Stochastic Control:
With Applications to Financial Mathematics*

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*Web: http://www.ricam.oeaw.ac.at/people/page/sass/teaching/stochcontrol/

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For more references regarding the "basics" in the appendix we refer to the lecture notes Stochastische Differentialgleichungen, available at

http://www.ricam.oeaw.ac.at/people/page/sass/teaching/sdes/

and the references therein. These are mainly based on [KS99, Ok00], more general results can be found in [Pr04] covering also non-continuous processes. A highly readable book is [St00].

Text books on stochastic control are always difficult to read. Short and good introductions are given in [K99, Ko99], the latter with an extensive overview of the applications in portfolio optimization. In addition, the lecture notes [To02] provide a good introduction to the concept of viscosity solutions. There are also some good review papers on applications of stochastic control methods in Finance, e.g. [Ru03]. The course will be based on the references made so far and to a certain extent on [ElK81, FR75, FS93, He94, Ok00, YZ99].

1 Introduction

We consider optimal control of Itô-type processes which satisfy a stochastic differential equation (SDE) w.r.t. some Wiener process.

1.1 Stochastic control problem

Let $(\Omega, \overline{\mathcal{F}}, P)$ be a probability space, $T > 0$ the terminal time, $\mathcal{F}$ a filtration satisfying the usual conditions and $W = (W_t)_{t\in[0,T]}$,

$$W_t = \begin{pmatrix} W_t^1 \\ \vdots \\ W_t^m \end{pmatrix}.$$

an m-dimensional Wiener process w.r.t. $\mathcal{F}$.

- A *control process (the actions)* is an $\mathcal{F}$-progressively measurable process $u = (u_t)_{t\in[0,T]}$ with values in some set $\mathcal{U} \subseteq I\!R^p$.
- The n-dimensional *controlled process (state of the system)* $X = (X_t)_{t\in[0,T]}$ is given by

  $$dX_t = b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dW_t, \quad X_0 = x_0, \tag{1.1}$$

  where

  $$b : [0,T] \times I\!R^n \times \mathcal{U} \to I\!R^n, \quad \sigma : [0,T] \times I\!R^n \times \mathcal{U} \to I\!R^{n\times m}$$

  are measurable. Further conditions have to be specified for each control problem separately. In particular conditions are needed which guarantee the existence of X. To emphasize the dependency of X on the control u we may write X_t^u when convenient.
- As *optimization/performance criterion* we use (in the beginning)

  $$J(t,x,u) = \mathrm{E}\left[\int_t^T \psi(t, X_t^u, u_t)dt + \Psi(T, X_T^u) \,\middle|\, X_t^u = x\right]. \tag{1.2}$$

- The set of *admissible controls* is denoted by $\mathcal{A}(t,x)$ and consists of all controls $(u_s)_{s\in[t,T]}$ for which at least a unique strong solution of (1.1) exists on $[t,T]$ given $X_t = x$ and for which the performance measure in (1.2) is well defined. Depending on the particular problem further conditions may be specified. We denote $\mathcal{A}(x_0) = \mathcal{A}(0, x_0)$.
- The *value function* of the control problem is then defined as

  $$V(t,x) = \sup_{u\in\mathcal{A}(t,x)} J(t,x,u).$$

- Our aim is to find $V(0,x_0)$ and a control strategy u^* for which this optimal value is attained, i.e. for which $V(0,x_0) = J(0,x_0,u^*)$. Then u^* will be called optimal.

1.2 Portfolio optimization: first example

We consider a financial market consisting of one bond with prices

$$dB_t = B_t r dt, \quad B_0 = 1, \quad \text{i.e.} \quad B_t = \mathrm{e}^{r\,t},$$

and one stock with prices evolving like

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t), \quad S_0 = s_0 > 0,$$

with trend parameter $\mu \in I\!R$ and volatility $\sigma > 0$. The unique solution of this SDE is

$$S_t = s_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma\, W_t \right\}$$

as can be verified by Itô's formula, Theorem C.3. The wealth (the portfolio value) of an investor with initial capital $x_0 > 0$ evolves like

$$dX_t = N_t^B dB_t + N_t^S dS_t, \quad X_0 = x_0,$$

where N_t^B and N_t^S are the number of bonds and stocks, respectively, held by the investor at time t. This definition corresponds to a self-financing portfolio since changes in the wealth are only due to changes in the bond or stock prices (there is no consumption or endowment).

As control at time t we may use the fraction u_t of the wealth which should be invested in the stocks. Then

$$N_t^B = \frac{(1-u_t)X_t}{B_t}, \quad N_t^S = \frac{u_t X_t}{S_t}$$

yielding

$$\begin{aligned} dX_t &= (1-u_t)X_t r\, dt + u_t X_t(\mu\, dt + \sigma\, dW_t) \\ &= X_t\left((r + u_t(\mu - r))dt + u_t \sigma\, dW_t\right). \end{aligned}$$

Guessing

$$X_t = x_0 \exp\left\{ \int_0^t g_s\, ds + \int_0^t h_s\, dW_s \right\},$$

applying Itô's formula, and comparing the coefficients yields

$$g_t + \frac{1}{2} h_t^2 = r + (\mu - r) u_t, \quad h_t = \sigma\, u_t.$$

Thus

$$X_t = \exp\left\{ \int_0^t (r + (\mu - r)u_s - \frac{1}{2}\sigma^2 u_s^2) ds + \int_0^t \sigma\, u_s dW_s \right\}.$$

We want to maximize (1.2) with $\psi \equiv 0$, $\Psi(T, x) = \log(x)$, i.e.

$$J(0, x_0, u) = \mathrm{E}[\log(X_T^u) \mid X_0 = x_0]$$

over all control strategies in

$$\mathcal{A}(x_0) = \{u : \mathrm{E}\int_0^T (|b\, u_t| + |\sigma u_t|^2) dt < \infty, X_t^u > 0, \mathrm{E}[(\log X_T)^-] < \infty\}.$$

These conditions imply that $\int \sigma\, u_t dW_t$ is a martingale, in particular

$$\mathrm{E}\left[\int_0^T \sigma\, u_t dW_t\right] = 0.$$

Therefore we obtain for $u \in \mathcal{A}(x_0)$

$$J(0, x_0, u) = \log x_0 + \mathrm{E}\left[\int_0^T (r + (\mu - r)u_t - \frac{1}{2}\sigma^2 u_t^2) dt\right]$$

Taking derivatives for the integrand yields

$$\frac{\partial}{\partial u}\left(r + (\mu - r)u - \frac{1}{2}\sigma^2 u^2\right) = \mu - r - \sigma^2 u$$
$$\frac{\partial^2}{\partial u^2}\left(r + (\mu - r)u - \frac{1}{2}\sigma^2 u^2\right) = -\sigma^2 < 0.$$

So a pointwise maximization yields that the best choice of u_t is always (setting the first line equal 0 and solving for u)

$$u_t^* = \pi^* := \frac{\mu - r}{\sigma^2} \quad \text{for all} \quad t \in [0, T]. \tag{1.3}$$

The value function is

$$V(0, x_0) = \sup_{u \in \mathcal{A}(x_0)} J(0, x_0, u) = J(0, x_0, u^*) = \log(x_0) + \left(r + \frac{1}{2}\frac{(\mu - r)}{\sigma^2}\right) T.$$

The strategy given by (1.3) is the *Merton strategy* (for logarithmic utility), and we call π^* *Merton fraction.* So, if e.g. $\pi^* = 0.4$, this means that an investor should always keep 40% percent of his money invested in the stock. Note that this strategy requires a lot of trading.

Remark 1.1 We can get a corresponding result for n stocks with prices $(S_t)_{t\in[0,T]}$,

$$S_t = \begin{pmatrix} S_t^1 \\ \vdots \\ S_t^n \end{pmatrix}.$$

with dynamics

$$dS_t = \mathrm{Diag}(S_t)(\mu\, dt + \sigma\, dW_t), \quad S_0 = s_0,$$

where $\mathrm{Diag}(S_t)$ is the diagonal matrix with diagonal S_t, and W is a n-dimensional Wiener process, $s_0^i > 0$ for $I = 1, \ldots, n$, $\mu \in I\!R^n$, and σ a non-singular volatility matrix in $I\!R^{n\times n}$. So for stock i we have dynamics

$$dS_t^i = S_t^i \left(\mu_i\, dt + \sum_{j=1}^n \sigma_{ij}\, dW_t^j \right), \quad i = 1, \ldots, n.$$

We can define controls as a n-dimensional process u, where the i-th component u_t^i corresponds to the fraction of wealth which is invested in stock i. We then get the optimal solution

$$u_t^* = \pi^* = (\sigma\sigma^\top)^{-1}(\mu - r), \quad t \in [0, T],$$

where $^\top$ denotes transposition, i.e. $\sigma^\top$ is the transposed matrix to σ.

2 Dynamic Programming

2.1 Itô diffusions and their generators

We consider a n-dimensional SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{2.4}$$

where W is a m-dimensional Wiener process and the measurable *drift* and *diffusion coefficients*

$$b : [0, \infty) \times I\!R^n \to I\!R^n, \quad \sigma : [0, \infty) \times I\!R^n \to I\!R^{n\times m}$$

satisfy for some constant $K > 0$ and for all $x, y \in I\!R^n$, $s, t \geq 0$

$$\|b(s, x) - b(t, y)\| + \|\sigma(s, x) - \sigma(t, y)\| \leq K(\|y - x\| + |t - s|), \tag{2.5}$$

$$\|b(t, x)\|^2 + \|\sigma(t, x)\|^2 \leq K^2(1 + \|x\|^2). \tag{2.6}$$

We consider the filtration $\mathcal{F}$ generated by W and augmented with the null sets. Under these conditions (2.4) has a unique and strong solution X which we call *Itô diffusion.* We call

$$a(t, x) := \sigma(t, x)\sigma(t, x)^\top$$

the *diffusion matrix* of X.
For a random variable Y we write

$$\mathrm{E}_{t,x}[Y] = \mathrm{E}[Y \mid X_t = x].$$

and $\mathrm{E}_x[Y] = \mathrm{E}_{0,x}[Y]$.

Theorem 2.1 *Suppose that X is a time-homogeneous Itô diffusion and $f : I\!R^n \to I\!R$ bounded and measurable.*

(i) For all $\omega \in \Omega$

$$\mathrm{E}_x[f(X_{t+s}) \,|\, \mathcal{F}_t](\omega) = \mathrm{E}_{X_t(\omega)}[f(X_s)], \quad t, s \geq 0.$$

(ii) If τ is a stopping time with $\tau < \infty$, then

$$\mathrm{E}_x[f(X_{\tau+s}) \,|\, \mathcal{F}_\tau](\omega) = \mathrm{E}_{X_\tau(\omega)}[f(X_s)], \quad s \geq 0, \quad \omega \in \Omega.$$

Proof: Theorems 7.1.2 und 7.2.4 in [Ok00]. □

Part (i) is called *Markov property*, part (ii) *strong Markov property*. The dependency on ω is usually not written explicitly. Hence $\mathrm{E}_{X_t}[f(X_s)]$ is a random variable

$$g(X_t), \quad \text{where} \quad g : I\!R^n \to I\!R, \quad g(x) = \mathrm{E}_x[f(X_s)].$$

From now on suppose that X is an Itô-Diffusion like in (2.4).

The *infinitesimal generator* L of X is defined as

$$Lf(s,x) = \lim_{t \searrow s} \frac{\mathrm{E}_{s,x}[f(t, X_t)] - f(s,x)}{t-s}$$

for all $s \geq 0$, $x \in I\!R^n$ and f in the *domain* $\mathcal{D}_L$ of L which is the class of functions $f : [0,\infty) \times I\!R^n \to I\!R$ for which the limit exists for all s, x.

Further we define a partial differential operator $\mathcal{L}$ by

$$\mathcal{L} := \frac{\partial}{\partial t} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \tag{2.7}$$

This operator can be applied to functions f in

$$\mathcal{C}^{1,2} := \{g = g(t,x) : [0,\infty) \times I\!R^n \to I\!R \,:\, g \text{ cont. dbl. in } t, \text{ twice cont. dbl. in } x\}$$

yielding

$$\begin{aligned}\mathcal{L}f(t,x) &= f_t(t,x) + \sum_{i=1}^{n} f_{x_i}(t,x)^\top b_i(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x) f_{x_i,x_j}(t,x) \\ &= f_t(t,x) + (D_x f(t,x))^\top b(t,x) + \frac{1}{2}\mathrm{tr}((D_{xx} f(t,x)) a(t,x))\end{aligned}$$

where $D_x f$ denotes the gradient of f, D_{xx} the Hessian of f, i.e. $(D_{xx} f)_{ij} = f_{x_i,x_j}$, and 'tr$(A)$' is the *trace* of matrix A, the sum of the diagonal elements of A.

Remark 2.2 (i) Note that Itô's formula can be written as

$$df(t, X_t) = \mathcal{L}f(t, X_t)\,dt + (D_x f(t, X_t))^\top \sigma(t, X_t) dW_t.$$

(ii) If there is no dependency on t and all processes are 1-dimensional, we simply have

$$\mathcal{L}f(x) = b(x)f'(x) + \frac{1}{2}a(x)f''(x).$$

Theorem 2.3 *Suppose that $f \in \mathcal{C}^{1,2}$ and for all $u \geq t \geq 0$, $x \in \mathbb{R}$*

$$\mathrm{E}_{t,x}\left[\int_t^u |\mathcal{L}f(s, X_s)| ds\right] < \infty, \quad \mathrm{E}_{t,x}\left[\int_t^u |(D_x f(s, X_s))^\top \sigma(s, X_s)|^2 ds\right] < \infty.$$

Then $f \in \mathcal{D}_L$ and $Lf(t, x) = \mathcal{L}f(t, x)$, $t \geq 0$, $x \in \mathbb{R}^n$.

Proof: Follows directly from Itô's formula, cf. Remark 2.2 (i). □

Naturally, $\mathcal{L}$ is also called *generator* of X. The conditions in Theorem 2.3 hold, if $f \in \mathcal{C}^{1,2}$ with compact support, i.e. $f(x) = 0$ for $x \notin C$ for some compact set $C \subset \mathbb{R}^n$.

Theorem 2.4 Dynkin's Formula.
Let $f \in \mathcal{C}^{1,2}$ be a function with compact support and τ a stopping time satisfying $\mathrm{E}_x[\tau] < \infty$. *Then*

$$\mathrm{E}_x[f(\tau, X_\tau)] = f(0, x) + \mathrm{E}_x\left[\int_0^\tau \mathcal{L}f(s, X_s)\, ds\right].$$

Proof: Theorem 7.4.1 in [Ok00]. □

If τ is a first exit time from a bounded set $A \subset \mathbb{R}^n$, then Dynkins Formula holds for any $f \in \mathcal{C}^{1,2}$, since $f|_A$ can be extended outside of A correspondingly.

Example 2.5 For a 1-dimensional Wiener process W we consider

$$X_t = x_0 + W_t, \quad t \geq 0,$$

and the first exit time τ of the interval (a, b) with $a \leq x_0 \leq b$,

$$\tau = \inf\{t \geq 0 \,:\, X_t \notin (a, b)\}.$$

We want to find $p_{x_0} := P(X_\tau = b \,|X_0 = x_0)$ and $\mathrm{E}_{x_0}[\tau]$. We can proceed as follows:

- Show that $P(\tau < \infty \,|X_0 = x_0) = 1$ for all x_0. This can be done using that $W_{t+\Delta t} - W_t$ is normally distributed with mean 0 and variance Δt. Then
$$\mathrm{E}_{x_0} f(X_\tau) = p_{x_0} f(b) + (1 - p_{x_0}) f(a). \tag{2.8}$$
- Find the generator of X. Since $dX_t = 0\,dt + 1\,dW_t$ we get from (2.7)
$$\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}.$$
- Solve $\mathcal{L} f_0 = 0$. This gives $f_0(x) = c_0 x + d_0$ for constants $c_0, d_0 \in I\!R$. Dynkin's formula then yields
$$\mathrm{E}_{x_0}[f_0(X_\tau)] = f_0(x_0).$$
A comparison with (2.8) yields
$$p_{x_0} = \frac{f_0(x_0) - f_0(a)}{f_0(b) - f_0(a)} = \frac{x_0 - a}{b - a}.$$
- Solve $\mathcal{L} f_1 = 1$. This yields $f_1(x) = x^2 + c_1 x + d_1$ for constants $c_1, d_1 \in I\!R$. From Dynkin's formula we get
$$\mathrm{E}_{x_0}[f_1(X_\tau)] = f_1(x_0) + \mathrm{E}_{x_0}[\tau].$$
Comparing with (2.8) and using the solution for p_{x_0} we get
$$\mathrm{E}_{x_0}[\tau] = (b - x_0)(x_0 - a).$$
For $x_0 = a$ or $x_0 = b$ we have $\mathrm{E}_{x_0}[\tau] = 0$ as expected. The maximum expected time we get for starting at the mean $x_0 = \frac{a+b}{2}$. This yields
$$p_{\frac{a+b}{2}} = \frac{1}{2} \quad \text{and} \quad \mathrm{E}_{\frac{a+b}{2}}[\tau] = \frac{(b-a)^2}{4}.$$

2.2 The idea of dynamic programming

To solve the stochastic control problem presented in Section 1.1 we may proceed as follows:

- Use the Bellman Principle (if it holds)
$$V(t, x) = \sup_{u \in \mathcal{A}(t,x)} \mathrm{E}_{t,x}\left[\int_t^{t_1} \psi(s, X_s^u, u_s) ds + V(t_t, X_{t_1}^u)\right].$$
This principle states that choosing an optimal control in $[t, t_1]$ yields an optimal control if we continue optimally at t_1. This principle has to be proved.

- Apply Itô's formula to V (if V is smooth enough, e.g. $V \in \mathcal{C}^{1,2}$), yielding

$$\begin{aligned} V(t,x) &= \sup_{u \in \mathcal{A}(t,x)} \mathrm{E}_{t,x}\bigg[\int_t^{t_1} \psi(s, X_s, u_s) ds + V(t, X_t) \\ &\quad + \int_t^{t_1} V_t(s, X_s) + (D_x V(s, X_s))^\top b(s, X_s, u_s)\, ds \\ &\quad + \int_t^{t_1} \frac{1}{2} \mathrm{tr}\left((D_{xx} V(s, X_s)) a(s, X_s, u_s)\right)\, ds \\ &\quad + \int_t^{t_1} (D_x V(s, X_s))^\top \sigma(s, X_s, u_s) dW_s \bigg], \end{aligned}$$

where a is the diffusion matrix

$$a(s, X_s, u_s) := \sigma(s, X_s, u_s)\sigma(s, X_s, u_s)^\top.$$

If $\int_t^{t_1} (D_x V(s, X_s))^\top \sigma(s, X_s, u_s) dW_s$, $t_1 \geq t$, is a martingale and hence its expectation equals 0, we obtain

$$\begin{aligned} V(t,x) &= \sup_{u \in \mathcal{A}(t,x)} \mathrm{E}_{t,x}\bigg[\int_t^{t_1} \psi(s, X_s, u_s) ds + V(t, X_t) \\ &\quad + \int_t^{t_1} V_t(s, X_s) + (D_x V(s, X_s))^\top b(s, X_s, u_s)\, ds \\ &\quad + \int_t^{t_1} \frac{1}{2} \mathrm{tr}\left((D_{xx} V(s, X_s)) a(s, X_s, u_s)\right)\, ds \bigg]. \end{aligned}$$

- Subtract $V(t,x)$ on both sides, divide by $t_1 - t$ and go to the limit $t_1 \searrow t$. If 'sup' and 'lim' and expectation and 'lim' can be interchanged we get – using $V(t, X_t) = V(t,x)$ under $\mathrm{E}_{t,x}$ and $u_t \in \mathcal{U}$ –

$$0 = \sup_{u \in \mathcal{U}} \left\{ \psi(t,x,u) + V_t(t,x) + (D_x V(t,x))^\top b(t,x,u) + \frac{1}{2} \mathrm{tr}\left((D_{xx} V(t,x)) a(t,x,u)\right) \right\} \tag{2.9}$$

Defining an operator depending on u by

$$\mathcal{L}^u f(t,x) = (D_x f(t,x))^\top b(t,x,u) + \frac{1}{2} \mathrm{tr}((D_{xx} f(t,x)) a(t,x,u))$$

we can write (2.9) as

$$0 = \sup_{u \in \mathcal{U}} \left\{ \psi(t,x,u) + \mathcal{L}^u V(t,x) \right\}. \tag{2.10}$$

The equation (2.9) (or equivalently (2.10)) is called *Hamilton Jacobi Bellman equation*, short *HJB equation*. The above reasoning shows that under certain conditions the value function solves the HJB equation, so it provides a necessary condition.
Vice versa we may ask, when a solution of the HJB equation is the value function of the correspondin control problem. To this end we proceed as follows to 'solve' the HJB equation.

2.6 Algorithm

1. Find an optimal $u = \hat{u}(t,x)$ in (2.9).

2. If it exists, $\hat{u}$ formally depends on the derivatives V_t, $D_x V$, $D_{xx}V$, i.e.

$$\hat{u}(t,x) = \tilde{u}(t,x,V_t(t,x), D_x V(t,x), D_{xx}V(t,x)).$$

   Substituting $\hat{u}$ in (2.9) leads to a partial differential equation for V which has to be solved with boundary condition $V(T,x) = \Psi(T,x)$ to find a candidate V^* for the optimal value function.

3. If V^* satisfies certain conditions (see below) and $u_t^* = \hat{u}(t, X_t^*)$, $t \in [0,T]$, is an admissible control strategy, then V^* is indeed the value function of the control problem and $u_t^* = \hat{u}(t, X_t^*)$ defines an optimal control strategy in *Markovian form.* Here X_t^* is the solution of (1.1) using the optimal control strategy u^* in $[0,t)$.

A theorem which provides a set of conditions on V, ψ, Ψ, b, σ and u such that a solution V of the HJB equation and the corresponding maximizer $\hat{u}$ found in steps 1 and 2 of the above algorithm provide indeed the value function and an optimal control strategy, is called a *verification theorem.*

2.3 A verification theorem

In Section 2.2 we saw that under certain conditions, the value function V satisfies the HJB equation. Here we shall provide a verification theorem which guarantees that vice versa the solution found by Algorithm 2.6, steps 1 and 2, indeed provides the value function and an optimal control strategy.
We consider the control problem of Section 1.1. First we have to specify when a control strategy is admissible. Say, $u \in \mathcal{A}(t,x)$ if

(A1) $u = (u_s)_{s\in[t,T]}$ is progressively measurable, has values in $\mathcal{U}$, and satisfies $\mathrm{E}\left[\int_t^T \|u_s\|^2 ds\right] < \infty$.

(A2) (1.1) has a unique strong solution $(X_s)_{s\in[t,T]}$ with $X_t = x$ and

$$\mathrm{E}_{t,x}[\sup_{t\le s\le T} \|X_s\|^2] < \infty$$

(A3) $J(t,x,u)$ is well defined.

Condition (A3) will be guaranteed by (A1) and the conditions of the following theorem.

Theorem 2.7 Verification Theorem
Suppose that $\|\sigma(t,x,u)\|^2 \le C_\sigma(1 + \|x\|^2 + \|u\|^2)$ *and that* ψ *is continuous with* $\|\psi(t,x,u)\|^2 \le C_\psi(1 + \|x\|^2 + \|u\|^2)$ *for some* $C_\sigma, C_\psi > 0$ *and all* $t \ge 0$, $x \in \mathbb{R}^n$, $u \in \mathcal{U}$.

(i) Suppose that Φ lies in $\mathcal{C}^{1,2}([0,T) \times I\!R^n)$, is continuous on $[0,T] \times I\!R^n$ with $\|\Phi(t,x)\| \le C_\Phi(1+\|x\|^2)$, and satisfies the HJB equation and the boundary condition, i.e.

$$\begin{aligned}&\sup_{u\in\mathcal{U}} \{\psi(t,x,u) + \mathcal{L}^u\Phi(t,x)\} = 0, \quad t \in [0,T), x \in I\!R^n \\ &\Phi(T,x) = \Psi(T,x), \quad x \in I\!R^n.\end{aligned}$$

Then for all $t \in [0,T]$, $x \in I\!R^n$

$$\Phi(t,x) \ge V(t,x).$$

(ii) If a maximizer $\hat{u}(t,x)$ of $u \mapsto \psi(t,x,u) + \mathcal{L}^u\Phi(t,x)$ exists such that $u^* = (u_t^*)_{t\in[0,T]}$, $u_t^* = \hat{u}(t,X_t^*)$ is admissible, then $\Phi(t,x) = V(t,x)$ for all $t \in [0,T]$, $x \in I\!R^n$ and u^* is an optimal control strategy, i.e. $V(t,x) = J(t,x,u^{t,x})$ where $u^{t,x} = (u_s^*)_{s\in[t,T]} \in \mathcal{A}(t,x)$. Here X_t^* is the solution of (1.1) using control u_s^* on $[0,t)$.

Proof: Keep some $t \in [0,T]$, $x \in I\!R^n$ fixed. For arguing with bounded processes, we introduce

$$\tau_n = T \wedge \inf\{s > t \,:\, \|X_s - X_t|\| \ge n\}, \quad n \in I\!N.$$

The Itô formula for $X_t = x$ and admissible u yields

$$\Phi(\tau_n, X_{\tau_n}) = \Phi(t,x) + \int_t^{\tau_n} \mathcal{L}^{u_s}\Phi(s,X_s)ds + \int_t^{\tau_n} \Phi_x(s,X_s)^\top \sigma(s,X_s,u_s)dW_s.$$

From the admissibility of u, the continuity of Φ and the boundedness of X on $[t,\tau_n]$ we get $\mathrm{E}_{t,x}\left[\int_t^{\tau_n} \|\Phi_x(s,X_s)^\top\sigma(X_s,u_s)\|^2 ds\right] < \infty$ and thus

$$\mathrm{E}_{t,x}\left[\int_t^{\tau_n} \Phi_x(s,X_s)^\top \sigma(X_s,u_s)dW_s\right] = 0.$$

Therefore we get

$$\begin{aligned}
&\mathrm{E}_{t,x}\left[\int_t^{\tau_n} \psi(s,X_s,u_s)ds + \Phi(\tau_n,X_{\tau_n})\right] \\
&= \mathrm{E}_{t,x}\left[\int_t^{\tau_n} \psi(s,X_s,u_s)ds + \Phi(t,x) + \int_t^{\tau_n} \mathcal{L}^{u_s}\Phi(s,X_s)ds\right] \\
&= \Phi(t,x) + \mathrm{E}_{t,x}\left[\int_t^{\tau_n} \underbrace{\left(\psi(s,X_s,u_s) + \mathcal{L}^{u_s}\Phi(s,X_s)\right)}_{\le 0} ds\right] \\
&\le \Phi(t,x) \qquad (2.11)
\end{aligned}$$

since Φ satisfies the HJB and $u_s \in \mathcal{U}$ for $s \in [t,T]$.

For $n \to \infty$ we get $\tau_n \to T$. The quadratic growth conditions for ψ and Φ and the admissibility of u implies

$$\left| \int_t^{\tau_n} \psi(s, X_s, u_s) ds + \Phi(\tau_n, X_{\tau_n}) \right| \leq C_\psi \int_t^T (1+\|X_s\|^2+\|u_s\|^2) ds + C_\Phi(1+\|X_T\|^2) \in L^1.$$

So we get by dominated convergence and the continuity of Φ

$$\mathrm{E}_{t,x}\left[\int_t^{\tau_n} \psi(s, X_s, u_s) ds + \Phi(\tau_n, X_{\tau_n})\right] \to J(t,x,u) \quad (n \to \infty).$$

Thus we obtain from (2.11) that also $J(t,x,u) \leq \Phi(t,x)$ and by taking the supremum finally $V(t,x) \leq \Phi(t,x)$. This proves part (i).
For (ii) only observe that we have equality in (2.11) if we can find a maximizer $\hat{u}(t,x)$ and consider the strategy $u_t^* = \hat{u}(t, X_t^*)$ defined by this maximizer. Since this was the only inequality we get with the same arguments $V(t,x) = J(t,x,u^*) = \Phi(t,x)$. □

Remark 2.8

(i) Under the conditions of Theorem 2.7 the *Bellman Principle* holds: For any stopping time τ with values in $[t,T]$

$$V(t,x) = \sup_{u \in \mathcal{A}(t,x)} \mathrm{E}_{t,x}\left[\int_t^{\tau} \psi(s, X_s^u, u_s) ds + V(\tau, X_\tau^u)\right].$$

(ii) By definition the value function is always unique. Therefore Theorem 2.7 shows that a solution of the HJB equation is unique in the class of $\mathcal{C}^{1,2}$-functions with quadratic growth. Note that a control strategy does not have to be unique.

(ii) Existence is difficult to show in general. A typical set of very strong conditions is

- $\mathcal{U}$ is compact,
- Ψ is three times continuously differentiable in x and is bounded,
- b, σ, ψ in $\mathcal{C}^{1,2}$ and bounded,
- The uniform parabolicity condition holds, i.e.

$$y^\top a(t,x,u) y \geq c\,\|y\|^2$$

for all $t \in [0,T]$, $x, y \in I\!R^n$, $u \in \mathcal{U}$.

Then the HJB equation has a unique bounded solution $V \in \mathcal{C}^{1,2}$.

Example 2.9 We consider the control of

$$X_t^u = x + \int_0^t u_s ds + W_t,$$

everything one-dimensional, with performance criterion (for $\alpha, \beta > 0$)

$$J(t,x,u) = \mathrm{E}_{t,x}\left[\beta\, X_t^u - \int_t^T \alpha\, u_s^2 ds\right]$$

and progressively measurable controls u with values in $I\!R$ and satisfying $\int_0^T u_s^2 ds < \infty$. The augmented generator of X^u is given by

$$\mathcal{L}^u f(t,x) = u\, f_x(t,x) + \frac{1}{2}\, f_{xx}(t,x).$$

So in the first step of Algorithm 2.6 we have to solve the maximization problem in

$$\sup_{u \in I\!R}\{\mathcal{L}^u V(t,x) - \alpha\, u^2\} = 0$$

yielding the maximizer $\hat{u}(t,x) = \frac{V_x(t,x)}{2\,\alpha}$. Plugging this in the HJB equation and using the boundary condition $V(T,x) = \beta\, X_T$ we now have to solve (step 2)

$$-\alpha\, \hat{u}^2(t,x) + \mathcal{L}^{\hat{u}(t,x)} V(t,x) = V_t(t,x) + \frac{1}{4\,\alpha} V_x^2(t,x) + \frac{1}{2} V_{xx}(t,x) = 0.$$

This yields

$$V(t,x) = \beta\, x + \frac{\beta^2}{4\,\alpha}(T-t).$$

Step 3 consists of checking the conditions of the verification theorem which can easily be done for this example.

2.4 Example: Optimal investment

We use a model like in Remark 1.1, only that we now allow for more Wiener processes than we have stocks in the market (so called *incomplete* market). We consider a financial market consisting of one bond with prices

$$dB_t = B_t r dt, \quad B_0 = 1, \quad \text{i.e.} \quad B_t = \mathrm{e}^{r\,t},$$

and n stocks with prices evolving like

$$dS_t = \mathrm{Diag}(S_t)(\mu\, dt + \sigma\, dW_t), \quad S_0 = s_0,$$

where $\mathrm{Diag}(S_t)$ is the diagonal matrix with diagonal S_t, and W is a m-dimensional Wiener process, $m \geq n$, $s_0^i > 0$ for $i = 1, \ldots, n$, $\mu \in I\!R^n$, and

σ a matrix in $I\!R^{n\times m}$ with maximal rank. The latter implies that the $I\!R^{n\times n}$-matrix $\sigma\sigma^\top$ is non-singular. So for stock i we have dynamics

$$dS_t^i = S_t^i\left(\mu_i\,dt + \sum_{j=1}^m \sigma_{ij}\,dW_t^j\right), \quad i = 1,\dots,n.$$

As controls $u = (\pi, c)$ we consider the vector of risky fractions $\pi_t = (\pi_t^1,\dots,\pi_t^n)^\top$, where π_t^i is the fraction of the wealth invested in stock i, and the consumption rate c_t. The corresponding wealth process satisfies

$$\begin{aligned} dX_t^u &= \sum_{i=1}^n X_t^u \pi_t^i dS_t^i/S_t^i + X_t^u(1-\pi_t)dB_t/B_t - c_t\,dt \\ &= \left(\left(r + \pi_t^\top(\mu - r\,\mathbf{1})\right)X_t^u - c_t\right)dt + X_t^u\pi_t^\top\sigma\,dW_t, \end{aligned}$$

where $\mathbf{1}$ denotes the n-dimensional vector $(1,\dots,1)^\top$. The investor assigns utility $U_1(c_t)$ to the payout given by the consumption rate c_t and utility $U_2(X_T^u)$ to the terminal wealth. We consider power utility functions

$$U(x) = \frac{x^\alpha}{\alpha}, \quad \alpha < 1, \alpha \neq 0$$

and a discounting factor $\mathrm{e}^{-\beta t}$, $\beta \geq 0$. Thus we want to maximize

$$J(t,x,u) = \mathrm{E}_{t,x}\left[\int_t^T \mathrm{e}^{-\beta t}U(c_t)\,dt + \mathrm{e}^{-\beta T}U(X_T^u)\right].$$

The augmented generator is given by

$$\mathcal{L}^{(\pi,c)}v(t,x) = v_t(t,x) + \left((r + \pi^\top(\mu - r\,\mathbf{1}))x - c\right)v_x(t,x) + \frac{1}{2}\pi^\top\sigma\sigma^\top\pi x^2 v_{xx}(t,x)$$

and the corresponding HJB equation reads as

$$\sup_{\pi,c}\left\{e^{-\beta t}\frac{c^\alpha}{\alpha} + \mathcal{L}^{(\pi,c)}V(t,x)\right\} = 0.$$

Step 1: If x is strictly positive and V is increasing and concave, then we get maximizers

$$\hat{c}(t,x) = \left(e^{\beta t}V_x(t,x)\right)^{\frac{1}{\alpha-1}} \tag{2.12}$$

$$\hat{\pi}(t,x) = -(\sigma\sigma^\top)^{-1}(\mu - r\,\mathbf{1})\frac{V_x(t,x)}{xV_{xx}(t,x)}. \tag{2.13}$$

Step 2: Putting this in the HJB equation, it remains to solve

$$\frac{1-\alpha}{\alpha}\,\mathrm{e}^{-\frac{\beta t}{1-\alpha}}V_x^{\frac{\alpha}{\alpha-1}} + V_t + r\,x\,V_x - \frac{1}{2}(\mu - r\,\mathbf{1})^\top(\sigma\sigma^\top)^{-1}(\mu - r\,\mathbf{1})\frac{V_x^2}{V_{xx}} = 0$$

with boundary condition $V(T,x) = \mathrm{e}^{-\beta T} \frac{x^\alpha}{\alpha}$. Making an ansatz

$$V(t,x) = h(t)^{1-\alpha} \, \frac{x^\alpha}{\alpha} \tag{2.14}$$

yields for h the boundary condition $h(T) = \mathrm{e}^{-\frac{\beta T}{1-\alpha}}$ and

$$\mathrm{e}^{-\frac{\beta t}{1-\alpha}} + c\, h(t) + h'(t) = 0,$$

where

$$c = \frac{\alpha}{1-\alpha} \left(r + \frac{1}{2(1-\alpha)} (\mu - r\,\mathbf{1})^\top (\sigma\sigma^\top)^{-1} (\mu - r\,\mathbf{1}) \right).$$

This can be solved using standard methods yielding

$$h(t) = \mathrm{e}^{-\frac{\beta t}{1-\alpha}} \mathrm{e}^{-c\,(T-t)} + \frac{(1-\alpha)\mathrm{e}^{-ct}}{\beta - (1-\alpha)c} \left\{ \mathrm{e}^{-\frac{\beta-(1-\alpha)c}{1-\alpha} t} - \mathrm{e}^{-\frac{\beta-(1-\alpha)c}{1-\alpha} T} \right\}$$

if $\beta - (1-\alpha)c \neq 0$ and

$$h(t) = \mathrm{e}^{-ct}(1 + T - t)$$

if $\beta - (1-\alpha)c = 0$.

Step 3: Note that $h(t)$ is always strictly positive. Hence the value function (2.14) is strictly positive if the SDE for the controlled process has a strictly positive unique solution when using controls given by the maximizers $\hat{\pi}$, $\hat{c}$. Then, V would clearly lie in $\mathcal{C}^{1,2}$ and be strictly increasing and concave since $V_x(t,x) = h(t)^{1-\alpha} x^{\alpha-1} > 0$ and $V_{xx}(t,x) = -(1-\alpha) h(t)^{1-\alpha} x^{\alpha-2} < 0$.

Note further that with (2.14) we get from (2.12), (2.13)

$$\begin{aligned} \hat{c}(t,x) &= \frac{\mathrm{e}^{-\frac{\beta t}{1-\alpha}}}{h(t)}\, x, \\ \hat{\pi}(t,x) &= \frac{1}{1-\alpha} (\sigma\sigma^\top)^{-1} (\mu - r\,\mathbf{1}). \end{aligned}$$

For controls

$$c_t^* = \hat{c}(t,x) = \frac{\mathrm{e}^{-\frac{\beta t}{1-\alpha}}}{h(t)}\, X_t^*, \tag{2.15}$$

$$\pi_t^* = \hat{\pi}(t,x) = \frac{1}{1-\alpha} (\sigma\sigma^\top)^{-1} (\mu - r\,\mathbf{1}) \tag{2.16}$$

the SDE for the controlled process $X^* := X^{(\pi^*,c^*)}$ is of the form $dX_t^* = X_t^*((c_1 + f_1(t))dt + c_2 dW_t)$ and hence admits a unique strong solution $X_t^* = x_0 \exp\{\mathit{something}\}$ which is strictly positive and (as a strong solution) satisfies the integrability conditions we need. Since π^* is constant and $h(t)^{-1}$ bounded the admissibility and growth conditions can be verified using a suitable set $\mathcal{U}$ for the controls (partly difficult).

Thus the value function is indeed given by (2.14) and an optimal control strategy by (2.15), (2.16).

2.5 Types of controls

In general a control at time t is given by some random variable $u_t(\omega)$ One can distinguish some special cases:

- *Open loop* or *deterministic* controls, if $u_t(w) = f(t)$ is only a function in t (non random).
- *Closed loop* or *feedback* controls: u is adapted to the filtration generated by the controlled process, i.e. u_t is $\sigma(X_s^u, s \leq t)$-measurable.
- A special case of the feedback controls are *Markovian* controls which are of the form $u_t(\omega) = f(t, X_t(\omega))$.

In particular the controls given by the maximizers $\hat{u}$ like in Algorithm 2.6 and in the Verification Theorem 2.7 are Markovian. The proof of Theorem 2.7 shows that a Markovian control will be at least as good as any feedback control, if the former exists.

3 Some extensions

3.1 Minimization

Suppose we want to minimize the performance criterion. Then the value function would be of the form

$$\tilde{V}(t,x) = \inf_u \mathrm{E}_{t,x}\left[\int_t^T \tilde{\psi}(s, X_s, u_s)ds + \tilde{\Psi}(T, X_T)\right].$$

Switching to the supremum and defining $\psi := -\tilde{\psi}$, $\Psi := -\tilde{\Psi}$ yields

$$\tilde{V}(t,x) = -\sup_u \mathrm{E}_{t,x}\left[\int_t^T \psi(s, X_s, u_s)ds + \Psi(T, X_T)\right] = -V(t,x),$$

where we can find V as before as the value function for maximizing the performance criterion with ψ and Ψ.
In Section 3.7 we will see an example.

3.2 Infinite time horizon

We shall assume that neither the coefficients $b(x,u)$ and $\sigma(x,u)$ of the controlled process X nor $\psi(x,u)$ depend explicitly on time t. So we look at dynamics

$$dX_t = b(X_t, u_t)dt + \sigma(X_t, u_t)dW_t, \quad X_0 = x_0, \tag{3.17}$$

and define the corresponding differential operator depending on u by

$$\mathcal{L}^u f(t,x) = (D_x f(x))^\top b(x,u) + \frac{1}{2}\mathrm{tr}((D_{xx} f(x))a(x,u)).$$

As performance criterion we use

$$J(x,u) = \mathrm{E}_x\left[\int_0^\infty \mathrm{e}^{-\beta s}\psi(X_s,u_s)ds\right]$$

with discount factor $\beta > 0$. The conditions on b and σ and the admissibility of control strategies – now described by class $\mathcal{A}(x)$ – are defined analogously to Section 2.3. The value function is

$$V(x) = \sup_{u\in\mathcal{A}(x)} J(x,u).$$

Theorem 3.1 Verification Theorem
Suppose that $\|\sigma(x,u)\|^2 \le C_\sigma(1+\|x\|^2+\|u\|^2)$ *and that* ψ *is continuous with* $\|\psi(x,u)\|^2 \le C_\psi(1+\|x\|^2+\|u\|^2)$ *for some* $C_\sigma, C_\psi > 0$ *and all* $x \in I\!R^n$, $u \in \mathcal{U}$.

(i) Suppose that Φ *lies in* $\mathcal{C}^2(I\!R^n)$ *with* $\|\Phi(x)\| \le C_\Phi(1+\|x\|^2)$*, and satisfies the HJB equation*

$$\sup_{u\in\mathcal{U}}\{\psi(x,u) + \mathcal{L}^u\Phi(x) - \beta\Phi(x)\} = 0, \quad t\in[0,T), x\in I\!R^n.$$

Then for all $x \in I\!R^n$

$$\Phi(x) \ge V(x).$$

(ii) If a maximizer $\hat{u}(x)$ *of* $u \mapsto \psi(x,u) + \mathcal{L}^u\Phi(x) - \beta\Phi(x)$ *exists such that* $u^* = (u_t^*)_{t\ge 0}$, $u_t^* = \hat{u}(X_t^*)$ *is admissible, then* $\Phi(x) = V(x)$ *for all* $x \in I\!R^n$ *and* u^* *is an optimal control strategy, i.e.* $V(x) = J(x,u^*)$. *Here* X_t^* *is the solution of (3.17) using control* u_s^* *on* $[0,t)$.

The proof is similar to the proof of Theorem 2.7.

3.3 Example: Discounted utility of consumption

In the model of Section 2.4 we now would like to maximize

$$J(x,u) = \mathrm{E}_x\left[\int_0^\infty \mathrm{e}^{-\beta t}\frac{1}{\alpha}\,c_t^\alpha\,dt\right]$$

over admissible controls $u = (\pi, c)$ which control

$$dX_t = \big((r + \pi_t^\top(\mu - r\,\mathbf{1}))X_t - c_t\big)\,dt + X_t\pi_t^\top\sigma\,dW_t, \quad X_0 = x.$$

We consider $\beta > 0$, $\alpha \in (0,1)$, $x > 0$ and we shall require as additional admissibility conditions $P(X_t > 0) = 1$ for all $t > 0$. The value function is

$$V(x) = \sup_{u\in\mathcal{A}(x)} J(x,u).$$

The HJB equation reads

$$\sup_{u\in I\!R^n\times[0,\infty)}\left\{\Big((r+\pi^\top(\mu-r\,\mathbf{1}))x-c\Big)\,V_x+\frac{1}{2}\pi^\top\sigma\sigma^\top\pi x^2V_{xx}-\beta\,V+\frac{c^\alpha}{\alpha}\right\}=0.$$

Step 1: Suppose $V_x>0$ and $V_{xx}<0$. Then we get maximizers

$$\begin{aligned}\hat\pi(x) &= -\eta\frac{V_x(x)}{x\,V_{xx}(x)}, \qquad \text{where} \quad \eta:=(\sigma\sigma^\top)^{-1}(\mu-r\,\mathbf{1}),\\ \hat c(x) &= V_x(x)^{\frac{1}{\alpha-1}}.\end{aligned}$$

Step 2: Plugging these in the HJB we get the differential equation

$$-\frac{1}{2}\eta^\top\sigma\sigma^\top\eta\frac{V_x^2}{V_{xx}}+r\,x\,V_x-\beta\,V+\frac{1-\alpha}{\alpha}\,V_x^{\frac{\alpha}{\alpha-1}}=0.$$

Making an ansatz $V(x)=A\frac{x^\alpha}{\alpha}$ for some $A>0$ we have $V_x(x)=A\,x^{\alpha-1}$, $V_{xx}(x)=-(1-\alpha)A\,x^{\alpha-2}$ and it remains to solve (if $x>0$)

$$\frac{1}{2(1-\alpha)}\eta^\top\sigma\sigma^\top\eta+r-\frac{\beta}{\alpha}+\frac{1-\alpha}{\alpha}\,A^{\frac{1}{\alpha-1}}=0$$

So we have to assume

$$\beta>\frac{\alpha}{2(1-\alpha)}\eta^\top\sigma\sigma^\top\eta+\alpha\,r$$

since then $A>0$. Solving for A yields

$$A=\left(\frac{\alpha}{1-\alpha}\left(\frac{\beta}{\alpha}-r-\frac{1}{2(1-\alpha)}\eta^\top\sigma\sigma^\top\eta\right)\right)^{\alpha-1}.$$

Step 3: As candidates for the optimal policy we thus get

$$\begin{aligned}\pi^*_t &= \frac{1}{1-\alpha}\eta=\frac{1}{1-\alpha}(\sigma\sigma^\top)^{-1}(\mu-r\,\mathbf{1}),\\ \gamma^*_t &= A^{\frac{1}{\alpha-1}}X^*_t\end{aligned}$$

and the wealth process X^* controlled by (π^*,c^*) satisfies

$$dX^*_t=\frac{1}{1-\alpha}X^*_t((1-\alpha)r+\eta^\top(\sigma\sigma^\top\eta-r\mathbf{1})-(1-\alpha)A^{\frac{1}{\alpha-1}})dt+\eta^\top\sigma dW_t).$$

This SDE has a unique strong solution

$$X^*_t=X_0\exp\left\{\left(\left(1-\frac{1}{1-\alpha}\eta^\top\mathbf{1}\right)r+\frac{1-2\alpha}{2(1-\alpha)^2}\eta^\top\sigma\sigma^\top\eta-A^{\frac{1}{\alpha-1}}\right)t+\frac{1}{1-\alpha}\eta^\top\sigma W_t\right\}$$

which is strictly positive. Thus we also get $V_x>0$, $V_{xx}<0$ and one can prove the integrability conditions using that X is an L^2-process. Obviously, V lies in $\mathcal{C}^2(0,\infty)$. Hence by Theorem 3.1 we have found the optimal solution.

But note that we have restricted the domain of V to $(0,\infty)$ by the admissibility condition on the control strategies (only strictly positive wealth processes were allowed). Another way would be to terminate the evaluation as soon as $X_t \le 0$. This can be modelled by a suitable stopping time τ. Then boundary conditions would have to be specified for the case that we stop early. This can be done similarly as we do it in the next section for a finite time horizon.

3.4 Stopping the state process

Let u and X be control and state process as defined in Section 1.1. Now the wealth process should be constrained to a certain set. To this end let Q be an open set in $[0,T]\times \mathbb{R}^n$ and ∂Q the boundary of Q. The process should be stopped when leaving Q as described by the following stopping time

$$\tau := \inf\{t > 0 \,:\, (t, X_t) \notin Q\}.$$

In particular $(\{T\}\times\mathbb{R})\cap\overline{Q}$, where $\overline{Q} = Q\cup\partial Q$, is part of the boundary, so $P_{t,x}(\tau\le T) = 1$ for all $(t,x)\in Q$. Now let $\partial^* Q$ be a subset of the boundary which satisfies

$$P_{t,x}((\tau, X_\tau)\in\partial^* Q) = 1 \quad \text{for all} \quad (t,x)\in Q.$$

As performance criterion we use

$$J(t,x,u) = \mathrm{E}_{t,x}\left[\int_t^\tau \psi(s,X_s,u_s)ds + \Psi(\tau, X_\tau)\right],$$

where we have to specify Ψ also for $t<T$ on the boundary $\partial^* Q$. As usual,

$$V(t,x) = \sup_{u\in\mathcal{A}(t,x)} J(t,x,u),$$

and we define the admissibility conditions and the assumptions on b and σ as in Section 2.3.

Theorem 3.2 Verification Theorem
Suppose that $\|\sigma(t,x,u)\|^2 \le C_\sigma(1+\|x\|^2+\|u\|^2)$ *and that* ψ *is continuous with* $\|\psi(t,x,u)\|^2 \le C_\psi(1+\|x\|^2+\|u\|^2)$ *for some* $C_\sigma, C_\psi > 0$ *and all* $(t,x)\in Q$, $u\in\mathcal{U}$.

(i) Suppose $\Phi\in\mathcal{C}^{1,2}(Q)\cap\mathcal{C}(\overline{Q})$, *with* $\|\Phi(t,x)\|\le C_\Phi(1+\|x\|^2)$ *satisfying the HJB equation and the boundary condition, i.e.*

$$\sup_{u\in\mathcal{U}}\{\psi(t,x,u)+\mathcal{L}^u\Phi(t,x)\} = 0, \quad (t,x)\in Q$$
$$\Phi(t,x) = \Psi(t,x), \quad (t,x)\in\partial^* Q.$$

Then $\Phi(t,x)\ge V(t,x)$ *for all* $(t,x)\in Q$.

(ii) *If* $\hat{u}(t,x)$ *is a maximizer of* $u \mapsto \psi(t,x,u) + \mathcal{L}^u\Phi(t,x)$ *on* Q *and* $u^* = (u_t^*)_{t\le\tau}$, $u_t^* = \hat{u}(t,X_t^*)$ *is admissible, then* $\Phi(t,x) = V(t,x)$ *for all* $(t,x) \in Q$ *and* u^* *is an optimal control strategy, i.e.* $V(t,x) = J(t,x,u^{t,x})$ *where* $u^{t,x} = (u_s^*)_{s\in[t,T]} \in \mathcal{A}(t,x)$.

The proof is essentially the same as the proof of Theorem 2.7. Instead of τ_n we have to use stopping times $\tau_n \wedge \tau$.

Example 3.3 We look at the market model of Section 2.4 but consider only investment without consumption, i.e. we use controls $u_t = \pi_t$, where π_t^i is the fraction of wealth invested in stock i. For $\alpha \in (0,1)$ we want to maximize expected power utility of terminal wealth

$$\mathrm{E}\left[\frac{1}{\alpha}X_t^\alpha\right] \quad \text{such that} \quad P(X_T \ge q) = 1.$$

This problem is called *portfolio insurer problem* since there is a lower boundary for the payoff. It is quite attractive, since the distribution of the optimal terminal wealth of the unconstrained problem can be very skew, allowing for losses with high probability, and big gains only with a very low probability. We have to distinguish 3 cases:

- If $x_0 < \mathrm{e}^{-rT}q$, we cannot reach the minimum payout at time T with probability 1 since by investing in the bond we only get $x_0\mathrm{e}^{rT} < q$.
- If $x_0 = \mathrm{e}^{-rT}q$, pure investment in the bond yields exactly the payout q. So we cannot invest in the stocks since then we would make losses with a strictly positive probability, so we would miss q with a strictly positive probability.
- If $x_0 > \mathrm{e}^{-rT}q$ investment in bond and stocks is possible.

So let us assume that $x_0 > \mathrm{e}^{-rT}q$.
The same considerations show that at time t we need at least wealth $X_t \ge \mathrm{e}^{-r(T-t)}$ to be able to reach q with probability 1.
Thus we may define

$$Q = \{(t,x) \in (0,T) \times I\!R \,:\, x > \mathrm{e}^{-r(T-t)}q\}.$$

Then

$$\begin{aligned}\partial Q &= (\{0\} \times [\mathrm{e}^{-rT}q, \infty)) \\ &\quad \cup \{(t,x) \,:\, t \in (0,T), x = \mathrm{e}^{-r(T-t)}q\} \\ &\quad \cup (\{T\} \times [q,\infty)).\end{aligned}$$

Assuming that we start at some $x_0 > \mathrm{e}^{-rT}q$, we can only stop at

$$\partial^* Q = \{(t,x) \,:\, t \in (0,T), x = \mathrm{e}^{-r(T-t)}q\} \cup (\{T\} \times [q,\infty)).$$

So we define boundary conditions

$$\Psi(t,x) = \begin{cases} \frac{1}{\alpha}\, q^\alpha, & (t,x) \in \partial^* Q, t < T, \\ \frac{1}{\alpha}\, x^\alpha, & t = T. \end{cases}$$

So for $\tau = \inf\{t > 0 \,:\, (t, X_t) \notin Q\}$ we define

$$J(t,x,u) = \mathrm{E}_{t,x}\left[\frac{1}{\alpha}\, X_\tau^\alpha\right], \quad (t,x) \in Q$$

and

$$V(t,x) = \sup_{u \in \mathcal{A}(t,x)} J(t,x,u),$$

the admissibility conditions defined similar as in Section 2.3. The HJB equation with boundary conditions then reads as

$$\sup_{\pi \in I\!R^n} \left\{ V_t + (r + \pi^\top(\mu - r\,\mathbf{1}))xV_x + \frac{1}{2}\pi^\top \sigma\sigma^\top \pi\, x^2 V_x x \right\} = 0, \qquad (t,x) \in Q,$$
$$V(t,x) = \Psi(t,x), \qquad (t,x) \in \partial^* Q.$$

3.5 Example: Portfolio Insurer

We shall have a closer look at Example 3.3, for simplicity only for one stock ($n = 1$). First, without consumption and without constraints the value function V^0 can be determined as in Example 2.4) yielding

$$V^0(t,x) = \mathrm{e}^{(\alpha\, r + c_0)(T-t)}\, \frac{x^\alpha}{\alpha}, \quad t \in [0,T], x \in I\!R, \tag{3.18}$$

where

$$c_0 = \frac{\alpha}{2(1-\alpha)} \left(\frac{\mu - r}{\sigma}\right)^2.$$

Further the optimal risky fraction is

$$\pi^0 = \frac{1}{1-\alpha}\, \frac{\mu - r}{\sigma^2}.$$

With constraint $P(X_T \geq q) = 1$ we have seen in Example 3.3 that we have to solve

$$\sup_{\pi \in I\!R} \left\{ V_t + (r + \pi(\mu - r))xV_x + \frac{1}{2}\pi^2\sigma^2 x^2 V_{xx} \right\} = 0, \qquad (t,x) \in Q,$$
$$V(t,x) = \Psi(t,x), \qquad (t,x) \in \partial^* Q$$

with $Q, \partial^* Q, \Psi$ as given in Example 3.3. Taking derivatives we get as candidate

$$\hat{\pi}(t,x) = -\frac{\mu - r}{\sigma^2}\, \frac{V_x(t,x)^2}{x\, V_{xx}(t,x)}.$$

So we have to solve

$$V_t(t,x) + r\,x V_x(t,x) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{V_x^2(t,x)}{V_{xx}(t,x)} = 0 \tag{3.19}$$

subject to

$$V(t,x) = \Psi(t,x) = \begin{cases} \frac{1}{\alpha}\,q^\alpha, & (t,x) \in \partial^* Q, t < T, \\ \frac{1}{\alpha}\,x^\alpha, & t = T. \end{cases}$$

A separation approach for V (factorization in $f(t)x^\alpha$ as in Example 2.4) won't work. To see this, note that using the terminal condition we only would get the solution V^0 of the unconstrained problem which does not satisfy the boundary conditions for $t < T$.

So we may try to use a finite difference method to find a numerical approximation of V. Therefore it is convenient to work on some grid. Unfortunately the lower boundary $t \mapsto \mathrm{e}^{-r(T-t)}q$ depends on time. To get a constant boundary we will make a change of variables

$$y = e^{-rt}x.$$

Then we get the domain

$$\tilde{Q} = (0,T) \times (e^{-rT}q, \infty)$$

with (interesting part of the) boundary

$$\partial^* \tilde{Q} = \big((0,T) \times \{e^{-rT}q\}\big) \cup \big(\{T\} \times [e^{-rT}q, \infty)\big)\,.$$

We then introduce

$$\tilde{V}(t,y) := V(t, e^{rt}y), \quad (t,y) \in \tilde{Q}.$$

Then

$$\begin{aligned} \tilde{V}_t(t,y) &= V_t(t, e^{rt}y) + r\,y\,e^{rt} V_x(t, e^{rt}y), \\ \tilde{V}_y(t,y) &= e^{rt} V_x(t, e^{rt}y), \\ \tilde{V}_{yy}(t,y) &= e^{2rt} V_{xx}(t, e^{rt}y). \end{aligned}$$

We then have for $x = \mathrm{e}^{rt}y$

$$\begin{aligned} \hat{\pi}(t,x) &= -\frac{\mu - r}{\sigma^2}\,\frac{V_x(t,x)^2}{x\,V_{xx}(t,x)} \\ &= -\frac{\mu - r}{\sigma^2}\,\frac{V_x(t,\mathrm{e}^{rt}y)^2}{\mathrm{e}^{rt}y\,V_{xx}(t,\mathrm{e}^{rt}y)} \\ &= -\frac{\mu - r}{\sigma^2}\,\frac{\tilde{V}_y(t,y)^2}{y\,\tilde{V}_{yy}(t,y)} \\ &=: \tilde{\pi}(t,y). \end{aligned}$$

So it remains to solve

$$\tilde{V}_t(t,y) - \frac{1}{2}\left(\frac{\mu - r}{\sigma^2}\right)^2 \frac{\tilde{V}_y(t,y)}{\tilde{V}_{yy}(t,y)} = 0$$

subject to

$$\tilde{V}(t,y) = V(t,\mathrm{e}^{rt}y) = \begin{cases} \mathrm{e}^{\alpha r T}\frac{y^\alpha}{\alpha} = \frac{q^\alpha}{\alpha}, & (t,y) \in \partial^*\tilde{Q}, t < T, \\ \mathrm{e}^{\alpha r T}\frac{y^\alpha}{\alpha} & (t,y) \in \partial^*\tilde{Q}, t = T. \end{cases}$$

We get these boundary conditions using that $(t,y) \in \partial^*\tilde{Q}$ if and only if $(t,\mathrm{e}^{rt}y) \in \partial^*Q$. For $t < T$ this implies $y = \mathrm{e}^{-rT}q$ and $V(t,\mathrm{e}^{rt}y) = q^\alpha/\alpha$.

Remark 3.4 This change of variables corresponds to controlling the discounted wealth process

$$Y_t = \mathrm{e}^{-rt}X_t$$

with performance criterion

$$\tilde{J}(t,x,\pi) = \mathrm{E}_{t,y}\left[\frac{1}{\alpha}\left(\mathrm{e}^{rT}Y_T^\pi\right)^\alpha\right].$$

But we still have a problem. We need an upper boundary. So we choose $\overline{y} >> \mathrm{e}^{-rT}q$ and make the reasonable assumption that $\tilde{V}(t,\overline{y}) = V(t,\mathrm{e}^{rt}\overline{y}) \approx V^0(t,\mathrm{e}^{rt}\overline{y})$, where V^0 is the value function (3.18) of the unconstrained problem. Then we get from (3.18) the additional boundary conditions

$$\tilde{V}(t,\overline{y}) = \mathrm{e}^{c_0(T-t)}\frac{(\mathrm{e}^{rT}\overline{y})^\alpha}{\alpha}, \quad t \in [0,T].$$

We shall use an explicit finite difference scheme on the grid

$$\begin{aligned} &t_0 = 0, \ldots, t_N = T, \quad t_i = i\Delta t, \quad \Delta t = \frac{T}{N}, \\ &y_0 = e^{-rT}q, \ldots, y_M = \overline{y}, \quad y_j = y_0 + j\Delta y, \quad \Delta y = \frac{\overline{y} - y_0}{M}. \end{aligned}$$

We approximate the derivatives by finite differences. So for $v_{i,j} := \tilde{V}(t_i,y_j)$ we may use

$$\begin{aligned} \tilde{V}_t(t_i,y_j) &\approx \frac{v_{i+1,j} - v_{i,j}}{\Delta t}, \\ \tilde{V}_y(t_i,y_j) &\approx \frac{v_{i,j+1} - v_{i,j}}{\Delta y}, \\ \tilde{V}_{yy}(t_i,y_j) &\approx \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta y^2}. \end{aligned}$$

Going backwards in time we set

$$v_{N,j} = \mathrm{e}^{\alpha r T}\frac{y_j^\alpha}{\alpha}$$

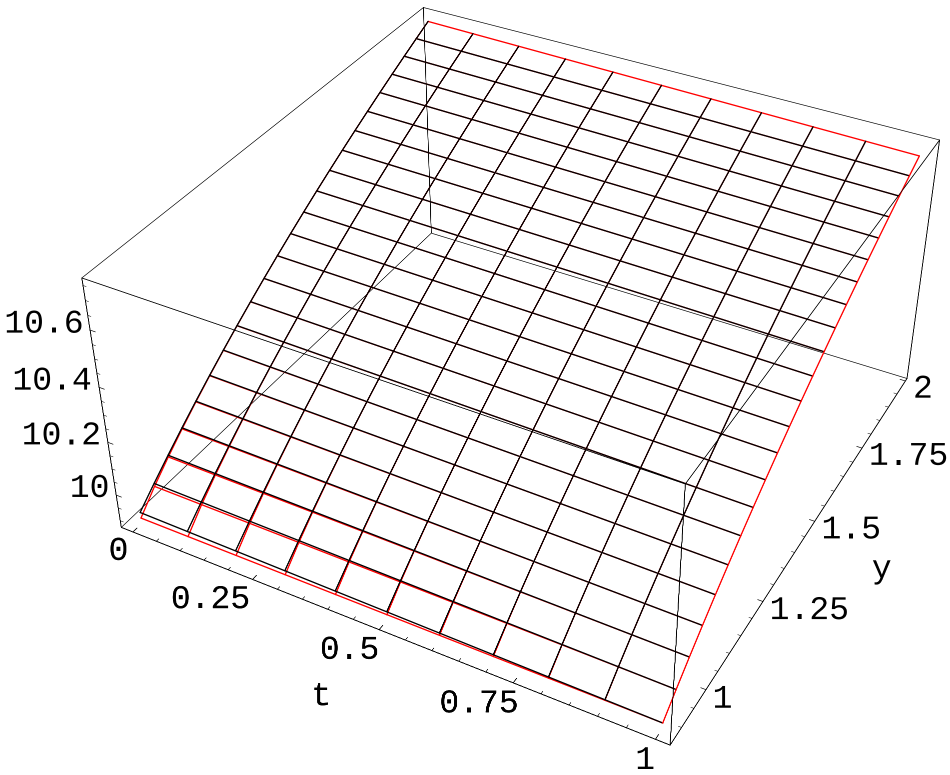

Figure 1: Value functions V^0 and $\tilde{V}$ (red) for Example 3.5

and for $i = N-1, \ldots, 0$ we determine $v_{i,j}$, $j = 0, \ldots, M$, by solving

$$\begin{aligned} v_{i,0} &= \tilde{V}(t_i, y_0) = \mathrm{e}^{-\alpha r(T-i\Delta t)} \frac{q^\alpha}{\alpha}, \\ v_{i,M} &= \tilde{V}(t_i, y_M) = \mathrm{e}^{c_0(T-i\Delta t)} \frac{(\mathrm{e}^{rT}\overline{y})^\alpha}{\alpha}, \\ 0 &= \frac{v_{i+1,j} - v_{i,j}}{\Delta t} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(v_{i,j+1} - v_{i,j})^2}{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}, \quad j = 1, \ldots, M-1. \end{aligned}$$

Example 3.5 We implement the algorithm for parameters $\alpha = 0.1$, $r = 0.02$, $\mu = 0.1$, $\sigma = 0.4$ and bound $q = 0.9$ for initial capital $x_0 = 1$. The numerical results (see Figures 1, 2) indicate

- $\tilde{V}(t, y) \leq V^0(t, \mathrm{e}^{rt} y)$
- $\tilde{\pi}$ and hence $\hat{\pi}$ are increasing in t and increasing in y with $\tilde{\pi} \in [0, \pi^0]$, where π^0 is the risky fraction corresponding to the unconstrained problem.
- Further $\tilde{\pi}(t, y) \searrow 0$ for $y \searrow e^{-rT} q$.

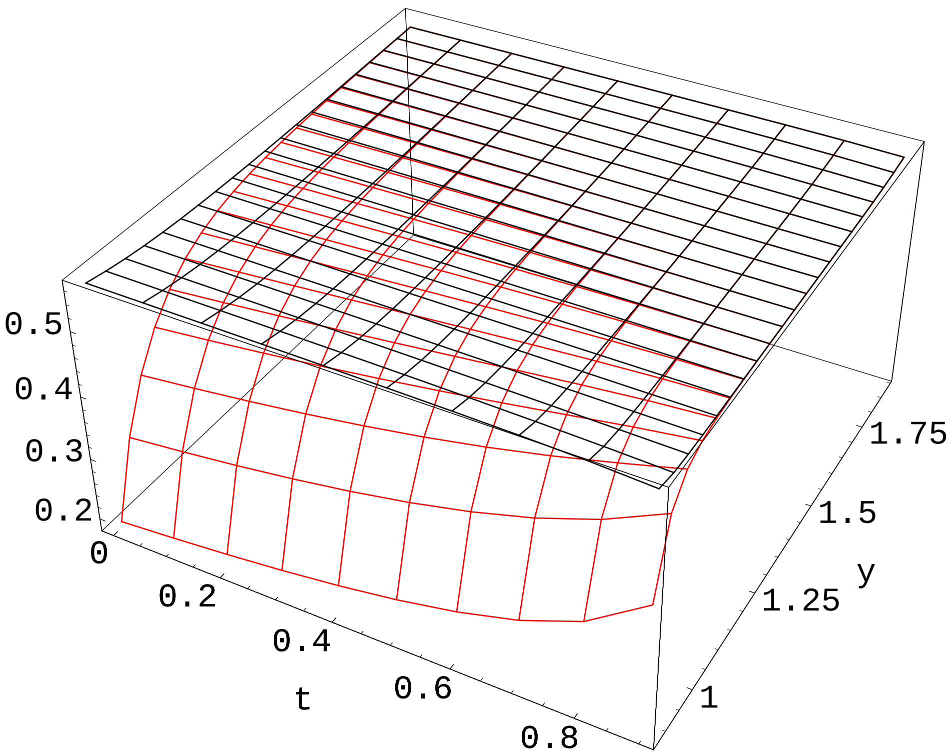

Figure 2: Optimal strategy π^0 and $\tilde{\pi}$ (red) for Example 3.5

Note that these results are only numerical without saying anything about existence. Further, since we only used an approximative boundary condition, there is less hope to show convergence to the true solution.

Even with a correct boundary condition it would be difficult to obtain convergence results since the equations for $v_{i,j}$ are non linear. But one can discretize the HJB equation itself to get a discrete time control problem which leads to an approximative solution. This will be discussed in a subsequent chapter.

3.6 Dynamic programming for deterministic optimal control problems

We consider a deterministic control problem, where the n-dimensional *state* $X(t)$ of the system evolves according to the PDE

$$\dot{X}(t) = b(t, X(t), u(t)), \quad X(0) = x_0 \tag{3.20}$$

and the controls are given by measurable functions

$$u : [0, T] \to \mathcal{U} \subseteq I\!R^p.$$

We use the notation $X(t)$ and $u(t)$ for deterministic functions of time while we reserve X_t and u_t ('t' as index) for stochastic processes. For a dynamic programming approach we also look at starting time t and consider the performance criterion

$$J(t, x, u) = \int_t^T \psi(s, X(s; t, x), u(s))ds + \Psi(T, X(T; t, x))$$

where $X(s; , t, x)$ is a solution of (3.20) starting at t with x. Proper conditions have to be imposed on ψ,Ψ and the controls u such that a solution of (3.20) exists and J is well defined. Suppose that for the latter these are specified in some admissibility set $\mathcal{A}(t, x)$.
We consider the minimization of J,

$$V(t, x) = \inf_{u \in \mathcal{A}(t,x)} J(t, x, u).$$

Remark 3.6 A problem like above with running cost ψ and terminal cost Ψ is called a *Bolza problem*, with ψ and without terminal cost a *Lagrange problem* and with Ψ and without running cost a *Mayer problem*. In determinstic optimal control these three problems are equivalent in the sense, that each one of them can be transformed in one of the other formulations.
E.g. transforming a Bolza problem to a Mayer problem can be done by introducing a further variable x_{n+1} with dynamics $\dot{X}_{n+1}(t) = \psi(t, X(t), u(t)) =:$

$b_{n+1}(t, X(t), X_{n+1}(t), u(t))$. Then we consider the $(n+1)$-dimensional controlled process $\tilde{X} = (X^\top, X_{n+1})^\top$ and define $\tilde{b} = (b^\top, b_{n+1})^\top$, $\tilde{J}(t, x, x_{n+1}, u) = X_{n+1}(T; t, x_{n+1}) + \Psi(T, X(T; t, x)$, $\tilde{V}(t, x, x_{n+1}) = x_{n+1} + V(t, x)$, yielding HJB

$$\tilde{V}_t + \sup_u \left\{ b^\top D_x \tilde{V} + b_{n+1} D_{x_{n+1}} \tilde{V} \right\} = 0$$

which is the same as for the non-reformulated problem. These transformations are in general not possible for stochastic control due to the conditional expectations involved.

Suitable conditions for the dynamic principle to work are in the determinstic case similar to the stochastic control problem. Usually we have two choices: Being more restrictive on the controls or more restrictive on the cost functions. Supposing that $\mathcal{U}$ is closed conditions for the first approach are e.g. that the control functions are piecewise continuous and b, ψ, Ψ are continuous and continuously differentiable such that a solution X always exists, cf. e.g. [FR75]. On the other hand we could only assume that the controls are measurabe functions and then need further boundedness conditions on $f = b, \psi, \Psi$ like

$$\|f(t, x, u) - f(t, y, u)\| \leq K \|x - y\|, \text{ for all } x, y, \quad \|f(t, 0, u)\| \leq K$$

for all t, u, cf. e.g. [YZ99].
Under such conditions one can show that if $V \in \mathcal{C}^{1,1}([0, T], I\!R^n)$, then V satisfies the HJB equation

$$V_t(t, x) + \inf_{u \in \mathcal{U}} \left\{ \psi(t, x, u) + b(t, x, u)^\top D_x V(t, x) \right\} = 0 \tag{3.21}$$

such that $V(T, x) = \Psi(T, x)$.
On the other hand, under suitable conditions, if $\Phi \in \mathcal{C}^{1,1}([0, T], I\!R^n)$ is a solution of (3.21) with $V(T, x) = \Phi(T, x)$ and u^* and the states X^* controlled by u^* solving (3.20) satisfy

$$\Phi_t(t, X^*(t)) + \psi(t, X^*(t), u^*(t)) + b(t, X^*(t), u^*(t))^\top D_x \Phi(t, X^*(t)) = 0$$

then u^* is an optimal control and $V = \Phi$.
So we have the same algorithm as for the stochastic counterpart:

1. Write down the HJB and find a minimizer $\hat{u}(t, x)$.

2. Solve the HJB after plugging in $\hat{u}$.

3. Verify the conditions on ψ, Ψ, u.

As we will see in the following example and in Section 3.7 we may mix or interchange steps 1 and 2 if it works.

Example 3.7 Linear regulator problem (LQ system)
Consider the n-dimensional controlled system (with p-dimensional controls)

$$\begin{aligned} \dot{X}(t) &= A(t)X(t) + B(t)u(t), \\ J(t,x,u) &= \int_t^T \big(X(s)^\top C(s)X(s) + u(s)^\top D(s)u(s)\big)\, ds + X(T)^\top R X(T), \end{aligned}$$

where A, B, C, D are continuous, A, C, R are $(n \times n)$-dimensional, B is $(n \times p)$-dimensional, D is $(p \times p)$-dimensional, C, D, R are symmetric, D positive definite, C, R non negative definite. The HJB equation reads

$$V_t(t,x) + \inf_{u \in \mathcal{U}} \left\{ (A(t)x + B(t)u)^\top D_x V(t,x) + x^\top C(t)x + u^\top D(t)u \right\} = 0$$

withboundary condition $V(T,x) = x^\top R x$. We make an ansatz $V(t,x) = x^\top K(t)x$, where K is $\mathcal{C}^1$ and symmetric and satisfies $K(T) = R$. Then $V_t(t,x) = x^\top \dot{K}(t)x$, $D_x V(t,x) = 2K(t)x$, $D_{xx} = 2K(t)$, yielding the HJB equation

$$x^\top \dot{K}(t)x + \inf_{u \in \mathcal{U}} \left\{ 2x^\top A(t)^\top K(t)x + 2u^\top B(t)^\top K(t)x + x^\top C(t)x + u^\top D(t)u \right\}.$$

A pointwise minimization yields

$$\hat{u}(t,x) = -D(t)^{-1}B(t)^\top K(t)x.$$

Plugging this in the HJB equation yields

$$x^\top \dot{K}x + x^\top (KA + A^\top K)x - x^\top KB(D^{-1})B^\top Kx + x^\top Cx = 0.$$

A sufficient condition is that K satisfies

$$\dot{K}(t) = -(K(t)A(t) + A(t)^\top K(t)) + K(t)B(t)(D(t)^{-1})B(t)^\top K(t) - C. \quad (3.22)$$

Under our conditions (C, D, R are symmetric, D positive definite, C, R non negative definite) a C^1-solution of this *matrix Riccati equation* with $K(T) = R$ exists, cf. [FR75, Theorem 5.2]. Then it can be shown that $V(t,x) = x^\top K(t)x$ and that an optimal control is given by $\hat{u}$.
So the optimally controlled system state satisfies

$$\frac{d}{dt}X^*(t) = A(t)X^*(t) - B(t)F(t)^{-1}B(t)^\top K(t)X^*(t),$$

which is a linear PDE and hence is relatively easy to handle. This model is often used in engineering, partly as approximation for more complex models. But for dimensions $n > 1$ there is no explicit solution of the matrix Riccati equation. Of course there is a bunch of numerical methods available to approximate the solution.

3.7 Stochastic linear regulator problem

We now want to look at a stochastic version of Example 3.7. The comparison will allow us to see what the influence of the additional noise really is. Consider with the same conditions on the still deterministic matrices A, B, C, D, R and for m-dimensional Brownian motion W and non-singular $\sigma\sigma^\top$

$$\begin{aligned} dX_t &= (A(t)X_t + B(t)u_t)dt + \sigma dW_t, \\ J(t,x,u) &= \mathrm{E}_{t,x}\left[\int_t^T \left(X_s^\top C(s)X_s + u_s^\top D(s)u_s\right)ds + X_T^\top R X_T\right] \end{aligned}$$

and $V(t,x) = \inf_u J(t,x,u)$. Based on Section 3.1 we get the HJB equation

$$0 = \inf_{u\in\mathcal{U}}\left\{x^\top C(t)x + u^\top D(t)u + (A(t)x + B(t)u)^\top D_x V(t,x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\top D_{xx}V(t,x))\right\}.$$

Due to the expectation we have to expect a more complicated dependency of the value function on time and hence make the ansatz

$$V(t,x) = k(t) + x^\top K(t)x, \quad k(T) = 0, \quad K(T) = R$$

with continuously differentiable k and K and symmetric K. Similar as in Example 3.7 we have $V_t(t,x) = \dot{k}(t) + x^\top\dot{K}(t)x$, $D_xV(t,x) = 2K(t)x$, $D_{xx} = 2K(t)$, yielding the HJB equation

$$\inf_{u\in\mathcal{U}}\left\{\dot{k}(t) + x^\top\dot{K}(t)x + 2x^\top A(t)^\top K(t)x + 2u^\top B(t)^\top K(t)x + x^\top C(t)x + u^\top D(t)u + \mathrm{tr}(\sigma\sigma^\top K(t))\right\}$$

and a pointwise minimization yields the same minimizer

$$\hat{u}(t,x) = -D(t)^{-1}B(t)^\top K(t)x.$$

Plugging this in the HJB equation we obtain

$$\dot{k} + x^\top\dot{K}x + x^\top(KA + A^\top K)x - \ x^\top KB(D^{-1})B^\top Kx + x^\top Cx\mathrm{tr}(\sigma\sigma^\top K) = 0.$$

A sufficient condition is that k and K satisfy

$$\begin{aligned} \dot{k}(t) &= -\mathrm{tr}(\sigma\sigma^\top K(t)), \\ \dot{K}(t) &= -(K(t)A(t) + A(t)^\top K(t)) + K(t)B(t)(D(t)^{-1})B(t)^\top K(t) - C. \end{aligned}$$

The latter is the same matrix Riccati equation as (3.22) and the differential equation for k simply yields for $k(T) = 0$

$$k(t) = \int_t^T \mathrm{tr}(\sigma\sigma^\top K(s))ds.$$

Since we minimize, this can be seen as the additional costs we have to pay for the noise. Apart from that the solution has absolutely the same structure as for the deterministic problem in Example 3.7.

4 Viscosity solutions and stochastic control

We have seen that the value function V solves the HJB equation if V is smooth, which means for determinstic control problems $V \in \mathcal{C}^{1,1}$ and for stochastic control roblems $V \in \mathcal{C}^{1,2}$. Often V is not smooth like the examples in Section 4.1, 4.3 will show. Then we need another concept of solutions of the HJB equation, these are viscosity solutions for which we can show in Section 4.6 that the value function solves the HJB equation in the viscosity sense.

4.1 A deterministic example

In the setting of Section 3.6, $\mathcal{U} = [-1,1]$, $b(x,u) = u$, $\mathcal{A}(t,x)$ measurable controls with values in $\mathcal{U}$. The dynamics of the controlled brocess is

$$\dot{X}(t) = b(X(t), u(t)),$$

i.e.

$$X(t) = x_0 + \int_0^t u(s)\,ds = X(t_0) + \int_{t_0}^t u(s)\,ds.$$

As performance criterion we use $J(t,x,u) = X(T;t,x)^2$ which we maximize:

$$V(t,x) = \sup_{u\in\mathcal{A}(t,x)} J(t,x,u) = \sup_{u\in\mathcal{A}(t,x)} \left(x + \int_t^T u(s)^2 ds\right)^2.$$

We can see directly

$$V(t,x) = \begin{cases} (x+T-t)^2, & x \geq 0, \\ (x-T+t)^2, & x \leq 0, \end{cases} \qquad \hat{u}(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x \leq 0. \end{cases}$$

This yields

$$V_x(t,x) = \begin{cases} 2(x+T-t), & x > 0, \\ 2(x-T+t), & x < 0, \end{cases}$$

so

$$V_x(t,0+) = 2(T-t) > 0 > -2(T-t) = V_x(t,0-).$$

Thus V is not smooth since not differentiable at $x=0$.

4.2 Upper and lower semi continuous functions

A real valued function f on $I\!R^n$ is called *upper semi continuous (u.s.c.)*, if

$$\limsup_{x_n\to x} f(x_n) = f(x), \quad x \in I\!R^n,$$

and *lower semi continuous (u.s.c.)*, if

$$\liminf_{x_n\to x} f(x_n) = f(x), \quad x \in I\!R^n.$$

So, if an u.s.c. function f has a jump, the upper point belongs to the graph of f, if f is l.s.c. the lower point. An u.s.c. function attains its supremum on compacta (maximum exists), a l.s.c. attains its infimum (minimum exists) on compacta.

4.3 A stochastic example

We consider $\mathcal{U} = I\!R$ and a two-dimensional controlled process $(X, Y)^\top$ with dynamics

$$\begin{aligned} dX_t &= Y_t \, dW_t^1, \\ dY_t &= u_t \, dt + dW_t^2 \end{aligned}$$

where $(W^1, W^2)^\top$ is a standard Wiener process. So we have coefficients

$$b(x, y, u) = \begin{pmatrix} 0 \\ u \end{pmatrix}, \quad \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}.$$

For any l.s.c. function g let

$$J(t, x, y, u) = \mathrm{E}_{t,x,y}[g(X_t)]$$

and $\mathcal{A}_{t,x,y} = \{u \,|\, X, Y \text{exist}, \mathrm{E}_{t,x,y}[g^-(X_t)] < K, X \text{ martingale}\}$,

$$V(t, x, y) = \sup_{u \in \mathcal{A}(t,x,y)} J(t, x, y, u).$$

Suppose that $V \in \mathcal{C}^{1,2}$. Then one can show analogously to Section 2.2 – using that we have continuous b and σ – that V satisfies

$$\sup_{u \in \mathcal{U}} \left\{ V_t + b^\top DV + \frac{1}{2} \mathrm{tr}\left(\sigma\sigma^\top D^2 V\right) \right\} \leq 0.$$

Computing the gradient DV and the Hessian D^2V we get on $[0, T) \times I\!R^2$

$$\sup_{u \in \mathcal{U}} \left\{ V_t + u \, V_y + \frac{1}{2} \left(y^2 V_{xx} + V_{yy}\right) \right\} \leq 0.$$

Since we can choose any $u \in \mathcal{U} = I\!R$ this implies $V_y(t, x, y) = 0$ for all $(t, x, y) \in [0, T) \times I\!R^2$. Therefore V is constant w.r.t. y and we may consider from now on $V(t, x) = V(t, x, y)$ for any $y \in I\!R$. We get

$$\sup_{u \in \mathcal{U}} \left\{ V_t + \frac{1}{2} \, y^2 V_{xx} \right\} \leq 0.$$

Choosing $y = 0$ yields $V_t(t, x) \leq 0$, so V is decreasing for $t \in [0, T)$. Looking at $y \to \infty$ we obtain $V_{xx}(t, x) \leq 0$ for all x, so V is concave in x for all t. Using these properties and that g is l.s.c. one can prove that

$$V(T-, x) \geq V(T, x) = g(x), \quad x \in I\!R.$$

So $V(t, x) \geq g(x)$ for all x and $V(t, x)$ concave in x, which implies that $V(t, x) \geq \breve{g}(x)$, where $\breve{g}$ is the conave envelope of g (the smallest concave function greater or equal to g).
On the other hand we have by Jensen's inequality and the martingale property of X

$$V(t, x) \leq \sup_{u \in \mathcal{A}(t,x)} \mathrm{E}_{t,x}[\breve{g}(X_T)] \leq \sup_{u \in \mathcal{A}(t,x)} \breve{g}(\mathrm{E}_{t,x}[X_T]) = \breve{g}(x).$$

Therefore $V(t, x) = \breve{g}(x)$. Now, if $\breve{g}$ is not twice continuously differentiable we have a contradiction to our assumption $V \in \mathcal{C}^{1,2}$, i.e. V cannot be smooth.

4.4 Viscosity solutions

Consider

$$F(x, v(x), D_x v(x), D_{xx} v(x)) = 0 \tag{4.23}$$

for $x \in \mathcal{O} \subset I\!R^n$ open. Let F be continuous, with values in $I\!R$, satisfying the ellipticity condition

$$F(x, r, p, A) \leq F(x, r, p, A') \quad \text{for all } A \geq A', \tag{4.24}$$

where '$A \geq A'$' means that $A - A'$ is positive definite.
Before defining viscosity solutions we shall look at classical (smooth) solutions to gain some insight.

Definition 4.1 $v : \mathcal{O} \to I\!R$ is a *classical supersolution (subsolution)* of 4.23 if $v \in \mathcal{C}^2(\mathcal{O})$ and

$$F(x, v(x), D_x v(x), D_{xx} v(x)) \geq 0 \quad (\leq 0) \quad \text{for all} \quad x \in \mathcal{O}.$$

Proposition 4.2 *Suppose $v \in \mathcal{C}^2(\mathcal{O})$. Then v is a classical supersolution (subsolution) if and only if for all $\hat{x} \in \mathcal{O}$, $w \in \mathcal{C}^2(\mathcal{O})$, for which $\hat{x}$ is a minimizer (maximizer) of $v - w$, we have*

$$F(\hat{x}, v(\hat{x}), D_x w(\hat{x}), D_{xx} w(\hat{x})) \geq 0 \quad (\leq 0). \tag{4.25}$$

Beweis. Given that (4.25) holds, we simply have to choose $w = v$. For the opposite direction suppose that v is a classical supersolution and that $\hat{x} \in \mathcal{O}$, $w \in \mathcal{C}^2(\mathcal{O})$, $\hat{x}$ a minimizer of $v - w$. Since $\hat{x}$ is a minimizer we have $D_x v(\hat{x}) = D_x w(\hat{x})$, $D_{xx} v(\hat{x}) \geq D_{xx} w(\hat{x})$, hence by (4.24)

$$\begin{aligned} F(\hat{x}, v(\hat{x}), D_x w(\hat{x}), D_{xx} w(\hat{x})) &= F(\hat{x}, v(\hat{x}), D_x v(\hat{x}), D_{xx} w(\hat{x})) \\ &\geq F(\hat{x}, v(\hat{x}), D_x v(\hat{x}), D_{xx} v(\hat{x})) \\ &\geq 0. \end{aligned}$$

□

Thinking of F as a HJB, the idea is to define (viscosity) sub- and supersolutions v by (4.25) holding for all $\hat{x}$ and smooth w and to weaken the conditions on v such that one can show that the value function of the control problem is a viscosity solution. Even if v is not smooth one can then work with smooth w. To this end define for a real valued g the lower semi continuous envelope

$$\underline{g}(x) = \liminf_{y \to x} g(y)$$

and the upper semi continuous envelope

$$\overline{g}(x) = \limsup_{y \to x} g(y).$$

Definition 4.3 suppose that $v : \mathcal{O} \to I\!R$ is locally bounded.

(i) v is a *(viscosity) supersolution* of (4.23) if

$$F(\hat{x}, \underline{v}(\hat{x}), D_x w(\hat{x}), D_{xx} w(\hat{x})) \geq 0$$

for all $\hat{x} \in \mathcal{O}$, $w \in \mathcal{C}^2(\mathcal{O})$ such that $\hat{x}$ is a minimizer of $\underline{v} - w$.

(ii) v is a *(viscosity) subsolution* of (4.23) if

$$F(\hat{x}, \overline{v}(\hat{x}), D_x w(\hat{x}), D_{xx} w(\hat{x})) \leq 0$$

for all $\hat{x} \in \mathcal{O}$, $w \in \mathcal{C}^2(\mathcal{O})$ such that $\hat{x}$ is a maximizer of $\overline{v} - w$.

(iii) v is a *viscosity solution* of (4.23) if v is both a super- and a subsolution of (4.23).

4.5 Properties

We start with a change of variables formula (proof not difficult):

Proposition 4.4 *Let v be a l.s.c. supersolution of (4.23). If $f \in \mathcal{C}^1(I\!R)$ with $f' \neq 0$ on $I\!R$ then*

$$\tilde{v} = f^{-1} \circ v$$

is a supersolution (subsolution) of

$$\tilde{F}(x, v(x), D_x v(x), D_{xx} v(x)) = 0$$

if $f' > 0$ (if $f' < 0$), and where

$$\tilde{F}(x, r, p, A) = F(x, f(r), f'(r)p, f''(r)pp^\top + f'(r)A).$$

The proof of the following proposition is relatively easy compared to other convergence results for PDEs. It is very important for numerical schemes.

Proposition 4.5 *(i) Let v_ε be a l.s.c. supersolution of $F_\varepsilon(x, D_x v(x), D_{xx} v(x)) = 0$, $\varepsilon > 0$, where F_ε is continuous and satisfies (4.24). Suppose $(\varepsilon, x) \mapsto v_\varepsilon(x)$, $(\varepsilon, x, p, A) \mapsto F_\varepsilon(x, p, A)$ are locally bounded. Define*

$$\underline{v}_0(x) = \liminf_{(\varepsilon,y)\to(0,x)} v_\varepsilon(y), \quad \overline{F}_0(x,p,A) = \limsup_{(\varepsilon,x',p',A')\to(0,x,p,A)} F_\varepsilon(x',p',A').$$

Then $\underline{v}_0$ is a l.s.c. supersolution of $\overline{F}_0(x, D_x v(x), D_{xx} v(x)) = 0$.
(ii) Let v_ε be a u.s.c. subsolution of $F_\varepsilon(x, D_x v(x), D_{xx} v(x)) = 0$, $\varepsilon > 0$, where F_ε is continuous and satisfies (4.24). Suppose $(\varepsilon, x) \mapsto v_\varepsilon(x)$, $(\varepsilon, x, p, A) \mapsto F_\varepsilon(x, p, A)$ are locally bounded. Define

$$\overline{v}_0(x) = \limsup_{(\varepsilon,y)\to(0,x)} v_\varepsilon(y), \quad \underline{F}_0(x,p,A) = \liminf_{(\varepsilon,x',p',A')\to(0,x,p,A)} F_\varepsilon(x',p',A').$$

Then $\overline{v}_0$ is a u.s.c. subsolution of $\underline{F}_0(x, D_x v(x), D_{xx} v(x)) = 0$.

Example 4.6 In Example 4.3 we can show by an approximation argument using Proposition 4.5 that $V(t, x) = \breve{g}(x)$ is a viscosity solution of the given HJB.

4.6 Viscosity solutions and HJB equations

Now let us consider also the dependency on time, i.e. F in the form

$$F(t,x,D_xV(t,x),D_{xx}V(t,x)) = -V_t(t,x) - \sup_{u\in\mathcal{U}}\{\psi(t,x,u)+\mathcal{L}^uV(t,x)\} = 0 \tag{4.26}$$

on $Q = [0,T)\times I\!R^n$.

Theorem 4.7 *Suppose b,σ,ψ are for any fixed u in $\mathcal{C}^1(Q)\cap\mathcal{C}(\overline{Q})$ and b,σ have unifomrly bounded derivatives w.r.t. t,x and are of linear growth in x and u. Then for any u.s.c. subsolution V_* and a l.s.c. supersolution of 4.26*

$$\sup_{(t,x)\in\overline{Q}}(V_*(t,x)-V^*(t,x)) = \sup_{x\in I\!R^n}(V_*(T,x)-V^*(T,x))$$

Theorem 4.8 *Suppose that the value function V is locally bounded on Q and that ψ is continuous in t,x for all fixed u and that*

$$(t,x)\mapsto \sup_{u\in\mathcal{U}}\{\psi(t,x,u)+\mathcal{L}^uV(t,x)\}$$

is continuous in t,x. Then V is a viscosity solution of (4.26).

4.7 Portfolio optimization under transaction costs

We consider the model of Section 2.4 with one stock and finite time horizon T. So we have a bond and a stock with prices following

$$\begin{aligned} dB_t &= B_t\,r\,dt, \quad B_0=1\\ dS_t &= S_t(\mu\,dt+\sigma\,dW_t), \quad S_0=1. \end{aligned}$$

Without costs we saw in Section 2.4 that for maximizing expected power utility x^α/α of terminal wealth it is optimal to keep a constant fraction

$$\pi_M = \frac{1}{1-\alpha}\,\frac{\mu-r}{\sigma^2}$$

of the portfolio value (wealth) invested in the stock. Such a strategy requires continuous trading since the postion has always to be adjusted when the stock does not evolve like the money market. Because the stock prices are not of finite variation, any reasonable costs for trading would lead to an immediate ruin of an investor trying to do so.

4.7.1 Wealth processes under transaction costs

We consider *proportional* transaction costs: An investor has to pay a fraction $\gamma\in(0,1)$ of his transaction Δ as fees, so he pays $\gamma|\Delta|$ where Δ is the amount

of money for which stocks are bought ($\Delta > 0$) or sold ($\Delta < 0$). The fees are paid from the bond (bank account).
To compute the costs we need two processes to describe the portfolio. We use the wealth X^0 in the bond and the wealth X^1 in the stock. It is not clear what the value of the portfolio should be. At the terminal time we distinguish two possibilities:

$$
\begin{aligned}
X_T &= X_T^0 + X_T^1 \quad \text{(total wealth (no liquidation costs))}, \\
\overline{X}_T &= X_T - \gamma |X_T^1| \quad \text{(wealth of the liquidated portfolio)}.
\end{aligned}
$$

To make sure that it is always possible to liquidate the position in the stocks and thus to end up with a (strictly) positive wealth after liquidation one has to ensure that the two-dimensional wealth process $(X_t^0, X_t^1)_{t\in[0,T]}$ stays in (the closure of) the solvency cone $\mathcal{S}$ given by

$$\mathcal{S} = \{(x_0, x_1) \in I\!R^2 \,:\, x_0 + x_1 - \gamma|x_1| > 0\}$$

which is the interior of a cone with boundaries $x_1(x_0) = -\frac{x_0}{1-\gamma}$ for $x_0 < 0$ and $x_1(x_0) = -\frac{x_0}{1+\gamma}$ for $x_0 \geq 0$.
As controls we consider the cumulated purchases L_t and the cumulated sales M_t up to time t which are assumed to be adapted, positive, increasing and right continuous. The controlled wealth processes are for $(x_0, x_1) \in \mathcal{S}$ then given by

$$X_t^0 = x_0 + \int_0^t rX_s^0 ds + (1-\gamma)M_t - (1+\gamma)L_t, \tag{4.27}$$

$$X_t^1 = x_1 + \int_0^t \mu X_s^1 ds + \int_0^t \sigma X_s^1 dW_s + L_t - M_t. \tag{4.28}$$

The controls L and M are admissible if X^0, X^1 are well defined and $(X_t^0, X_t^1) \in \mathcal{S}$ for all $t \in [0, T]$.

4.7.2 Value function

For $(x_0, x_1) \in \mathcal{S}$ we consider the value functions

$$
\begin{aligned}
V(t, x_0, x_1) &= \sup_{L,M} \mathrm{E}_{t,x_0,x_1}\left[\frac{1}{\alpha}\, X_T^\alpha\right], \\
\overline{V}(t, x_0, x_1) &= \sup_{L,M} \mathrm{E}_{t,x_0,x_1}\left[\frac{1}{\alpha}\, \overline{X}_T^\alpha\right],
\end{aligned}
$$

where the supremum is taken over all admissible control processes L, M starting at t with $L_{t-} = 0$, $M_{t-} = 0$, the class which we denote by $\mathcal{A}(t, x_0, x_1)$.

Proposition 4.9 *V and $\overline{V}$ are continuous on $\overline{\mathcal{S}}$, concave, strictly increasing in x_0, x_1, and satisfy the homotheticity property*

$$V(t, y\,x_0, y\,x_1) = y^\alpha V(t, x_0, x_1) \quad \textit{for all} \quad y > 0.$$

Proof: We shall only give the argument for the homotheticity property. The proof for the concavity uses similar arguments based on the elementary definition of a concave function.
Due to the linearity of (4.27), (4.28) one can easily verify that at t the controls L , M are admissible for $(x_0, x_1) \in \mathcal{S}$ if and only if yL, yM are admissible for yx_0, yx_1 and that

$$\begin{aligned} X_T^0(t, yx_0, yx_1, yL, yM) &= yX_T^0(t, x_0, x_1, L, M), \\ X_T^1(t, yx_0, yx_1, yL, yM) &= yX_T^1(t, x_0, x_1, L, M), \end{aligned}$$

where e.g. $X_T^0(t, x_0, x_1, L, M)$ denotes the terminal wealth in the bond for starting at t with x_0, x_1 and using $L, M \in \mathcal{A}(t, x_0, x_1)$. Therefore

$$\begin{aligned} X_T(t, yx_0, yx_1, yL, yM) &= X_T^0(t, yx_0, yx_1, yL, yM) + X_T^1(t, yx_0, yx_1, yL, yM) \\ &= yX_T(t, x_0, x_1, L, M) \end{aligned}$$

and

$$V(t, yx_0, yx_1) = \sup_{(L,M)\in\mathcal{A}(t,x_0,x_1)} \mathrm{E}_{t,x_0,x_1}\left[\frac{y^\alpha}{\alpha} X_T(t, x_0, x_1, L, M)^\alpha\right] = y^\alpha V(t, x_0, x_1).$$

□

4.7.3 Heuristic derivation of the HJB equation

In general, the controlled processes might have jumps. So we are no longer in the situation we considered before. The correct methods are provided by *singular stochastic control theory*, see [FS93]. Here we will outline a heuristic approach which allows to conjecture the HJB equation and the form of optimal control strategies and then requires suitable verification results.
Suppose that L and M are absolutely continuous, i.e.

$$L_t = \int_0^t l_s\, ds, \quad M_t = \int_0^t m_s\, ds \tag{4.29}$$

where $0 \le l_t, m_t \le \kappa$ for some maximum rate $\kappa > 0$. Then we can rewrite (4.27), (4.28) as

$$\begin{aligned} dX_t^0 &= (rX_t^0 + (1-\gamma)m_t - (1+\gamma)l_t)dt, \\ dX_t^1 &= (\mu X_t^1 + l_t - m_t)dt + \sigma X_t^1 dW_t. \end{aligned}$$

and under further regularity conditions we obtain the HJB

$$\sup_{l,m} \{V_t(t, x_0, x_1) + \mathcal{L}V(t, x_0, x_1) + \mathcal{L}_B V(t, x_0, x_1)\, l + \mathcal{L}_S V(t, x_0, x_1)\, m\} = 0,$$

where

$$\begin{aligned}
\mathcal{L}V &= rx_0V_{x_0} + \mu x_1 V_{x_1} + \frac{1}{2}\sigma^2 x_1^2 V_{x_1x_1},\\
\mathcal{L}_B V &= V_{x_1} - (1+\gamma)V_{x_0},\\
\mathcal{L}_S V &= (1-\gamma)V_{x_0} - V_{x_1}.
\end{aligned}$$

Then, if V is smooth enough, maximizers are given by

$$\hat{l}(t,x_0,x_1) = \begin{cases} 0, & \text{if } \mathcal{L}_B V(t,x_0,x_1) < 0,\\ \kappa, & \text{if } \mathcal{L}_B V(t,x_0,x_1) \geq 0,\end{cases}$$

and

$$\hat{m}(t,x_0,x_1) = \begin{cases} 0, & \text{if } \mathcal{L}_S V(t,x_0,x_1) < 0,\\ \kappa, & \text{if } \mathcal{L}_S V(t,x_0,x_1) \geq 0.\end{cases}$$

Thus on the *no trading region* NT, defined by

$$NT = \{(t,x_0,x_1) \in [0,T]\times\mathcal{S} \,:\, \hat{l}(t,x_0,x_1) = 0, \hat{m}(t,x_0,x_1) = 0\},$$

we have $V_t + \mathcal{L}V = 0$. Further we can introduce the *buy region* B and the sell region S where it is optimal to buy and to sell respectively,

$$\begin{aligned}
B &= \{(t,x_0,x_1) \in [0,T]\times\mathcal{S} \,:\, \hat{l}(t,x_0,x_1) > 0\},\\
S &= \{(t,x_0,x_1) \in [0,T]\times\mathcal{S} \,:\, \hat{m}(t,x_0,x_1) > 0\}.
\end{aligned}$$

Due to Proposition 4.9 V_{x_0} is strictly positive and thus $(1+\gamma)V_{x_0} > (1-\gamma)V_{x_0}$. Therefore the condition $\mathcal{L}_B \geq 0$ implies $\mathcal{L}_S < 0$ and $\mathcal{L}_S \geq 0$ implies $\mathcal{L}_B < 0$, hence the regions may equivalently be defined by

$$\begin{aligned}
NT &= \{(t,x_0,x_1) \,:\, V_t(t,x_0,x_1) + \mathcal{L}V(t,x_0,x_1) = 0\},\\
B &= \{(t,x_0,x_1) \,:\, \mathcal{L}_B V(t,x_0,x_1) \geq 0\},\\
S &= \{(t,x_0,x_1) \,:\, \mathcal{L}_S V(t,x_0,x_1) \geq 0\}.
\end{aligned} \tag{4.30}$$

So they provide a partition of $[0,T]\times\mathcal{S}$, i.e. $NT \cup B \cup S = [0,T]\times\mathcal{S}$ and $NT \cap B \cap S = \emptyset$.
We shall assume that the slices $NT(t)$, $B(t)$, $S(t)$ for given t have an interval structure with $NT(t) = (\underline{x}_1(t,x_0), \overline{x}_1(t,x_0))$, $S = [\overline{x}_1(t,x_0), \infty)$, $B = (-x_0/(1-\gamma), \underline{x}_1(t,x_0)]$ for $x_0 < 0$ and $B = (-x_0/(1+\gamma), \underline{x}_1(t,x_0)]$ for $x_0 \geq 0$, where $\underline{x}_1(t,x_0) \leq \overline{x}_1(t,x_0)$ and $(x_0, \underline{x}_l(t,x_0)), (x_0, \overline{x}_1(t,x_0)) \in \mathcal{S}$. Further suppose that we start at some $(x_0,x_1) \in NT(0)$. Then the process (X_t^0, X_t^1) will only hit the boundary of B and S, never the interior. Assuming that V is continuously differentiable we have $\mathcal{L}_B V = 0$ and $\mathcal{L}_S V = 0$ on the boundaries $\partial B \cap \partial NT$ and $\partial S \cap \partial NT$, respectively, and $\mathcal{L}_B V > 0$, $\mathcal{L}_S V > 0$ on NT. Thus we might extend V as a solution of $\mathcal{L}_B V = 0$ on B and of $\mathcal{L}_S V = 0$ on S without changing the optimal policy. This leads to the variational inequalities

$$\max\{V_t + \mathcal{L}V, \mathcal{L}_B V, \mathcal{L}_S V\} = 0 \tag{4.31}$$

whose solution provides us with the optimal trading regions. According to (4.30) we have equality $V_t + \mathcal{L}V = 0$, $\mathcal{L}_B V = 0$, $\mathcal{L}_S V = 0$ on NT, B, S, respectively. Note that this is not a HJB equation in the sense that we maximize over possible strategies. Rather it is a set of variational inequalities and we have to find the free boundaries between the regions where one of the inequalities is *active* (= 0). Solving this free boundary problems yields the trading regions NT, B, S. One may then view the maximum in (4.31) as a supremum over the 3 possible strategies which choose one of the inequalities, so the maximum of these choices corresponds to the optimal action (hold, buy or sell stocks).

Using the homotheticity property in Lemma 4.9 and assuming that V is continuously differentiable we get for $y > 0$

$$\frac{\partial V(t, y\,x_0, y\,x_1)}{\partial x_0} = \frac{\partial y^\alpha V(t, x_0, x_1)}{\partial x_0} = y^\alpha V_{x_0}(t, x_0, x_1)$$

and on the other hand – not using the homotheticity property – by the chain rule

$$\frac{\partial V(t, y\,x_0, y\,x_1)}{\partial x_0} = y\,V_{x_0}(t, y\,x_0, y\,x_1).$$

Comparing these two we have (the same holds for the partial derivative w.r.t. x_1)

$$V_{x_0}(t, y\,x_0, y\,x_1) = y^{\alpha-1} V_{x_0}(t, x_0, x_1), \quad V_{x_1}(t, y\,x_0, y\,x_1) = y^{\alpha-1} V_{x_1}(t, x_0, x_1). \tag{4.32}$$

Since $\mathcal{L}_B$ is linear in V_{x_0} and V_{x_1} this means that if we have found a point $(t, x_0, x_1) \in B$, then $(t, y\,x_0, y\,x_1) \in B$ for all $y > 0$. So at t the whole ray $y(x_0, x_1)$, $y > 0$, belongs to B. This shows that $NT(t)$, $B(t)$, $S(t)$ are cones. So we know quite a bit about the trading regions. What about the strategy?

Remark 4.10 In similar problems it can then be shown that controls L and M exists such that $(X_t^0, X_t^1) \in \overline{NT(t)}$ and

$$L_t = \int_0^t \mathbf{1}_{\{(X_s^0, X_s^1) \in \partial B(s)\}}(s)\, dL_s, \quad M_t = \int_0^t \mathbf{1}_{\{(X_s^0, X_s^1) \in \partial S(s)\}}(s)\, dM_s,$$

compare [Ko99] and the references therein. So trading occurs only on the boundary. Further it can be shown that only that much is traded that the process stays on the boundary. Mathematically the controlled process $(X_t^0, X_t^1)_{t \in [0,T]}$ is a continuous reflected diffusion process, reflected at the boundaries of NT, and trading only occurs with infintesimal small transactions at the local time on the boundary.

For the form of a verification theorem which still has to be shown to guarantee that the optimal strategy is of the conjectured form, we refer for a similar problem to [Ko99] and the references therein.

4.7.4 No short selling, no borrowing

If, in addition we require in the admissibility conditions, that no short selling takes place ($X_t^1 \geq 0$) and no borrowing is allowed ($X_t^0 \geq 0$) then we consider instead of the solvency region the domain $\mathcal{D} = [0,\infty)^2 \setminus \{0,0\}$ and define the trading regions as subsets of $\mathcal{D}$. Then it might happen that one of the trading regions is empty. Further, if $x_0 = 0$ ($x_1 = 0$) we should exclude the second (third) inequality in (4.31) since buying (selling) is not admissible. This leads to the following theorem for which we refer to Akian et al. (1996)[1].

Theorem 4.11 *V is a concave and continuous viscosity solution of*

$$\begin{aligned}
&\max\{V_t + \mathcal{L}V, \mathcal{L}_B V, \mathcal{L}_S V\} = 0 \quad \text{on} \quad [0,T) \times \mathcal{D} \setminus (\{0\} \times (0,\infty) \cup (0,\infty) \times \{0\}),\\
&\max\{V_t + \mathcal{L}V, \mathcal{L}_S V\} = 0 \quad \text{on} \quad [0,T) \times \{0\} \times (0,\infty),\\
&\max\{V_t + \mathcal{L}V, \mathcal{L}_B V\} = 0 \quad \text{on} \quad [0,T) \times (0,\infty) \times \{0\}.
\end{aligned}$$

with $V(T,x_0,x_1) = \frac{1}{\alpha}(x_0+x_1)^\alpha$.
Further V is unique in the class of continuous functions satisfying $|h(t,x_0,x_1)| \leq K(1+(x_0^2+x_1^2)^\alpha)$ for all $(x_0,x_1) \in \mathcal{D}$, $t \in [0,T]$, and some constant K. The same is true for $\overline{V}$ with boundary condition $\overline{V}(T,x_0,x_1) = \frac{1}{\alpha}(x_0+(1-\gamma)x_1)^\alpha$.

The proof in Akian et al. (1996) is based on the derivation of a weak dynamic programming principle leading to (4.31). The uniqueness is shown following the Ishii technique, see [CIL92].

4.7.5 Reduction of the dimension

By homotheticity we have for the total wealth $x = x_0 + x_1$ and the risky fraction $\pi = x_1/x$

$$V(t,x_0,x_1) = x^\alpha V(t,x_0/x,x_1/x) = x^\alpha V(t,1-\pi,\pi) =: x^\alpha \Phi(t,\pi)$$

So we may try to reparameterize the problem as a control problem for the risky fractions $\pi_t = \frac{X_t^1}{X_t^0+X_t^1}$. We have for $V(t,x_0,x_1) = (x_0+x_1)^\alpha \Phi(t, \frac{x_1}{x_0+x_1})$

$$\begin{aligned}
V_t &= x^\alpha \Phi_t,\\
V_{x_0} &= x^{\alpha-1}(\alpha\Phi - \pi\Phi_\pi),\\
V_{x_1} &= x^{\alpha-1}(\alpha\Phi + (1-\pi)\Phi_\pi),\\
V_{x_1,x_1} &= x^{\alpha-2}\left((1-\pi)^2\Phi_{\pi,\pi} - (1-\pi)(1-2\alpha)\Phi_\pi - \alpha(1-\alpha)\Phi\right),
\end{aligned}$$

where we used $x = x_0 + x_1$ and $\pi = x_1/x$ as above. Plugging this into the definition of the operators we get

$$\mathcal{L}V = x^\alpha \mathcal{L}^\pi \Phi, \quad \mathcal{L}_B V = x^{\alpha-1}\mathcal{L}_B^\pi \Phi, \quad \mathcal{L}_S V = x^{\alpha-1}\mathcal{L}_S^\pi \Phi,$$

[1] M. Akian, A. Sulem, P. Séquier (1996): A finite horizon multidimensional portfolio selection problem with singular transactions. In: Proceedings of the 34th Conference on Decisions & Control, New Orleans, 2193–2197.

where

$$\begin{aligned}
\mathcal{L}^\pi\Phi(t,\pi) &= \alpha\left(r+\pi(\mu-r)-\frac{1}{2}\sigma^2(1-\alpha)\pi^2\right)\Phi(t,\pi) \\
&\quad + (\mu-r-(1-2\alpha)\pi)\,\pi(1-\pi)\Phi_\pi(t,\pi) \\
&\quad +\frac{1}{2}\sigma^2\pi^2(1-\pi)^2\Phi_{\pi,\pi}(t,\pi), \\
\mathcal{L}^\pi_B(t,\pi) &= (1+\gamma\pi)\Phi_\pi(t,\pi)-\alpha\gamma\Phi(t,\pi), \\
\mathcal{L}^\pi_S(t,\pi) &= -(1-\gamma\pi)\Phi_\pi(t,\pi)-\alpha\gamma\Phi(t,\pi).
\end{aligned}$$

Since $x>0$ we thus get the variational inequalities

$$\max\{\mathcal{L}^\pi\Phi(t,\pi),\mathcal{L}^\pi_B\Phi(t,\pi),\mathcal{L}^\pi_S\Phi(t,\pi)\}=0 \tag{4.33}$$

for $(t,\pi)\in[0,T)\times(0,1)$. Note that $x_0\geq 0, x_1\geq 0, x>0$ imply $\pi\in[0,1]$. We shall denote the corresponding trading regions in terms of π by NT^π, B^π, S^π. The boundary conditions at terminal time T now read

$$\Phi(T,\pi)=\frac{1}{\alpha} \quad \text{and} \quad \overline{\Phi}(T,\pi)=\frac{1}{\alpha}(1-\gamma\pi)^\alpha.$$

On B^π we can solve $\mathcal{L}^\pi_B\Phi=0$ yielding

$$\Phi(t,\pi)=C_B(t)(1+\gamma\pi)^\alpha \tag{4.34}$$

with an unknown strictly positive function C_B. On S^π we get correspondingly

$$\Phi(t,\pi)=C_S(t)(1-\gamma\pi)^\alpha. \tag{4.35}$$

We still have to discuss what happens on the boundaries $\pi=0$ (corresponding to $x_1=0$ in Theorem 4.11) and $\pi=1$ (corresponding to $x_0=0$). We assume that π_M, the optimal fraction without costs, lies in $(0,1)$. Then we can expect $NT^\pi(t)\neq\emptyset$ for $t<T$, so at $t<T$ it can happen that $(t,0)\in NT^\pi$ or that $(t,0)\in B^\pi$. In the latter case we have the boundary condition $\mathcal{L}^\pi_B\Phi(t,0)=0$. In the first case we have

$$\Phi_t(t,0)+\alpha r\Phi(t,0)=0. \tag{4.36}$$

For $\pi=1$ we have the boundary condition $\mathcal{L}^\pi_S\Phi(t,1)=0$ if $(t,1)\in S^\pi$ and

$$\Phi_t(t,1)+\alpha(\mu-\frac{1}{2}\sigma^2(1-\alpha))\Phi(t,1)=0. \tag{4.37}$$

if $(t,1)\in NT^\pi$.
Now we have everything at hand we need to set up a good numerical procedure to find the free boundaries

$$a(t)=\inf NT(t)\ ,\quad b(t)=\sup NT(t). \tag{4.38}$$

4.7.6 A Semi-Smooth Newton Method

The algorithm we present to solve (4.33) is based on a primal-dual active set strategy, compare Hintermüller et al. (2003)[2] and Ito and Kunisch (2006)[3]. Here we face two free boundaries and a different type of constraints and have to adapt their algorithm. We now work in the setting of Section 4.7.5 but no longer use the superscript $^{\pi}$.

Problem (4.33) is equivalent to solving

$$\Phi_t + \mathcal{L}\Phi + \lambda_B + \lambda_S = 0\ , \tag{4.39}$$

$$\mathcal{L}_B\Phi \leq 0,\ \lambda_B \geq 0,\ \lambda_B\mathcal{L}_B\Phi = 0, \tag{4.40}$$

$$\mathcal{L}_S\Phi \leq 0,\ \lambda_S \geq 0,\ \lambda_S\mathcal{L}_S\Phi = 0\ . \tag{4.41}$$

The two complementarity problems in (4.40), (4.41) can be written as

$$\lambda_B = \max\{0, \lambda_B + c\,\mathcal{L}_B\Phi\}\ , \quad \lambda_S = \max\{0, \lambda_S + c\,\mathcal{L}_S\Phi\} \tag{4.42}$$

for any constant $c > 0$. So we have to solve (4.39), (4.42). At T the trading regions are given by $S(T) = [0,1]$ for $\overline{\Phi}$ and $NT(T) = [0,1]$ for Φ. We split $[0,T]$ in N intervals and go backwards in time with $t_N = T$, $t_n = t_{n+1} - \Delta t$, $\Delta t = T/N$. Having computed $\Phi(t_{n+1},\cdot)$ and the corresponding regions we use the following algorithm to compute $v = \Phi(t_n,\cdot)$ and $NT(t_n)$:

0. Set $\overline{v} = \Phi(t_{n+1},\cdot)$, $k = 0$, choose an interval NT_0 in $[0,1]$, constant $c > 0$.
1. Define the boundaries a_k and b_k of NT_k as in (4.38).
2. On $[a_k, b_k]$ solve (numerically) $\frac{1}{\Delta t}(\overline{v} - v) + \mathcal{L}v = 0$ using the boundary conditions $\mathcal{L}_B v = 0$ if $a_k \notin NT_k$, (4.36) if $a_k \in NT_k$ (implying $a_k = 0$) and $\mathcal{L}_S v = 0$ if $b_k \notin NT_k$, (4.37) if $b_k \in NT_k$ (implying $b_k = 1$).
3. If $a_k \neq 0$ define v on $[0, a_k)$ by (4.34). If $b_k \neq 1$ define v on $(b_k, 1]$ by the second equation in (4.35). Choose C_B and C_S such that v is continuous in a_k and b_k. So $v_{k+1} = v$ is continuously differentiable.
4. Set

$$\lambda_B^{k+1} = \begin{cases} 0 & \text{on} \quad (a_k, 1] \\ -\frac{1}{\Delta t}(\overline{v} - v_{k+1}) - \mathcal{L}v_{k+1} & \text{on} \quad [0, a_k] \end{cases}$$

and

$$\lambda_S^{k+1} = \begin{cases} 0 & \text{on} \quad [0, b_k) \\ -\frac{1}{\Delta t}(\overline{v} - v_{k+1}) - \mathcal{L}v_{k+1} & \text{on} \quad [b_k, 1]. \end{cases}$$

[2]M. Hintermüller, K. Ito, K. Kunisch (2003): The primal-dual active set strategy as a semismooth Newton method. SIAM Journal on Optimization 13, 865–888.

[3]K. Ito, K. Kunisch (2006): Parabolic variational inequalities: The Lagrange multiplier approach. Journal de Mathḿatiques Pures et Appliquées (9) 85, 415–449.

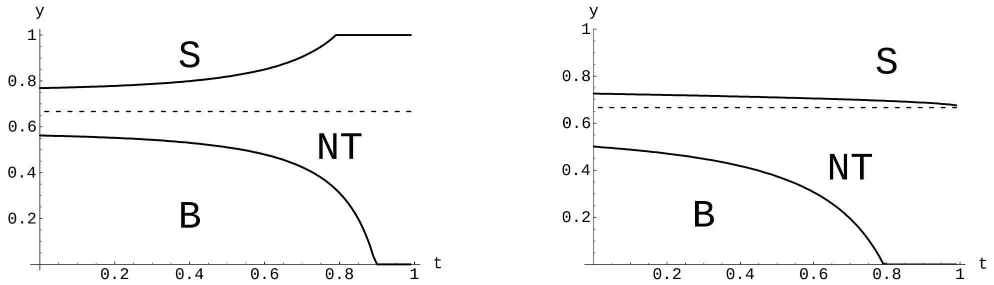

Figure 3: Trading regions for $\alpha = 0.1$ for Φ and $\overline{\Phi}$

5. Introduce the active sets

$$\begin{aligned} B_{k+1} &= \overline{\{y \in [0,1] \,:\, \lambda_B^{k+1}(y) + c\,\mathcal{L}_B v_{k+1}(y) > 0\}}, \\ S_{k+1} &= \overline{\{y \in [0,1] \,:\, \lambda_S^{k+1}(y) + c\,\mathcal{L}_S v_{k+1}(y) > 0\}} \end{aligned}$$

   and set $NT_{k+1} = [0,1] \setminus (B_{k+1} \cup S_{k+1})$. Verify that the interval structure holds and define the boundaries a_{k+1} and b_{k+1} by (4.38).

6. If $a_{k+1} = a_k$ and $b_{k+1} = b_k$ then set $NT(t_n) = (a_{k+1}, b_{k+1})$, $\Phi(t_n, \cdot) = v_{k+1}$ and STOP; otherwise increase k by 1 and continue with step 1.

Example 4.12
We consider a bond and a stock with parameters $r = 0$, $\mu = 0.096$, $\sigma = 0.4$, and horizon $T = 1$. We use mesh sizes $\Delta t = 0.01$ and $\Delta y = 0.001$, choose $c = 1$, and at t_{N-1} use $NT_0 = (0.1, 0.8)$, and at all other time steps t_n use $NT_0 = NT(t_{n+1})$. For the utility function we consider both $\alpha = 0.1$ and the more risk averse parameter $\alpha = -1$. These yield without transaction costs optimal risky fractions 0.667 and (dotted lines in Figure 3). We consider proportional costs $\gamma = 0.01$. In Figure 3 we look at $\alpha = 0.1$, left-hand at V with liquidation at the end, right-hand at $\tilde{V}$. We see that the liquidation costs we have to pay at T imply that we also trade close to the terminal time, while without liquidation this is never optimal. Using Mathematica the computation time for each graph was about $18\,s$.

4.7.7 More complex transaction costs

Typical transaction costs considered in portfolio theory are constant costs, fixed costs (proportional to the portfolio value), and proportional costs (proportional to the transaction volume). While the latter penalizes the size of the transaction, the first two punish the frequency of trading. So a combination of both types is of interest, as well from a practical as a theoretical point of view. Then the trading strategies of interest are so called impulse control strategies consisting of a sequence of stopping times at which trading takes place and the transactions at those times. Optimal impulse control strategies can then be described as solutions of quasi variational inequalities which differ

from variational inequalities like (4.31) by inclusion of a maximization problem in the inequalities for the buy and sell region which determines the optimal transaction when trading.
The first type of costs considered were purely proportional costs – like we did above – for which the optimal solution is given by a cone in which it is optimal not to trade at all, and which corresponds to an interval for the risky fraction. When reaching the boundaries, infinitesimal trading occurs in such a way that the wealth process just stays in the cone.
Adding a constant component to the transaction costs punishes very frequent trading and so will avoid the occurrence of infinitesimal trading at the boundary. An investor now has to choose discrete trading times and optimal transactions at these times so that the methodology of optimal impulse control comes into play. The insight is that there is still some no-transaction region, but reaching the boundary transactions will be done in such a way that the wealth process restarts at some curve between boundary and Merton line. For constant costs the trading regions depend also on the total wealth.
A simpler approach is possible when considering fixed costs instead of constant costs. For purely fixed costs we get a constant new risky fraction close to the Merton fraction π_M. If combined with proportional costs we get two different new risky fractions after buying and after selling.

5 Optimal Stopping

We will cite some results on optimal stopping in Section 5.1 and then look at the pricing of American options as an important application of the theory of optimal stopping.

5.1 Some results on optimal stopping

For the results cited below we refer to [KS98, Appendix D], [Ok00, Chapter 10], and [PS06].
Suppose that $Y = (Y_t)_{t\in[0,T]}$ is a right continuous, positive, stochastic process w.r.t. some Filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions and with trivial $\mathcal{F}_0$. We would like to find

$$V(0) = \sup_{\tau\in\mathcal{S}_{0,T}} \mathrm{E}[Y_\tau]$$

where $\mathcal{S}_{s,t}$ is the class of stopping times with values in $[s,t]$, and an optimal stopping time τ^* for which $V(0)$ is attained, i.e. $V(0) = \mathrm{E}[Y_{\tau^*}]$. We assume that $V(0) \in (0,\infty)$. The main idea to solve the optimal stopping problem is the introduction of the *Snell envelope*

$$\overline{Y}_t = \operatorname{esssup}_{\tau\in\mathcal{S}_{t,T}} \mathrm{E}[Y_\tau \,|\, \mathcal{F}_t], \quad t\in[0,T].$$

The *essential supremum* $X^* = \text{esssup}\mathcal{X}$ of a family of random variables $\mathcal{X}$ is characterized by two conditions: (i) For all $X \in \mathcal{X}$ we have $X \leq X^*$ ($a.s.$) and if $X \leq \overline{X}$ ($a.s.$) for all $X \in \mathcal{X}$, then $X^* \leq \overline{X}$ ($a.s.$).
If the essential supremum is taken over a countable number of random variables or over expectations (real numbers) it coincides ($a.s.$) with the supremum. Therefore $\overline{Y}_0 = V(0)$. By definition we also have $\overline{Y}_T = Y_T$.

Theorem 5.1 *The Snell envelope $\overline{Y}$ is the smallest supermartingale majorant of Y. In particular we have for any $t \in [0,T]$, $\tau \in \mathcal{S}_{t,T}$*

$$\mathrm{E}[\overline{Y}_\tau \,|\, \mathcal{F}_t] \leq \overline{Y}_t.$$

For the characterization of an optimal stopping time the following theorem is of uttermost importance.

Theorem 5.2 *τ^* is optimal if and only if (i) $(Y_{\tau^* \wedge t})_{t \in [0,T]}$ is a martingale and (ii) $Y_{\tau^*} = \overline{Y}_{\tau^*}$.*

For the existence we need the

Assumption 5.3 $\mathrm{E}[\sup_{0 \leq t \leq T} Y_t] < \infty$.

Theorem 5.4 *Suppose Y satisfies the conditions above, in particular Assumption 5.3. Then*

$$\tau^* = \inf\{t \in [0,T] \,:\, Y_t = \overline{Y}_t\}$$

is optimal.

Remark 5.5 In discrete time the Snell envelope can be introduced similarly and constructed by backward induction. Say we have a stochastic process $(Y_n)_{n=0,\ldots,N}$, adapted to a filtration $(\mathcal{F}_n)_{n=0,\ldots,N}$ and we want to find the value

$$v_n = \sup_{\tau \in \mathcal{S}_{n,N}} \mathrm{E}[Y_\tau].$$

and find an optimal stopping time τ_n^* in the class $\mathcal{S}_{n,N}$ of stopping time with values in $n, \ldots, N$, i.e. for which $\mathrm{E}[Y_{\tau_n^*}] = v_n$. In particular we are interested in $n = 0$. Defining

$$\overline{Y}_N = Y_N$$

and backwards for $n = N-1, \ldots, 0$

$$\overline{Y}_n = \max\{Y_n, \mathrm{E}[\overline{Y}_{n+1} \,|\, \mathcal{F}_n]\}.$$

The resulting process $(\overline{Y}_n)_{n=0,\ldots,N}$ will be the smallest supermartingale majorant of $(Y_n)_{n=0,\ldots,N}$, the Snell envelope. Further by induction it follows that

$$\tau_n^* = \min\{m = n, \ldots, N \,:\, \overline{Y}_m = Y_m\} = \min\{m = n, \ldots, N \,:\, \overline{Y}_m \leq Y_m\}$$

is optimal with $v_n = \mathrm{E}[Y_{\tau_n^*}] = \mathrm{E}[\overline{Y}_n]$.

Example 5.6 The *problem of best choice* (formerly the *secretary problem*): A company wants to hire a mathematician and has invited N applicants for a job interview. They are interviewed sequentially and after each candidate it has to be decided whether s/he is hired or not. The aim is to maximize the probability to get the best candidate. The rank of candidate n is given by a random variable X_n and we assume that all permutations of ranks have the same probability. What can be observed are the relative ranks R_n with values in $\{1, \dots, n\}$ up to time n. So we consider the filtration $\mathcal{F}_n = \sigma(R_1, \dots, R_n)$, $n = 1, \dots, N$. The company faces the optimal stopping problem

$$v_n := \sup_{\tau \in \mathcal{S}_{n,N}} \mathrm{E}[Y_\tau], \quad n = 1, \dots, N,$$

where

$$Y_n = P(X_n = 1 \,|\, R_1, \dots, R_n) = P(X_n = 1 \,|\, \mathcal{F}_n)$$

and is interested in the optimal stopping times τ_n^* at n, in particular for $n = 1$. It can be shown that $R_1, \dots, R_N$ are independent with

$$P(R_n = k) = \frac{1}{n}, \quad k = 1, \dots, n,$$

and that

$$Y_n = \frac{n}{N} \mathbf{1}_{\{R_n = 1\}}.$$

Using that we have due to independence $\mathrm{E}[\overline{Y}_{n+1} \,|\, \mathcal{F}_n] = v_{n+1}$, we can compute the Snell envelope $\overline{Y}_n$, τ_n^*, and v_n by backward induction and show that the optimal strategy can be described by

$$\begin{aligned} \tau_1^* &= \inf\{n \geq \sigma^* \,:\, R_n = 1 \text{ or } n = N\}, \\ \sigma^* &= \inf\{k \in \{1, \dots, N\} \,:\, k/N \geq v_{k+1}\}. \end{aligned}$$

Thus – as long as σ^* is not reached – the decision is always *no*, and from σ^* onwards the first candidate, who is the best of those seen so far, gets the job. Further it can be shown that for looking at σ^* and v_1 as functions of the number of candidates N, we get

$$\lim_{n \to \infty} \frac{\sigma^*(N)}{N} = \frac{1}{\mathrm{e}}, \quad \lim_{n \to \infty} v_1(N) = \frac{1}{\mathrm{e}},$$

i.e., it is asymptotically optimal to send home N/e candidates after their interview and then take the next relatively best candidate. This yields approximately a probability of $1/e$ to get the best of all candidates. For more details we refer e.g. to [Ir03].

5.2 Optimal Stopping for underlying Markov processes

Suppose now that $Y_t = g(t, X_t)$ where $(X_t)_{t\in[0,T]}$ is a strong Markov process with values in $\mathbb{R}^n$, e.g. an Itô diffusion like defined in (2.4).
Then the optimal stopping problem reads

$$V(t,x) = \sup_{\tau\in\mathcal{S}_{t,T}} \mathrm{E}_{t,x}[g(\tau, X_\tau)], \tag{5.43}$$

where V is the *value function* of the optimal stopping problem. Assumption 5.3 now reads $\mathrm{E}_{0,x}[\sup_{t\in[0,T]} g(t, X_t)] < \infty$. Denoting by $X_s(t,x)$ the process starting at t with x, we get corresponding to Theorem 5.4

Theorem 5.7 *Suppose that Assumption 5.3 holds, that V is lower semi continuous and g upper semi continuous. Then*

$$\tau^*_{t,x} = \inf\{s \in [t,T] \,:\, V(s, X_s(t,x)) = g(s, X_s(t,x))\}$$

is optimal for (5.43).

Furthermore, one can introduce the continuation set $\mathcal{C}$ and the stopping set $\mathcal{D}$

$$\begin{aligned}
\mathcal{C} &= \{(t,x) \in [0,T]\times\mathbb{R}^n \,:\, V(t,x) > g(t,x)\},\\
\mathcal{D} &= \{(t,x) \in [0,T]\times\mathbb{R}^n \,:\, V(t,x) = g(t,x)\}.
\end{aligned}$$

Then for $(0,x) \in \mathcal{C}$, the optimal stopping time $\tau^*_{0,x} = \tau_{\mathcal{D}}$ is the first entry time of (t, X_t) into $\mathcal{D}$.

5.3 Reminder on pricing of European options

We consider the Black-Scholes model for a financial market consisting of one bond with prices

$$dB_t = B_t r dt, \quad B_0 = 1, \quad \text{i.e.} \quad B_t = \mathrm{e}^{r\,t}, \quad t \in [0,T],$$

and one stock with prices evolving according to

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t), \quad S_0 = s_0 > 0,$$

where W is a Wiener process, $\mu \in \mathbb{R}$, $\sigma > 0$, and interest rate $r > 0$. Risk neutral pricing requires a change of measure to a new probability measure under which the discounted price processes become martingales. This *risk neutral measure* $\tilde{P}$ can be defined by a Radon-Nikodym density $Z_T = \frac{d\tilde{P}}{dP}$ in the following way:

$$\tilde{P}(A) := \mathrm{E}[Z_T \mathbf{1}_A], \quad A \in \mathcal{F}_T,$$

where

$$Z_T = \exp\left\{ -\frac{\mu - r}{\sigma} W_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T \right\}.$$

Note that the *density process* $(Z_t)_{t\in[0,T]}$, defined by

$$Z_t := \mathrm{E}[Z_T \,|\, \mathcal{F}_t] = \exp\left\{ -\frac{\mu - r}{\sigma} W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t \right\},$$

is a martingale under P. The expectation and the conditional expectation under $\tilde{P}$ can then be computed by

$$\tilde{\mathrm{E}}[X] = \mathrm{E}[Z_T X], \quad \tilde{\mathrm{E}}[X \,|\, \mathcal{F}_t] = \frac{\mathrm{E}[Z_T X \,|\, \mathcal{F}_t]}{\mathrm{E}[Z_T \,|\, \mathcal{F}_t]} = Z_t^{-1}\mathrm{E}[Z_T X \,|\, \mathcal{F}_t].$$

By the Girsanov Theorem

$$\tilde{W}_t := W_t + \frac{\mu - r}{\sigma} t, \quad t \in [0, T]$$

defines a Wiener process under $\tilde{P}$. Rewriting the stock dynamics in terms of $\tilde{W}$ we get

$$dS_t = S_t(r\,dt + \sigma\,d\tilde{W}_t),$$

and applying Itô's formula to the discounted stock prices $\tilde{S}_t := S_t/B_t$ yields

$$d\tilde{S}_t = \tilde{S}_t \sigma\,d\tilde{W}_t,$$

hence $\tilde{S}$ is a martingale under $\tilde{P}$. If we consider a self-financing trading strategy $\pi = (\pi_t)_{t\in[0,T]}$ where π_t denotes the fraction of wealth X_t invested in the stocks, we get for the wealth process

$$dX_t = rX_t\,dt + \pi_t X_t \sigma\,d\tilde{W}_t$$

and for the discounted wealth $\tilde{X}_t := X_t/B_t$

$$d\tilde{X}_t = \pi_t \tilde{X}_t d\tilde{W}_t.$$

Thus also the wealth process is a $\tilde{P}$ (local) martingale. This was to be expected since we can only invest in the two martingales $B_t/B_t = 1$ and $\tilde{S}$.
Say we have a financial derivative which pays at the terminal time the amount C to its buyer. The arbitrage free price $p(C)$ of this contingent claim C is given by x_0 if we can find initial capital x_0 and a self-financing trading strategy π such that the corresponding discounted wealth process $\tilde{X} = \tilde{X}^\pi$ satisfies

$$\frac{C}{B_T} = x_0 + \int_0^T \pi_t \tilde{X}_t \sigma\,d\tilde{W}_t. \tag{5.44}$$

The Black-Scholes market is a so called *complete* market model in which we know that every $\mathcal{F}_T$-measurable, square-integrable claim C can be hedged by a trading strategy like in (5.44). If the wealth process is indeed a martingale, we get from (5.44)

$$p(C) = \tilde{\mathrm{E}}[\tilde{X}_T^\pi] = \tilde{\mathrm{E}}[C/B_T],$$

where the latter does no longer depend directly on the trading strategy, so we can simply compute the price by taking expectation of the discounted claim with respect to the risk neutral measure $\tilde{P}$.
Furthermore one can show that under these conditions the arbitrage free price of C at time t is given by

$$p_t(C) = B_t\,\tilde{\mathrm{E}}[C/B_T]\,|\,\mathcal{F}_t].$$

There are several approaches to find $p_t(C)$. If we have a financial derivative which pays at terminal time

$$C = \Phi(S_T)$$

one can show under some integrability conditions on Φ that prices are of the form

$$p_t(C) = f(t, S_t)$$

by using the Markov property of the underlying stock prices. Applying a result like e.g. the Feynman-Kac Formula (see e.g. [St00]) we get under suitable conditions that f satisfies for $t \in [0, T)$, $x > 0$

$$\begin{aligned} f_t(t,x) + rxf_x(t,x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(t,x) - r\,f(t,x) &= 0, \\ f(T,x) &= \Phi(x). \end{aligned}$$

For example, for $\Phi(x) = (x-K)^+$ we have a European Call option which offers the right to buy the stock for *strike price* $K > 0$ at time T. The solution of the corresponding PDE above yields the so called *Black-Scholes formula* for the call price, cf. any textbook on financial mathematics.
The price for a put option given by $\Phi(x) = (K - x)^+$ can then be determined by the *put-call-parity*, cf. e.g. [KS98, Example 2.4.3].

5.4 American options

Using the same model and notation as in Section 5.3 we shall now look at claims of American type which guarantee a certain payout C_t at time t if exercised. Such an American contingent claim consists of a continuous process $(C_t)_{t\in[0,T]}$ of $\mathcal{F}$-adapted possible payouts and the investor chooses a time τ when to exercise. Since he can at t only use the information $\mathcal{F}_t$, τ should be modelled as stopping time. So for the buyer of an American contingent claim we have a problem of optimal stopping. For the meaning of *optimal* see the discussion after Theorem 5.9.

So the buyer chooses a stopping time in $\mathcal{S}_{0,T}$. In particular, if he has not exercised before τ, he receives the payout C_T at terminal time. At time 0 the maximum the buyer is willing to pay is

$$p_B(C.) = \sup\{y : \text{There exist } \tau \in \mathcal{S}_{0,T}, \pi \text{ s.t. } X^\pi_\tau(0, -y) + C_\tau \geq 0\}.$$

On the other hand, the seller needs at least

$$p_S(C.) = \inf\{z : \text{There exists } \pi \text{ s.t. } X^\pi_t(0, z) - C_t \geq 0 \text{ for all } t \in [0, T]\}.$$

We need the following assumption:

Assumption 5.8 $sup_{\tau\in\mathcal{S}_{0,T}} \tilde{\mathrm{E}}[C_\tau/B_\tau] < \infty$.

Then we get

Theorem 5.9 $p_B(C.) = \sup_{\tau\in\mathcal{S}_{0,T}} \tilde{\mathrm{E}}[C_\tau/B_\tau] = p_S(C.)$. *Furthermore optimal* τ^* *and a corresponding hedging strategy* π^* *exist, satisfying*

$$\sup_{\tau\in\mathcal{S}_{0,T}} \tilde{\mathrm{E}}[C_\tau/B_\tau] = \tilde{\mathrm{E}}[C_{\tau^*}/B_{\tau^*}]$$

and

$$\frac{C_{\tau^*}}{B_{\tau^*}} = p_B(C.) + \int_0^{\tau^*} \pi^*_t \tilde{X}^{\pi^*}_t \sigma \, d\tilde{W}_t.$$

Proof: This is a combination of Theorem 12.3.8 (b) in [Ok00] and Theorem 2.5.3 in [KS98]. □

So the theorem says, that the arbitrage free price

$$p(C.) = p_B(C.) = P_S(C.)$$

is unique. It allows the buyer to find a stopping time τ and a trading strategy for his initial debt $-p(C.)$ such that he will make no losses almost surely. For the seller it guarantees that he can cover the claim any time when the investor exercises. The buyer will also be interested to know the optimal stopping time τ^* for which

$$\sup_{\tau\in\mathcal{S}_{0,T}} \tilde{\mathrm{E}}[C_\tau/B_\tau] = \tilde{\mathrm{E}}[C_{\tau^*}/B_{\tau^*}],$$

because choosing τ^* guarantees that he makes no losses if he hedges the payout. Only in this sense τ^* is optimal for the buyer. Note that the supremum in Theorem 5.9 is over expectations under $\tilde{P}$ and not under the original measure P and therefore the strategy τ^* is not optimal in the sense of best expected payoff.

The proof of Theorem 5.9 shows that we have as arbitrage free price at time t

$$p_t(C.) = B_t \text{esssup}_{\tau \in \mathcal{S}_{t,T}} \tilde{\mathrm{E}}[C_\tau / B_\tau \mid \mathcal{F}_t].$$

This is B_t times the Snell envelope $\overline{Y}_t$ of $Y = (Y_s)_{s\in[0,T]}$, $Y_s = C_s/B_s$. Thus, under the conditions on Y in Section 5.1 those results carry over directly – now applied under measure $\tilde{P}$ – and they yield the existence of τ^*. So the main part in the proof of Theorem 5.9 is about the existence of π^* and the equivalence of the prices. By Theorem 5.4 we know

$$\tau^* = \inf\{s \in [t,T] \,:\, p_t(C.)/B_t = Y_t\} = \inf\{s \in [t,T] \,:\, p_t(C.) = C_t\}.$$

Let us now consider the Markovian case, i.e. assume that

$$C_t = \psi(t, S_t)$$

for suitable ψ. Then the price of the American contingent claim C is given by the value function V, i.e. $p_t(C.) = V(t,x)$ on $\{S_t = x\}$, where

$$V(t,x) = B_t \sup_{\tau \in \mathcal{S}_{t,T}} \tilde{\mathrm{E}}_{t,x}[\psi(\tau, S_\tau)/B_\tau]$$

for $t \in [0,T)$ and $V(T,x) = \psi(T,x)$ for $x > 0$. Note that $\tilde{\mathrm{E}}_{t,x}[\,\cdot\,] = \tilde{\mathrm{E}}[\,\cdot \mid S_t = x]$. Following Section 5.2 the continuation region is

$$\mathcal{C} = \{(t,x) \in [0,T] \times (0,\infty) \,:\, V(t,x) > \psi(t,x)\}, \tag{5.45}$$

the stopping region is

$$\mathcal{D} = \{(t,x) \in [0,T] \times (0,\infty) \,:\, V(t,x) = \psi(t,x)\} \tag{5.46}$$

and the optimal stopping time is $\tau^* = \tau_{\mathcal{D}}$.

Example 5.10 For the American call option we have $\psi(t,x) = (x-K)^+$ for some strike price $K > 0$. Then for $\tau \in \mathcal{S}_{t,T}$, Jensen's inequality, Theorem 5.1 and $r \geq 0$ imply

$$\begin{aligned} \tilde{\mathrm{E}}_{t,x}[(S_\tau - K)^+/B_\tau] &\geq \Big(\tilde{\mathrm{E}}_{t,x}[\tilde{S}_\tau] - \tilde{\mathrm{E}}_{t,x}[K/B_\tau]\Big)^+ \\ &= \frac{1}{B_t}\Big(x - K\,\tilde{\mathrm{E}}_{t,x}[\mathrm{e}^{-r(\tau-t)}]\Big)^+ \\ &\geq \frac{1}{B_t}(x-K)^+ = \frac{1}{B_t}\psi(t,x), \end{aligned}$$

where the last inequality is strict if $t < T$ and $\tilde{P}_{t,x}(\tau = T) > 0$. Therefore, $V(t,x) > \psi(t,x)$ for $t < T$ and $\mathcal{D} = \{T\} \times (0,\infty)$, i.e. it is optimal to exercise at the terminal time. Thus the American call has the same (optimal) payout as the European call. This is no longer true if one considers e.g. dividend payments.

5.5 The American put option

In the notation of the preceding section we now look at

$$\psi(t,x) = (K-x)^+$$

for some $K > 0$. This corresponds to the American put option with strike price K.
Following [PS06, Section 25], the optimal stopping problem can be transformed to a free boundary problem to find the boundary between the continuation and stopping regions. We sketch the procedure:

Step 1: From Section 5.4 we know that the arbitrage free price is given by

$$V(t,x) = B_t \sup_{\tau \in \mathcal{S}_{t,T}} \tilde{\mathrm{E}}_{t,x}[(K - S_\tau)^+ / B_\tau],$$

and that we continue (do not stop) if $(u, S_u(t,x))_{u\in[t,T]}$ lies in the continuation region

$$\mathcal{C} = \{(t,x) \in [0,T] \times (0,\infty) \,:\, V(t,x) > (K-x)^+\},$$

and the optimal stopping time τ^* is the first entry time of $(u, S_u(t,x))_{u\in[t,T]}$ in the stopping region

$$\mathcal{D} = \{(t,x) \in [0,T] \times (0,\infty) \,:\, V(t,x) = (K-x)^+\}$$

Step 2: All points (t,x) with $x \geq K$, $t < T$ belong to $\mathcal{C}$. Further one can show that all points with $0 < x < b_\infty$ belong to $\mathcal{D}$, where $b_\infty < K$ is the constant boundary of the stopping region for the corresponding infinite time horizon problem, where an explicit solution can be derived, see e.g. [PS06, Section 25]. Showing that $x \mapsto V(t,x)$ is convex on $(0,\infty)$ it follows that a function $t \mapsto b(t)$ (boundary between $\mathcal{C}$ and $\mathcal{D}$) exists such that

$$\mathcal{C} = \{(t,x) \in [0,T) \times (0,\infty) : x > b(t)\}$$

and

$$\mathcal{D} = \{(t,x) \in [0,T] \times (0,\infty) : x \leq b(t)\} \cup (\{T\} \times (b(T),\infty)).$$

Since $(K-x)^+$ does not depend on time, $t \mapsto V(t,x)$ is decreasing and therefore $t \mapsto b(t)$ increasing.

Step 3: V is continuous.

Step 4: The smooth fit condition for $x \mapsto V(t,x)$ holds at $b(t)$, i.e.

$$V_x(t, b(t)) = -1.$$

Step 5: b is continuous on $[0,T)$ and satisfies $b(T-) = K$. So we may set $b(T) = K$.

Step 6: Arguments like we used to derive the HJB equation lead to the fact that V is $C^{1,2}$ on $\mathcal{C}$ and satisfies

$$V_t + \mathcal{L}_S V - r\,V = 0$$

on $\mathcal{C}$, where $\mathcal{L}_S$ is the generator of $\mathcal{S}$ under $\tilde{P}$,

$$\mathcal{L}_S f(x) = r\,x f_x(x) + \frac{1}{2}\sigma^2 x^2 f_{xx}(x).$$

Steps 1–6 together yield the following free boundary problem for the value function V and the unknown boundary b:

$$\begin{aligned}
&V_t(t,x) + \mathcal{L}_S V(t,x) - r\,V(t,x) = 0, \quad (t,x) \in \mathcal{C},\\
&V(t,x) > (K-x)^+, \quad (t,x) \in \mathcal{C},\\
&V(t,x) = (K-x)^+, \quad (t,x) \in \mathcal{D},\\
&V_x(t,x) = -1, \quad x = b(t).
\end{aligned}$$

It can then be proved (Theorem 25.3 in [PS06]):

Theorem 5.11 *The boundary $b(t)$ is the unique solution of*

$$\begin{aligned}
&\mathrm{e}^{-r(T-t)} \int_0^K \Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{K-y}{b(t)} - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)\right) dy\\
&+rK \int_t^T \mathrm{e}^{-ru}\Phi\left(\frac{1}{\sigma\sqrt{u-t}}\left(\log\frac{b(u)}{b(t)} - \left(r - \frac{\sigma^2}{2}\right)(u-t)\right)\right) du = K - b(t)
\end{aligned}$$

in the class of continuous increasing functions $f : [0,T] \to (0,\infty)$ satisfying $f(t) < K$ for $t < T$ (Here Φ denotes the cumulative standard normal distribution).

So far, it does not seem to be possible to be more explicit and numerical schemes, e.g. like those we used in Section 4.7.6, have to be used. Alternatively one may approximate the Black Scholes model by a binomial tree model and proceed as in Remark 5.5 to find an approximation for the value function and the optimal stopping time (or the free boundary).

Appendix

The results in the appendix are formulated for stochastic processes with infinite time horizon, i.e. $t \geq 0$, and can easily be adapted to a finite time horizon, i.e. $t \in [0,T]$. Further all (in)equalities for random variables are to be understood P-almost surely (a.s.). Only to emphasize this we may write '$P-a.s.$' in some places.

A Conditional Expectation

The conditional expectation is the expected value of a random variable given the available information, which can be described by a σ-algebra. It is again a random variable!

A.1 Conditional expectation and conditional probability

Let $(\Omega, \overline{\mathcal{F}}, P)$ be a probability space, $X : \Omega \to \mathbb{R}$ a random variable with $\mathrm{E}|X| < \infty$, and $\mathcal{G} \subseteq \overline{\mathcal{F}}$ a sub-σ-algebra.

A random variable Z with values in $\mathbb{R}$ is (a version of) the *conditional expectation* of X given $\mathcal{G}$, if

(i) Z is $\mathcal{G}$-measurable and

(ii) $\mathrm{E}[\mathbf{1}_G Z] = \mathrm{E}[\mathbf{1}_G X]$ for all $G \in \mathcal{G}$.

It is denoted by $Z = \mathrm{E}[X \,|\, \mathcal{G}]$.

In fact, $\mathrm{E}[X \,|\, \mathcal{G}]$ is the projection of X to $L^2(\mathcal{G})$, This can be used to show existence and (a.s.) uniqueness of $\mathrm{E}[X \,|\, \mathcal{G}]$.

For $X = 1_A$, $A \in \mathcal{A}$, the *conditional probability* given $\mathcal{G}$ is

$$P(A \,|\, \mathcal{G}) = \mathrm{E}[1_A \,|\, \mathcal{G}].$$

Since the conditional expectation is a random variable, all (in)equalities below are to be understood P-almost surely (a.s.).

A.2 Some properties

(B1) $\mathrm{E}[\mathrm{E}[X \,|\, \mathcal{G}]] = \mathrm{E}[X]$.

(B2) Linearity: $\mathrm{E}[\alpha X_1 + \beta X_2 \,|\, \mathcal{G}] = \alpha \mathrm{E}[X_1 \,|\, \mathcal{G}] + \beta \mathrm{E}[X_2 \,|\, \mathcal{G}]$ for $\alpha, \beta \in \mathbb{R}$.

(B3) $\mathrm{E}[X_1 \,|\, \mathcal{G}] \leq \mathrm{E}[X_2 \,|\, \mathcal{G}]$ for $X_1 \leq X_2$.

(B4) $\mathrm{E}[X \mid \mathcal{G}] = \mathrm{E}[X]$ if X is independent of $\mathcal{G}$.

(B5) $\mathrm{E}[Y X \mid \mathcal{G}] = Y\,\mathrm{E}[X \mid \mathcal{G}]$ for $\mathcal{G}$-measurable Y.

(B6) Tower property: $\mathrm{E}[\mathrm{E}[X \mid \mathcal{G}_2] \mid \mathcal{G}_1] = \mathrm{E}[X \mid \mathcal{G}_1]$ for $\mathcal{G}_1 \subseteq \mathcal{G}_2$.

(B7) Jensen's Inequality: If f is convex with $\mathrm{E}|f(X)| < \infty$, then $f(\mathrm{E}[X \mid \mathcal{G}]) \leq \mathrm{E}[f(X) \mid \mathcal{G}]$.

B Stochastic Processes in Continuous Time

B.1 Stochastic processes

Let $(\Omega, \overline{\mathcal{F}}, P)$ be a suitable probability space. A *stochastic process* is a family of random variables $X = (X_t)_{t \geq 0}$ with values in the state space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borelean σ-algebra in $\mathbb{R}$. The index t is usually interpreted as time. For each $\omega \in \Omega$ the map $t \mapsto X_t(\omega)$ is called a *path* of X.

A stochastic process $(Y_t)_{t \geq 0}$ is a *version* of X, if $P(X_t = Y_t) = 1$ for all $t \geq 0$.

If the paths of X are (P-a.s.) continuous (left continuous, right continuous), we call X continuous (left continuous, right continuous).

If all X_t are integrable (i.e. lie in L^1) or square integrable (in L^2), then we call the process *integrable* or *square integrable*, respectively.

B.2 Filtrations

A *filtration* $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ is an increasing family of σ-algebras in $\overline{\mathcal{F}}$, i.e. for $s < t$ we have $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \overline{\mathcal{F}}$. $\mathcal{F}_t$ may be seen as the information which is available at time t, i.e. for each event $A \in \mathcal{F}_t$ it can be decided at time t if it has occurred or not.

A stochastic process X is $\mathcal{F}$-*adapted*, if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

An important filtration is the filtration generated by a process X which we denote by $\mathcal{F}^X$. It is defined as

$$\mathcal{F}_t^X = \sigma(X_s, s \in [0, t]).$$

So $\mathcal{F}_t^X$ is the smallest σ-algebra for which all X_s, $0 \leq s \leq t$, are measurable. In particular, X is $\mathcal{F}^X$-adapted.

A filtration $\mathcal{F}$ is called right continuous, if $\mathcal{F}_t = \bigcap_{u > t} \mathcal{F}_u$. A filtration $\mathcal{F}$ is said to fulfill the *usual conditions*, if $\mathcal{F}$ is right continuous and $\mathcal{F}_0$ contains all P-null sets.

A stochastic process X is *progressively measurable* w.r.t. a filtration $\mathcal{F}$, if for all $t \geq 0$ the maps $(s, \omega) \mapsto X_s(\omega)$ on $[0,t] \times \Omega$ are measurable w.r.t $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$.

In particular, every progressively mesasurable process is adapted.

Proposition B.1 *Left or right continuous adapted processes are progressively measurable.*

B.3 Stopping times

A random variable $\tau : \Omega \to [0, \infty]$ is called *stopping time* w.r.t. the filtration $\mathcal{F}$ or simply $\mathcal{F}$*-stopping time*, if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in [0, \infty]$. If X is $\mathcal{F}$-adapted and τ an $\mathcal{F}$-stopping time, we set $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$. We might have to define a suitable X_∞.

For an $\mathcal{F}$-stopping time τ we have $\{\tau < t\}$, $\{\tau = t\}$, $\{\tau > t\} \in \mathcal{F}_t$.

Lemma B.2 *If* τ_1, τ_2 *are stopping times, so are* $\tau_1 + \tau_2$, $\tau_1 \wedge \tau_2 := \min\{\tau_1, \tau_2\}$, $\tau_1 \vee \tau_2 := \max\{\tau_1, \tau_2\}$.

Lemma B.3 *If* $\tau_1, \tau_2, \ldots$ *are stopping times, so is* $\sup_{n \in \mathbb{N}} \tau_n$.

For an $\mathcal{F}$-stopping time τ, the *σ-algebra $\mathcal{F}_\tau$ of the events determined before τ* consists of all events A for which $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

If X is $\mathcal{F}$-progressively measurable and τ an $\mathcal{F}$-stopping time, it can be shown that X_τ is $\mathcal{F}_\tau$-measurable.

B.4 Martingales

An $\mathcal{F}$-adapted process X is a *martingale*, if $\mathrm{E}|X_t| < \infty$ for all $t \geq 0$ and

$$\mathrm{E}[X_t \,|\, \mathcal{F}_s] = X_s \quad (P - a.s.)$$

for all $0 \leq s \leq t$.

Example B.4 Wiener process
A *Wiener process* or a *Brownian motion* (w.r.t. $\mathcal{F}$) is an $\mathcal{F}$-adapted process $W = (W_t)_{t \geq 0}$ which satisfies

(W1) $W_0 = 0$,

(W2) $W_t - W_s$ is independent of $\mathcal{F}_s$, $t > s \geq 0$,

(W3) $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$, $t > s \geq 0$.

(W4) W is continuous.

The processes

$$W, \quad (W_t^2 - t)_{t\geq 0}, \quad \left(\exp\left(a\,W_t - \tfrac{1}{2}a^2\,t\right)\right)_{t\geq 0} \text{ for } a > 0$$

are martingales.

As in the example, the filtration under consideration might not always be mentioned explicitly. But W is always a Wiener process w.r.t. $\mathcal{F}^W$.
A stochastic process X is called *uniformly integrable*, if

$$\sup_{t\geq 0} \mathrm{E}|X_t| < \infty \quad \text{and} \quad \sup_{t\geq 0} \mathrm{E}|1_A X_t| \to 0 \text{ for } P(A) \to 0.$$

Lemma B.5 *Let X be a $\mathbb{R}$-valued random variable with $\mathrm{E}|X| < \infty$ and $\mathcal{F}$ a filtration. Then*

$$(\mathrm{E}[X \mid \mathcal{F}_t])_{t\geq 0}$$

is a uniformly integrable martingale.

Theorem B.6 Optional Sampling

(i) Let $M = (M_t)_{t\geq 0}$ be a right continuous $\mathcal{F}$-martingale and τ a bounded $\mathcal{F}$-stopping time. Then $\mathrm{E}M_\tau = \mathrm{E}M_0$.

(ii) Let $M = (M_t)_{t\geq 0}$ be a right continuous, uniformly integrable $\mathcal{F}$-martingale and τ any $\mathcal{F}$-stopping time. Then $\mathrm{E}M_\tau = \mathrm{E}M_0$.

C The Stochastic Integral

From now on we assume that the filtration $\mathcal{F}$ satisfies the usual conditions and that W is a Wiener process w.r.t. $\mathcal{F}$. For a stochastic process X we consider norms

$$\begin{aligned} \|X_T\|_2 &= \left(\mathrm{E}[X_T^2]\right)^{\frac{1}{2}}, \\ \|X\|_{L^2,T} &= \left(\int_0^T X_t^2 dt\right)^{\frac{1}{2}}. \end{aligned}$$

C.1 Stochastic integral for simple processes

A *simple process* X is of the form

$$X_t = \xi_0 \mathbf{1}_{\{0\}}(t) + \sum_{j=1}^{\infty} \xi_j \mathbf{1}_{(t_j, t_{j+1}]}(t), \tag{C.1}$$

where ξ_j is $\mathcal{F}_{t_j}$-measurable and $\sup_{j\in\mathbb{N}} |\xi_j(\omega)| < C$ for all $\omega \in \Omega$ and some $C \in \mathbb{R}$. Furthermore,

$$0 = t_0 < t_1 < \ldots, \quad \lim_{n\to\infty} t_n = \infty.$$

We denote the class of simple processes by $\mathcal{P}_0$.

The *stochastic integral* of $X \in \mathcal{P}_0$ (using representation (C.1) is defined as

$$I_t(X) := \sum_{j=0}^{n-1} \xi_j (W_{t_{j+1}} - W_{t_j}) + \xi_n (W_t - W_{t_n}).$$

Thus the stochastic integral at time t is a random variable and hence $I(X) = (I_t(X))_{t \geq 0}$ is a stochastic process!

Proposition C.1 *For $t \geq s \geq 0$, $X, Y \in \mathcal{P}_0$, $\alpha, \beta \in I\!R$*

(I1) $I_0(X) = 0$.

(I2) $I_t(\alpha X + \beta Y) = \alpha I_t(X) + \beta I_t(Y)$.

(I3) $\mathrm{E}[I_t(X) \,|\, \mathcal{F}_s] = I_s(X)$.

(I4) $\mathrm{E}[(I_t(X) - I_s(X))^2 \,|\, \mathcal{F}_s] = \mathrm{E}\left[\int_s^t X_u^2 du \,|\mathcal{F}_s\right]$.

(I5) $\mathrm{E}[(I_t(X))^2] = \mathrm{E}\left[\int_0^t X_u^2 du\right]$.

(I6) $\|I(X)\|_t = \|X\|_{L^2,t}$ *for all* $t \geq 0$.

C.2 The stochastic integral

Suppose $X \in \mathcal{P}$, the class of progressively measurable processes which satisfy

$$\|X\|_{L^2} = \sum_{n=1}^{\infty} 2^{-n} (1 \wedge \|X\|_{L^2,n}) < \infty.$$

One can show that $\mathcal{P}_0$ is dense in $\mathcal{P}$. Thus there exist simple processes $(X^n)_{n \in I\!N}$ satisfying

$$\lim_{n \to \infty} \|X^n - X\|_M = 0.$$

Proposition C.1(I3,I5) shows that $I(X^n)$ is a square integrable martingale. One can show that the class of square integrable martingales M is a complete mertic space under the metric

$$M \mapsto \sum_{n=1}^{\infty} 2^{-n} (1 \wedge \|M_n\|_2)$$

Using also the isometry in Proposition C.1 (I6) this implies that the limit is well defined and the *stochastic integral* can be set as

$$\int_0^t X_u \, dW_u := I_t(X), \quad t \geq 0.$$

We also write

$$\int X_u\,dW_u = \left(\int_0^t X_u\,dW_u\right)_{t\geq 0},$$

if we consider the stochastic integral as process.

Proposition C.2 *The stochastic integral*

$$I(X) = \int X_t\,dW_t$$

satisfies $(I1),\ldots,(I6)$ *and hence is a square integrable martingale.*

Example C.3

$$\int_0^t W_s\,dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t, \quad t \geq 0.$$

C.3 A generalization

The stochastic integral can be extended to cover integrands in $\mathcal{P}^*$, the class of progressively measurable processes X which satisfy

$$\int_0^t X_s^2\,ds < \infty \quad (P-a.s.) \quad \text{for all} \quad T \geq 0.$$

This can be done by defining stopping times

$$\tau_n = \inf\{t \geq 0 \,:\, \int_0^t X_s^2\,ds \geq n\} \wedge n$$

These satisfy $\tau_n \to \infty$ and we have $X_t^{(n)} := X_t \mathbf{1}_{\{\tau_n \geq t\}} \in \mathcal{P}$. Thus

$$I_t^{(n)} := \int_0^t X_s^{(n)} ds.$$

exists and we can define

$$I_t(X) := I_t^{(n)} \quad \text{on} \quad 0 \leq t \leq \tau_n, \quad n \in \mathbb{N}.$$

We use the same notation

$$\int_0^t X_s dW_s = I_t(X), \quad t \geq 0.$$

Remark C.4 Local martingales
Note that for $X \in \mathcal{P}^*$ the properties $(I3)-(I6)$ might no longer hold!

But $I(X)$ is still a continuous local martingale. A stochastic process M is a *local martingale* if there exist stopping times $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \to \infty$ such that the processes $(M_{t\wedge\tau_n})_{t\geq 0}$ are martingales.

C.4 Itô Formula

A (one-dimensional) *Itô process* is a stochastic process which admits a representation of the form

$$X_t = \xi + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s, \quad t \geq 0, \tag{C.2}$$

where ξ is $\mathcal{F}_0$-measurable and $(b_t)_{t\geq 0}$ and $(\sigma_t)_{t\geq 0}$ are $\mathcal{F}$- progressively measurable with

$$\int_0^t (|b_s| + |s_s|^2) ds < \infty \quad \text{for all} \quad t > 0.$$

In particular, X is continuous and $\mathcal{F}$-adapted. In differential form we may write

$$dX_t = b_t \, dt + \sigma_t \, dW_t, \quad X_0 = \xi.$$

Theorem C.5 Itô Formula
Suppose X is an Itô-process as given in (C.2) and $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Then,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d[X]_s, \tag{C.3}$$

where the quadratic variation $[X]$ of X is defined as

$$[X]_t = \int_0^t \sigma_s^2 \, ds.$$

For a shorter notation, we usually write only

$$df(X_t) = f'(X_t) dX_t + \frac{1}{2} f''(X_t) d[X]_t.$$

C.5 The multidimensional case

An m-dimensional Wiener process $W = (W_t)_{t\geq 0}$ with components

$$W_t = \begin{pmatrix} W_t^1 \\ \vdots \\ W_t^m \end{pmatrix}.$$

consists of independent Wiener processes $W^1, \ldots, W^m$, all adapted to the same filtration $\mathcal{F}$. The stochastic integral can be define componentwise. So for $n \times m$-matrices A_t with components in $\mathcal{P}^*$ we have

$$\int_0^t A_s dW_s = \begin{pmatrix} \sum_{j=1}^m \int_0^t A_s^{1j} dW_s^j \\ \vdots \\ \sum_{j=1}^m \int_0^t A_s^{nj} dW_s^j \end{pmatrix}.$$

An n-dimensional stochastic process which admits a representation of the form

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s, \quad t \geq 0, \tag{C.4}$$

where X_0 is $\mathcal{F}_0$-measurable and $(b_t)_{t\geq 0}$ and $(\sigma_t)_{t\geq 0}$ are $I\!R^n$-valued and $I\!R^{n\times m}$-valued, respectively, as well as $\mathcal{F}$- progressively measurable with

$$\int_0^t \left(\sum_{i=1}^n |b_s^i| + \sum_{i=1}^n \sum_{j=1}^m |s_s^{ij}|^2 \right) ds < \infty \quad \text{for all } t > 0$$

is called (multi-dimensional) *Itô process*. Again X is continuous and $\mathcal{F}$-adapted.

Theorem C.6 Multidimensional Itô Formula
Suppose X is an Itô-process as given in (C.4) and $f : [0,\infty) \times I\!R^n \to I\!R$ is continuously differentiable in the first component with derivative $f_t = \frac{\partial}{\partial t} f$, and twice continuously differentiable in the other components with partial derivatives $f_{x_i} := \frac{\partial}{\partial x_i} f$, $i = 1, \ldots, n$ and $f_{x_i,x_j} = \frac{\partial^2}{\partial x_j \partial x_i} f$, $i,j = 1, \ldots, n$). Then,

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f_t(s, X_s) ds \\ &\quad + \sum_{i=1}^n \int_0^t f_{x_i}(s, X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^n \int_0^t f_{x_i,x_j}(s, X_s) d[X^i, X^j]_s, \end{aligned}$$

where the covariations $[X^i, X^j]$ of X^i and X^j are defined as

$$[X^i, X^j]_t = \int_0^t (\sigma_s \sigma_s^\top)_{ij} ds,$$

where $^\top$ denotes transposition, so

$$(\sigma_t \sigma_t^\top)_{ij} = \sum_{k=1}^m \sigma_t^{ik} \sigma_t^{jk}.$$

.

Corollary C.7 Product Rule.
For (one-dimensional) Itô processes X, Y w.r.t. to the same Wiener process and with diffusion coefficients σ^X, σ^Y we have

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t.$$

where

$$[X, Y]_t = \int_0^t \sigma_s^X \sigma_s^Y ds, \quad t \geq 0.$$

If X and Y were defined w.r.t. independent Wiener processes, we had $[X, Y]_t = 0$.

D Stochastic Differential Equations

D.1 Problem formulation

We want to solve the *stochastic differential equation (SDE)*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{D.5}$$

where

$$b : [0, T] \times I\!R^n \to I\!R^n, \quad \sigma : [0, T] \times I\!R^n \to I\!R^{n \times m}$$

are measurable and W is an m-dimensionaler Wiener process w.r.t. to some filtration $\mathcal{F}$. X is n-dimensional, so (D.5) consists of the SDEs

$$dX_t^i = b_i(t, X_t)dt + \sum_{j=1}^{m} \sigma_{ij}(t, X_t)dW_t^j, \quad i = 1, \ldots, n.$$

Usually we also require that some initial condition $X_0 = \xi$ holds, where ξ is a $\mathcal{F}$-measurable random variable independent of W.

A continuous process $X = (X_t)_{t \in [0,T]}$ is a *solution* of (D.5), if it is $\mathcal{F}$-adapted, $X_0 = \xi$, which satisfies (D.5) and P-a.s.

$$\int_0^t |b_i(s, X_s)|ds < \infty, \quad \int_0^t \sigma_{ij}(s, X_s)|^2 ds < \infty.$$

The SDE is said to have a *strong solution*, if for given probability space $(\Omega, \overline{\mathcal{F}}, P)$, initial condition ξ and Wiener process W a solution X can be found which is adapted to the filtration generated by W and ξ (augmented with the null sets).

A *weak solution* means that a probability space $(\Omega, \overline{\mathcal{F}}, P)$, a Wiener process W and a filtration $\mathcal{F}$ can be found such that a solution exists.

D.2 Uniqueness and existence

To show uniqueness we need that the one sided derivatives of the coefficients b and σ are bounded. This follows from a Lipschitz condition like formulated in the following Theorem.

Theorem D.1 Uniqueness *Suppose that b, σ are Lipschitz continuous in x, i.e. there exists a constant $K > 0$ such that*

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leq K \|x - y\|, \quad t \in [0, T], x, y \in I\!R^n.$$

Then any two solutions $X, \tilde{X}$ of (D.5) satisfy

$$P(X_t = \tilde{X} \textit{ for all } t \in [0, T]) = 1.$$

The condition in Theorem D.1 is not restrictive enough to avoid so called explosions of the process. Therefore we also need the additional condition in the following theorem.

Theorem D.2 *Suppose that b, σ are Lipschitz continuous and satisfy a linear growth condition, i.e. there exists a constant $K > 0$ such that for all $t \in [0,T]$, $x, y \in I\!R^n$,*

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq K\,\|x - y\|, \tag{D.6}$$
$$\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 \leq K^2(1 + \|x\|^2). \tag{D.7}$$

Let the initial condition be given by some random variable ξ which is independent of W and satisfies $\mathrm{E}\|\xi\|^2 < \infty$.
Then (D.5) has a continuous, strong solution X with

$$\mathrm{E}\left[\int_0^T \|X_t\|^2\,dt\right] < \infty.$$