MEC-E8001 Finite Element Analysis; Formulae collection

GENERAL

Displacement: $\vec{u} = u_X\vec{I} + u_Y\vec{J} + u_Z\vec{K} = u_x\vec{i} + u_y\vec{j} + u_z\vec{k} = u\vec{i} + v\vec{j} + w\vec{k}$

Rotation (small): $\vec{\theta} = \theta_X\vec{I} + \theta_Y\vec{J} + \theta_Z\vec{K} = \theta_x\vec{i} + \theta_y\vec{j} + \theta_z\vec{k} = \phi\vec{i} + \theta\vec{j} + \psi\vec{k}$

Coordinate systems: $\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} i_X & i_Y & i_Z \\ j_X & j_Y & j_Z \\ k_X & k_Y & k_Z \end{bmatrix} \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix} = \{\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\}^{\mathrm{T}} \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix}, \quad \mathbf{i} = \frac{1}{h}\begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix}$

Strain-stress: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G}\begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}, \quad G = \frac{E}{2(1+\nu)}$ or

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} \equiv [E] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = G \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \quad [E]_\varepsilon = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

Linear strain: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$

Green-Lagrange: $\begin{Bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} (\partial u_x/\partial x)^2 + (\partial u_y/\partial x)^2 + (\partial u_z/\partial x)^2 \\ (\partial u_x/\partial y)^2 + (\partial u_y/\partial y)^2 + (\partial u_z/\partial y)^2 \\ (\partial u_x/\partial z)^2 + (\partial u_y/\partial z)^2 + (\partial u_z/\partial z)^2 \end{Bmatrix},$

$$\begin{Bmatrix} E_{xy} \\ E_{yz} \\ E_{zx} \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} (\partial u_x/\partial x)(\partial u_x/\partial y) + (\partial u_y/\partial x)(\partial u_y/\partial y) + (\partial u_z/\partial x)(\partial u_z/\partial y) \\ (\partial u_x/\partial y)(\partial u_x/\partial z) + (\partial u_y/\partial y)(\partial u_y/\partial z) + (\partial u_z/\partial y)(\partial u_z/\partial z) \\ (\partial u_x/\partial z)(\partial u_x/\partial x) + (\partial u_y/\partial z)(\partial u_y/\partial x) + (\partial u_z/\partial z)(\partial u_z/\partial x) \end{Bmatrix}$$

PRINCIPLE OF VIRTUAL WORK

$$\delta W = \sum\nolimits_{e\in E} \delta W^e = 0 \quad \forall \delta\mathbf{a}, \quad \delta W = \int_\Omega \delta w d\Omega$$

Virtual work density: $\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$, $\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}$

$$\delta w_V^{\text{ine}} = -\begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \rho \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix},\ \delta w_{V^\circ}^{\text{int}} = -\begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ \delta E_{zz} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix} - 2\begin{Bmatrix} \delta E_{xy} \\ \delta E_{yz} \\ \delta E_{zx} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix}$$

Bar mode (*x*): $\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx}$, $\delta w_\Omega^{\text{ext}} = \delta u f_x$, $\delta w_\Omega^{\text{ine}} = -\delta u \rho A \ddot{u}$, $\delta w_\Omega^{\text{cpl}} = \frac{d\delta u}{dx} EA\alpha\Delta\vartheta$,

$$\delta w_{\Omega^\circ}^{\text{int}} = -(\frac{d\delta u}{dx} + \frac{d\delta u}{dx}\frac{du}{dx} + \frac{d\delta v}{dx}\frac{dv}{dx} + \frac{d\delta w}{dx}\frac{dw}{dx})CA^\circ(\frac{du}{dx} + \frac{1}{2}\frac{du}{dx}\frac{du}{dx} + \frac{1}{2}\frac{dv}{dx}\frac{dv}{dx} + \frac{1}{2}\frac{dw}{dx}\frac{dw}{dx}),$$

$$\delta p_\Omega^{\text{int}} = -\frac{d\delta\vartheta}{dx} kA \frac{d\vartheta}{dx},\ \delta p_\Omega^{\text{ext}} = \delta\vartheta s\ .$$

Torsion mode (*x*): $\delta w_\Omega^{\text{int}} = -\frac{d\delta\phi}{dx} GI_{rr} \frac{d\phi}{dx}$, $\delta w_\Omega^{\text{ext}} = \delta\phi m_x$, $\delta w_\Omega^{\text{ine}} = -\delta\phi\rho I_{rr}\ddot{\phi}$

Bending mode (*xz*): $\delta w_\Omega^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2}$, $\delta w_\Omega^{\text{ext}} = \delta w f_z$,

$$\delta w_\Omega^{\text{ine}} = -\delta w \rho A \ddot{w} - \frac{d\delta w}{dx}\rho I_{yy}\frac{d\ddot{w}}{dx},\ \delta w_\Omega^{\text{sta}} = -\frac{d\delta w}{dx} N \frac{dw}{dx} \text{ in which } N = EA\frac{du}{dx}.$$

Bending mode (*xy*): $\delta w_\Omega^{\text{int}} = -\frac{d^2\delta v}{dx^2} EI_{zz} \frac{d^2 v}{dx^2}$, $\delta w_\Omega^{\text{ext}} = \delta v f_y$,

$$\delta w_\Omega^{\text{ine}} = -\delta v \rho A \ddot{v} - \frac{d\delta v}{dx}\rho I_{zz}\frac{d\ddot{v}}{dx},\ \delta w_\Omega^{\text{sta}} = -\frac{d\delta v}{dx} N \frac{dv}{dx} \text{ in which } N = EA\frac{du}{dx}.$$

Thin-slab mode (*xy*):

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\text{T}} t[E]_\sigma \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix},\ \delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix},$$

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\text{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix},\ \delta w_\Omega^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\text{T}} t\rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \end{Bmatrix},\ \delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ 2\delta E_{xy} \end{Bmatrix}^{\text{T}} t^\circ[C]_\sigma \begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix},$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\text{T}} \rho^\circ t^\circ \begin{Bmatrix} g_x \\ g_y \end{Bmatrix},\ \delta w_\Omega^{\text{cpl}} = \begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial x \end{Bmatrix}^{\text{T}} \frac{E\alpha t}{1-\nu}\Delta\vartheta \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},\ \delta p_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta\vartheta/\partial x \\ \partial\delta\vartheta/\partial y \end{Bmatrix}^{\text{T}} tk \begin{Bmatrix} \partial\vartheta/\partial x \\ \partial\vartheta/\partial y \end{Bmatrix},$$

$$\delta p_\Omega^{\text{ext}} = \delta\vartheta s$$

Bending mode (*xy*):

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_{\sigma} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix}, \ \delta w_{\Omega}^{\text{ext}} = \delta w f_z$$

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \partial \delta w / \partial x \\ \partial \delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} \rho \begin{Bmatrix} \partial \ddot{w} / \partial x \\ \partial \ddot{w} / \partial y \end{Bmatrix} - \delta w t \rho \ddot{w}$$

$$\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} \partial \delta w / \partial x \\ \partial \delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix} \text{ where } \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}$$

Solid:

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial \delta u / \partial x \\ \partial \delta v / \partial y \\ \partial \delta w / \partial z \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial w / \partial z \end{Bmatrix} - \begin{Bmatrix} \partial \delta u / \partial y + \partial \delta v / \partial x \\ \partial \delta v / \partial z + \partial \delta w / \partial y \\ \partial \delta w / \partial x + \partial \delta u / \partial z \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} \partial u / \partial y + \partial v / \partial x \\ \partial v / \partial z + \partial w / \partial y \\ \partial w / \partial x + \partial u / \partial z \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \ \delta w_{\Omega^{\circ}}^{\text{int}} = -\begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ \delta E_{zz} \end{Bmatrix}^{\mathrm{T}} [C] \begin{Bmatrix} E_{xx} \\ E_{yy} \\ E_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta E_{xy} \\ \delta E_{yz} \\ \delta E_{zx} \end{Bmatrix}^{\mathrm{T}} 4G \begin{Bmatrix} E_{xy} \\ E_{yz} \\ E_{zx} \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} \partial \delta u / \partial x \\ \partial \delta v / \partial y \\ \partial \delta w / \partial z \end{Bmatrix}^{\mathrm{T}} \frac{E\alpha}{1-2\nu} \Delta \vartheta \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \ \delta p_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial \delta \vartheta / \partial x \\ \partial \delta \vartheta / \partial y \\ \partial \delta \vartheta / \partial z \end{Bmatrix}^{\mathrm{T}} k \begin{Bmatrix} \partial \vartheta / \partial x \\ \partial \vartheta / \partial y \\ \partial \vartheta / \partial z \end{Bmatrix}, \ \delta p_{\Omega}^{\text{ext}} = \delta \vartheta s$$

APPROXIMATIONS (some) $u = \mathbf{N}^{\mathrm{T}} \mathbf{a}$, $\xi = \dfrac{x}{h}$

Quadratic (line): $\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1 - 3\xi + 2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi - 1) \end{Bmatrix}$, $\mathbf{a} = \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix}$ (bar)

Cubic (line): $\mathbf{N} = \begin{Bmatrix} N_{10} \\ N_{11} \\ N_{20} \\ N_{21} \end{Bmatrix} = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2 \xi \\ (3-2\xi)\xi^2 \\ h\xi^2(\xi - 1) \end{Bmatrix}$, $\mathbf{a} = \begin{Bmatrix} u_{10} \\ u_{11} \\ u_{20} \\ u_{21} \end{Bmatrix} (= \begin{Bmatrix} u_{z1} \\ -\theta_{y1} \\ u_{z2} \\ -\theta_{y2} \end{Bmatrix}$ beam *xz*-plane bending)

Linear (triangle): $\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$

VIRTUAL WORK EXPRESSIONS

Rigid body/point force: $\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix},$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{y1} \\ \delta u_{z1} \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{y1} \\ \ddot{u}_{z1} \end{Bmatrix} - \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{y1} \\ \delta\theta_{z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} J_{xx}\ddot{\theta}_{x1} \\ J_{yy}\ddot{\theta}_{y1} \\ J_{zz}\ddot{\theta}_{z1} \end{Bmatrix}$$

Bar mode: $\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho A h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}, \quad \delta W^{\text{cpl}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \Delta\vartheta_1 \\ \Delta\vartheta_2 \end{Bmatrix},$$

$$\delta P^{\text{int}} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{kA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix}, \quad \delta P^{\text{ext}} = \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{sh}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Torsion mode: $\delta W^{\text{int}} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{GI_{rr}}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho I_{rr} h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_{x1} \\ \ddot{\theta}_{x2} \end{Bmatrix}$$

Bending mode (xz):

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{\rho I_{yy}}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} + \frac{\rho A h}{420} \begin{bmatrix} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{bmatrix} \right) \begin{Bmatrix} \ddot{u}_{z1} \\ \ddot{\theta}_{y1} \\ \ddot{u}_{z2} \\ \ddot{\theta}_{y2} \end{Bmatrix}$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \; N = EA(\frac{u_{x2}-u_{x1}}{h})$$

Bending mode (*xy*):

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{zz}}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix}, \; \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{f_y h}{12} \begin{Bmatrix} 6 \\ h \\ 6 \\ -h \end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} (\frac{\rho I_{zz}}{30h} \begin{bmatrix} 36 & 3h & -36 & 3h \\ 3h & 4h^2 & -3h & -h^2 \\ -36 & -3h & 36 & -3h \\ 3h & -h^2 & -3h & 4h^2 \end{bmatrix} + \frac{\rho Ah}{420} \begin{bmatrix} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{bmatrix}) \begin{Bmatrix} \ddot{u}_{y1} \\ \ddot{\theta}_{z1} \\ \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \end{Bmatrix},$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & 3h & -36 & 3h \\ 3h & 4h^2 & -3h & -h^2 \\ -36 & -3h & 36 & -3h \\ 3h & -h^2 & -3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix}, \; N = EA(\frac{u_{x2}-u_{x1}}{h})$$

CONSTRAINTS

Frictionless contact: $\vec{n} \cdot \vec{u}_{\mathrm{A}} = 0$

Joint: $\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}}$

Rigid body (link): $\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}} + \vec{\theta}_{\mathrm{A}} \times \vec{\rho}_{\mathrm{AB}}$, $\vec{\theta}_{\mathrm{B}} = \vec{\theta}_{\mathrm{A}}$.

MATHEMATICS

Polar representation: $e^{i\alpha} = \cos\alpha + i\sin\alpha$, $\sin i\alpha = i\sinh\alpha$, $\cos i\alpha = \cosh\alpha$, $i^2 = -1$

Eigenvalue decomposition: $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$

Matrix function: If $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$, then $f(\mathbf{A}) = \mathbf{X}f(\boldsymbol{\lambda})\mathbf{X}^{-1}$

Newton's method: If $\mathbf{a} = \mathbf{a} - (\frac{\partial \mathbf{R}(\mathbf{a})}{\partial \mathbf{a}})^{-1}\mathbf{R}(\mathbf{a}) \equiv \mathbf{G}(\mathbf{a})$, then $\mathbf{R}(\mathbf{a}) = 0$

Taylor series: $f(x+\mathrm{a}) = \sum_{i=0}^{n} \frac{1}{i!}(\mathrm{a}\frac{d}{dx})^i f(x) + \frac{1}{(n+1)!} f^{(n+1)}(\underline{x})\mathrm{a}^{n+1} \quad \underline{x} \in [x, x+\mathrm{a}]$

INTEGRATION IN TIME (free vibrations)

Crank-Nicholson: $$\begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i+1)} = \begin{bmatrix} \mathbf{I} & -\mathbf{I}/2 \\ \boldsymbol{\alpha}/2 & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & \mathbf{I}/2 \\ -\boldsymbol{\alpha}/2 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i)}, \quad \boldsymbol{\alpha} = \mathbf{M}^{-1}\mathbf{K}\Delta t^2$$

Disc. Galerkin: $$\begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i+1)} = \begin{bmatrix} \boldsymbol{\alpha} & \mathbf{I} - \boldsymbol{\alpha}/2 \\ -\mathbf{I} - \boldsymbol{\alpha}/2 & \boldsymbol{\alpha}/3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i)}, \quad \boldsymbol{\alpha} = \mathbf{M}^{-1}\mathbf{K}\Delta t^2$$