_t we get

$$\frac{\partial f}{\partial x}(t, B_t) = \sigma S_t,$$

and

$$\frac{\partial f}{\partial t}(t, B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t) = \mu S_t.$$

Using the relation $S_t = f(t, B_t)$ these two equations rewrite as

$$\frac{\partial f}{\partial x}(t, B_t) = \sigma f(t, B_t),$$

and

$$\frac{\partial f}{\partial t}(t, B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t) = \mu f(t, B_t).$$

Since B_t is a Gaussian random variable taking all possible values in $\mathbb{R}$, the equations should hold for all $x \in \mathbb{R}$, as follows:

$$\frac{\partial f}{\partial x}(t, x) = \sigma f(t, x), \tag{4.23}$$

and

$$\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = \mu f(t,x). \tag{4.24}$$

Letting $g(t,x) = \log f(t,x)$, the first equation (4.23) shows that

$$\frac{\partial g}{\partial x}(t,x) = \frac{\partial \log f}{\partial x}(t,x) = \frac{1}{f(t,x)}\frac{\partial f}{\partial x}(t,x) = \sigma,$$

i.e.

$$\frac{\partial g}{\partial x}(t,x) = \sigma,$$

which is solved as

$$g(t,x) = g(t,0) + \sigma x,$$

hence

$$f(t,x) = e^{g(t,0)}e^{\sigma x} = f(t,0)e^{\sigma x}.$$

Plugging back this expression into the second equation (4.24) yields

$$e^{\sigma x}\frac{\partial f}{\partial t}(t,0) + \frac{1}{2}\sigma^2 e^{\sigma x} f(t,0) = \mu f(t,0)e^{\sigma x},$$

i.e. after division by $e^{\sigma x}$:

$$\frac{\partial f}{\partial t}(t,0) = \left(\mu - \sigma^2/2\right) f(t,0),$$

or

$$\frac{\partial g}{\partial t}(t,0) = \mu - \sigma^2/2,$$

i.e.

$$g(t,0) = g(0,0) + \left(\mu - \sigma^2/2\right) t,$$

and

$$\begin{aligned} f(t,x) &= e^{g(t,x)} \\ &= e^{g(t,0)+\sigma x} \\ &= e^{g(0,0)+\sigma x+\left(\mu-\sigma^2/2\right)t} \\ &= f(0,0)e^{\sigma x+\left(\mu-\sigma^2/2\right)t}, \end{aligned}$$

hence

$$S_t = f(t,B_t) = f(0,0)e^{\sigma B_t+\left(\mu-\sigma^2/2\right)t},$$

and the solution to (4.21) is given by

$$S_t = S_0 e^{\sigma B_t+\left(\mu-\sigma^2/2\right)t} \qquad t \in \mathbb{R}_+.$$

□

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Fig. 4.5: Geometric Brownian motion started at 1.*

Conversely, taking $S_t = f(t, B_t)$ with $f(t,x) = S_0 e^{\sigma x - \sigma^2 t/2 + \mu t}$ we may apply Itô's formula to check that

$$\begin{aligned}
dS_t &= df(t, B_t) \\
&= \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)dt \\
&= \left(\mu - \sigma^2/2\right) S_0 e^{\sigma B_t + \left(\mu - \sigma^2/2\right)t} dt + \sigma S_0 e^{\sigma B_t + \left(\mu - \sigma^2/2\right)t} dB_t \\
&\quad + \frac{1}{2}\sigma^2 S_0 e^{\sigma B_t + \left(\mu - \sigma^2/2\right)t} dt \\
&= \mu S_0 e^{\sigma B_t + \left(\mu - \sigma^2/2\right)t} dt + \sigma S_0 e^{\sigma B_t + \left(\mu - \sigma^2/2\right)t} dB_t \\
&= \mu S_t dt + \sigma S_t dB_t.
\end{aligned}$$

4.7 Stochastic Differential Equations

In addition to geometric Brownian motion there exists a large family of stochastic differential equations that can be studied, although most of the time they cannot be explicitly solved. Let now

$$\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^d \otimes \mathbb{R}^n$$

where $\mathbb{R}^d \otimes \mathbb{R}^n$ denotes the space of $d \times n$ matrices, and

$$b : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$$

* The animation works in Acrobat reader on the entire file.

satisfy the global Lipschitz condition

$$\|\sigma(t,x)-\sigma(t,y)\|^2+\|b(t,x)-b(t,y)\|^2 \leq K^2\|x-y\|^2,$$

$t \in \mathbb{R}_+$, $x, y \in \mathbb{R}^n$. Then there exists a unique strong solution to the stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(s, X_s)dB_s + \int_0^t b(s, X_s)ds, \qquad t \in \mathbb{R}_+,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a d-dimensional Brownian motion, see e.g. [71], Theorem V-7.

Next we consider a few examples of stochastic differential equations that can be solved explicitly using Itô calculus.

Examples

1. Consider the stochastic differential equation

$$dX_t = -\alpha X_t dt + \sigma dB_t, \quad X_0 = x_0,$$

with $\alpha > 0$ and $\sigma > 0$.

Looking for a solution of the form

$$X_t = a(t)\left(x_0 + \int_0^t b(s)dB_s\right)$$

where $a(\cdot)$ and $b(\cdot)$ are deterministic functions, yields

$$X_t = x_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s, \qquad t > 0,$$

after applying Theorem 4.1 to the Itô process $x_0 + \int_0^t b(s)dB_s$ of the form (4.15) with $u_t = b(t)$ and $v(t) = 0$, and to the function $f(t,x) = a(t)x$.

Remark: the solution of this equation *cannot* be written as a function $f(t, B_t)$ of t and B_t as in the proof of Proposition 4.3.

2. Consider the stochastic differential equation

$$dX_t = tX_t dt + e^{t^2/2} dB_t, \qquad X_0 = x_0.$$

Looking for a solution of the form $X_t = a(t)\left(X_0 + \int_0^t b(s)dB_s\right)$, where $a(\cdot)$ and $b(\cdot)$ are deterministic functions we get $a'(t)/a(t) = t$ and $a(t)b(t) = e^{t^2/2}$, hence $a(t) = e^{t^2/2}$ and $b(t) = 1$, which yields $X_t =$

$e^{t^2/2}(X_0 + B_t)$, $t \in \mathbb{R}_+$.

3. Consider the stochastic differential equation

$$dY_t = (2\mu Y_t + \sigma^2)dt + 2\sigma\sqrt{Y_t}dB_t,$$

where $\mu, \sigma > 0$.

Letting $X_t = \sqrt{Y_t}$ we have $dX_t = \mu X_t dt + \sigma dB_t$, hence

$$Y_t = \left(e^{\mu t}\sqrt{Y_0} + \sigma\int_0^t e^{\mu(t-s)}dB_s\right)^2.$$

Exercises

Exercise 4.1 Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion.

1. Let $c > 0$. Among the following processes, tell which is a standard Brownian motion and which is not. Justify your answer.

   a. $(B_{c+t} - B_c)_{t\in\mathbb{R}_+}$.
   b. $(cB_{t/c^2})_{t\in\mathbb{R}_+}$.
   c. $(B_{ct^2})_{t\in\mathbb{R}_+}$.

2. Compute the stochastic integrals

$$\int_0^T 2dB_t \qquad \text{and} \qquad \int_0^T (2\times \mathbf{1}_{[0,T/2]}(t) + \mathbf{1}_{(T/2,T]}(t))dB_t$$

   and determine their probability laws (including mean and variance).
3. Determine the probability law (including mean and variance) of the stochastic integral

$$\int_0^{2\pi}\sin(t)\,dB_t.$$

4. Compute $\mathbb{E}[B_t B_s]$ in terms of $s, t \geq 0$.
5. Let $T > 0$. Show that if f is a differentiable function with $f(0) = f(T) = 0$ we have

$$\int_0^T f(t)dB_t = -\int_0^T f'(t)B_t dt.$$

   Hint: Apply Itô's calculus to $t \mapsto f(t)B_t$.

Exercise 4.2 Let $f \in L^2([0,T])$. Compute the conditional expectation

$$E\left[e^{\int_0^T f(s)dB_s}\Big|\mathcal{F}_t\right], \qquad 0 \leq t \leq T,$$

where $(\mathcal{F}_t)_{t\in[0,T]}$ denotes the filtration generated by $(B_t)_{t\in[0,T]}$.

Exercise 4.3 Compute the expectation

$$E\left[\exp\left(\beta\int_0^T B_t dB_t\right)\right]$$

for all $\beta < 1/T$. Hint: expand $(B_T)^2$ using Itô's formula.

Exercise 4.4 Solve the ordinary differential equation $df(t) = cf(t)dt$ and the stochastic differential equation $dS_t = rS_t dt + \sigma S_t dB_t$, $t \in \mathbb{R}_+$, where $r, \sigma \in \mathbb{R}$ are constants and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

Exercise 4.5 Given $T > 0$, let $(X_t^T)_{t\in[0,T]}$ denote the solution of the stochastic differential equation

$$dX_t^T = \sigma dB_t - \frac{X_t^T}{T-t}dt, \qquad t \in [0,T], \tag{4.25}$$

under the initial condition $X_0^T = 0$ and $\sigma > 0$.

1. Show that

$$X_t^T = \sigma(T-t)\int_0^t \frac{1}{T-s}dB_s, \qquad t \in [0,T].$$

   Hint: start by computing $d(X_t^T/(T-t))$ using Itô's calculus.
2. Show that $\mathbb{E}[X_t^T] = 0$ for all $t \in [0,T]$.
3. Show that $\mathrm{Var}[X_t^T] = \sigma^2 t(T-t)/T$ for all $t \in [0,T]$.
4. Show that $X_T^T = 0$. The process $(X_t^T)_{t\in[0,T]}$ is called a *Brownian bridge.*

Exercise 4.6 Exponential Vasicek model. Consider a short term rate interest rate proces $(r_t)_{t\in\mathbb{R}_+}$ in the exponential Vasicek model:

$$dr_t = r_t(\eta - a\log r_t)dt + \sigma r_t dB_t, \tag{4.26}$$

where η, a, σ are positive parameters.

1. Find the solution $(z_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation

$$dz_t = -az_t dt + \sigma dB_t$$

   as a function of the initial condition z_0, where a and σ are positive parameters.
2. Find the solution $(y_t)_{t\in\mathbb{R}_+}$ of the stochastic differential equation

$$dy_t = (\theta - ay_t)dt + \sigma dB_t \tag{4.27}$$

   as a function of the initial condition y_0. Hint: let $z_t = y_t - \theta/a$.

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3. Let $x_t = e^{y_t}$, $t \in \mathbb{R}_+$. Determine the stochastic differential equation satisfied by $(x_t)_{t\in\mathbb{R}_+}$.
4. Find the solution $(r_t)_{t\in\mathbb{R}_+}$ of (4.26) in terms of the initial condition r_0.
5. Compute the mean* $\mathbb{E}[r_t]$ of r_t, $t \geq 0$.
6. Compute the asymptotic mean $\lim_{t\to\infty} \mathbb{E}[r_t]$.

Exercise 4.7 Cox-Ingerson-Ross model. Consider the equation

$$dr_t = (\alpha - \beta r_t)dt + \sigma\sqrt{r_t}dB_t \tag{4.28}$$

modeling the variations of a short term interest rate process r_t, where α, β, σ and r_0 are positive parameters.

1. Write down the equation (4.28) in integral form.
2. Let $u(t) = \mathbb{E}[r_t]$. Show, using the integral form of (4.28), that $u(t)$ satisfies the differential equation

$$u'(t) = \alpha - \beta u(t).$$

3. By an application of Itô's formula to r_t^2, show that

$$dr_t^2 = r_t(2\alpha + \sigma^2 - 2\beta r_t)dt + 2\sigma r_t^{3/2} dB_t. \tag{4.29}$$

4. Using the integral form of (4.29), find a differential equation satisfied by $v(t) = \mathbb{E}[r_t^2]$.

Exercise 4.8 Let $(B_t)_{t\in\mathbb{R}_+}$ denote a standard Brownian motion generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

1. Consider the Itô formula

$$f(X_t) = f(X_0) + \int_0^t u_s \frac{\partial f}{\partial x}(X_s)dB_s + \int_0^t v_s \frac{\partial f}{\partial x}(X_s)ds + \frac{1}{2}\int_0^t u_s^2 \frac{\partial^2 f}{\partial x^2}(X_s)ds, \tag{4.30}$$

where $X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds$.

Compute $S_t := e^{X_t}$ by the Itô formula (4.30) applied to $f(x) = e^x$ and $X_t = \sigma B_t + \nu t$, $\sigma > 0$, $\nu \in \mathbb{R}$.

2. Let $r > 0$. For which value of ν does $(S_t)_{t\in\mathbb{R}_+}$ satisfy the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t \quad ?$$

* You will need to use the generating function $\mathbb{E}[e^X] = e^{\alpha^2/2}$ for $X \simeq \mathcal{N}(0, \alpha^2)$.

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3. Let the process $(S_t)_{t\in\mathbb{R}_+}$ be defined by $S_t = S_0 e^{\sigma B_t + \nu t}$, $t \in \mathbb{R}_+$. Using the result of Exercise 16.2, show that the conditional probability $P(S_T > K \mid S_t = x)$ is given by

$$P(S_T > K \mid S_t = x) = \Phi\left(\frac{\log(x/K) + \nu\tau}{\sigma\sqrt{\tau}}\right),$$

where $\tau = T - t$. Hint: use the decomposition $S_T = S_t e^{\sigma(B_T - B_t) + \nu\tau}$.
4. Given $0 \le t \le T$ and $\sigma > 0$, let

$$X = \sigma(B_T - B_t) \qquad \text{and} \qquad \eta^2 = \mathrm{Var}[X], \qquad \eta > 0.$$

What is η equal to ?

Exercise 4.9 Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion generating the information flow $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

1. Let $0 \le t \le T$. What is the probability law of $B_T - B_t$?
2. From the answer to Exercise 16.5, show that

$$\mathbb{E}[(B_T)^+ \mid \mathcal{F}_t] = \sqrt{\frac{\tau}{2\pi}} e^{-\frac{B_t^2}{2\tau}} + B_t \Phi\left(\frac{B_t}{\sqrt{\tau}}\right),$$

$0 \le t \le T$, where $\tau = T - t$. Hint: write $B_T = B_T - B_t + B_t$.
3. Let $\sigma > 0$, $\nu \in \mathbb{R}$, and $X_t = \sigma B_t + \nu t$. Compute e^{X_t} using the Itô formula

$$f(X_t) = f(X_0) + \int_0^t u_s \frac{\partial f}{\partial x}(X_s) dB_s + \int_0^t v_s \frac{\partial f}{\partial x}(X_s) ds + \frac{1}{2}\int_0^t u_s^2 \frac{\partial^2 f}{\partial x^2}(X_s) ds$$

stated here for a process $X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds$, $t \in \mathbb{R}_+$, and applied to $f(x) = e^x$.
4. Let $S_t = e^{X_t}$, $t \in \mathbb{R}_+$, and $r > 0$. For which value of ν does $(S_t)_{t\in\mathbb{R}_+}$ satisfy the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t \quad ?$$

Exercise 4.10 From the answer to Exercise 16.4-(2), show that

$$\mathbb{E}[(\beta - B_T)^+ \mid \mathcal{F}_t] = \sqrt{\frac{\tau}{2\pi}} e^{-\frac{(\beta - B_t)^2}{2\tau}} + (\beta - B_t)\Phi\left(\frac{\beta - B_t}{\sqrt{\tau}}\right), \qquad 0 \le t \le T,$$

where $\tau = T - t$. Hint: write $B_T = B_T - B_t + B_t$.

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