Nonlinear dynamics & chaos Bifurcations

Lecture II

Recap

- ordinary differential equations are transformed to the form

$$\dot{x}_1 = f_1(x_1, \ldots, x_n)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, \ldots, x_n)$$

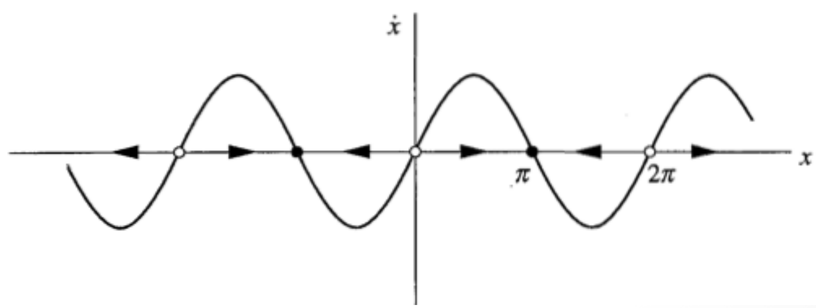
$\left(\dot{x}_i = \frac{dx_i}{dt}\right)$ - trick: $x_1 = x; \quad x_2 = \dot{x}_1$

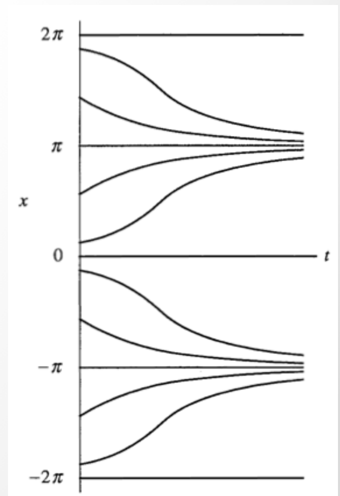
$$\rightarrow \frac{d^2x}{dt^2} = \dot{x}_2$$

- one-dimensional flows: $\dot{x} = f(x)$; linear stability analysis via $f'(x)$

- **vector field**: how the velocity of the particle depends on its position

- **fixed points**; stable and unstable





- time dependence $x(t)$

- potential V: $f(x) = -\frac{dV}{dx}$

Bifurcations

Question: One-dimensional motion is pretty simple (solutions settle down to equilibrium or head out to infinity); why bother?

Answer: Dependence on parameters, which may change the behaviour dramatically!

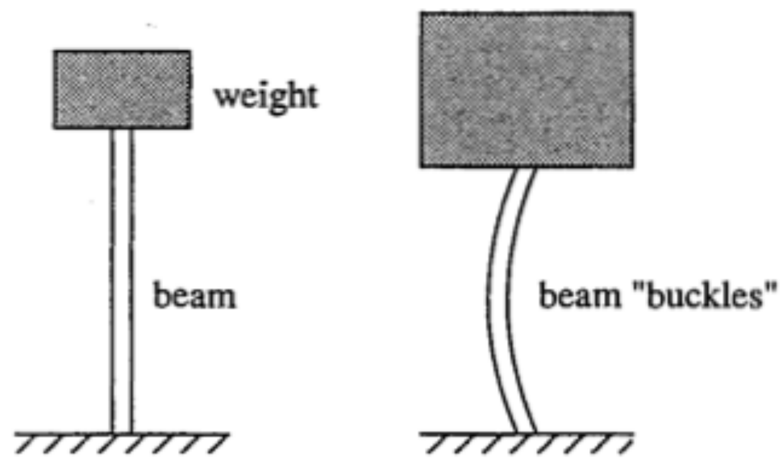
Bifurcation: Qualitative change of the dynamics due to variation of a (control) parameter.

The qualitative changes: creation and destruction or a change in the stability of fixed points.

Bifurcations

("Bifurcation" means "splitting into two".)

Example. Buckling of a beam.



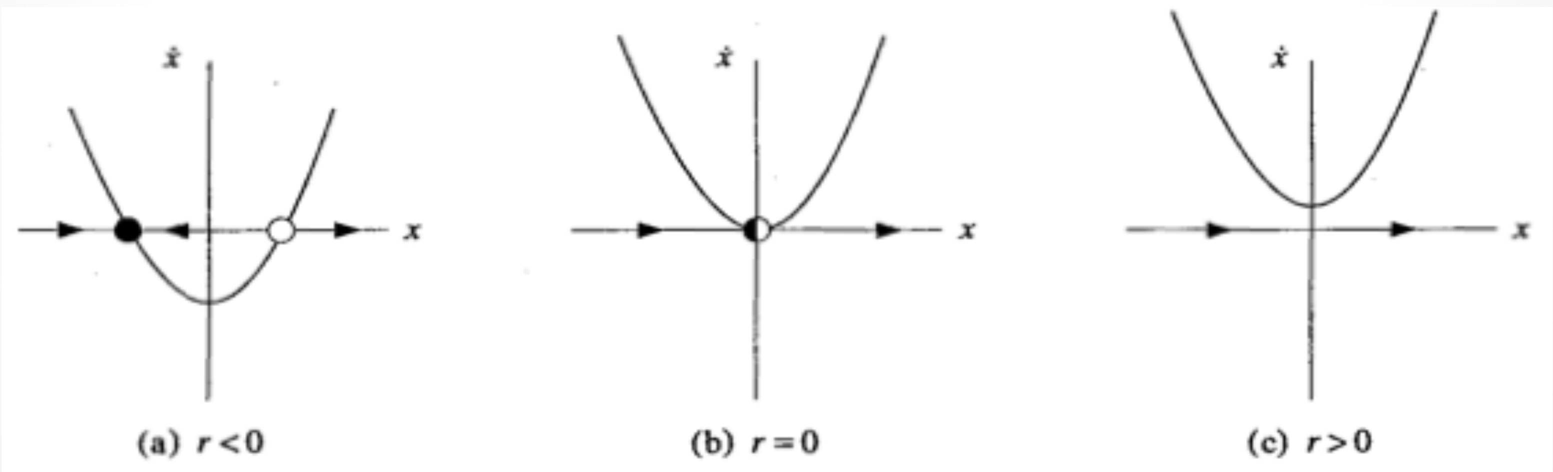
Dynamical variable x: deflection of the beam from vertical.
Control parameter: mass of the weight placed on the top.

Learn bifurcations first in the simplest case, i.e. on the line.

Saddle-node bifurcations

The most fundamental bifurcation.

As a parameter varies two fixed points move toward each other, collide and mutually annihilate, or, varying the parameter in the opposite direction, a FP is created. The prototypical example: $\dot{x} = r + x^2$



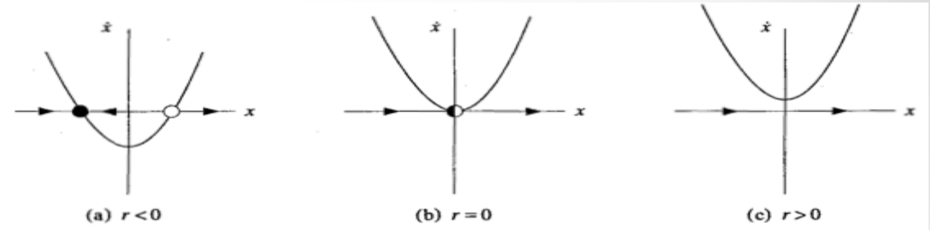
$x^*_{1,2} = \pm\sqrt{-r}$ $\quad$ $x^* = 0$ $\quad$ No fixed points.

Bifurcation at $r = 0$: the vector fields for $r < 0$ and $r > 0$ are qualitatively different.

Saddle-node bifurcations

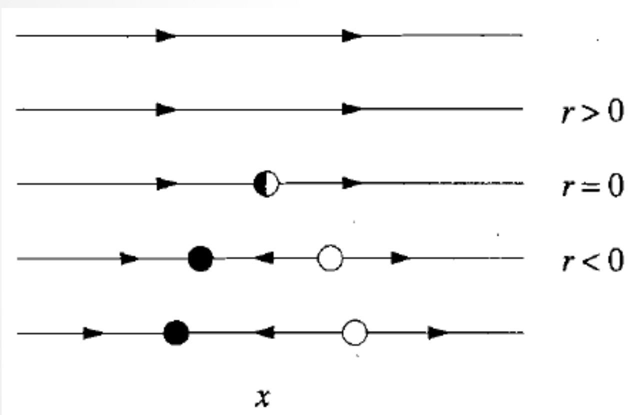
$$\dot{x} = r + x^2$$

Graphical conventions



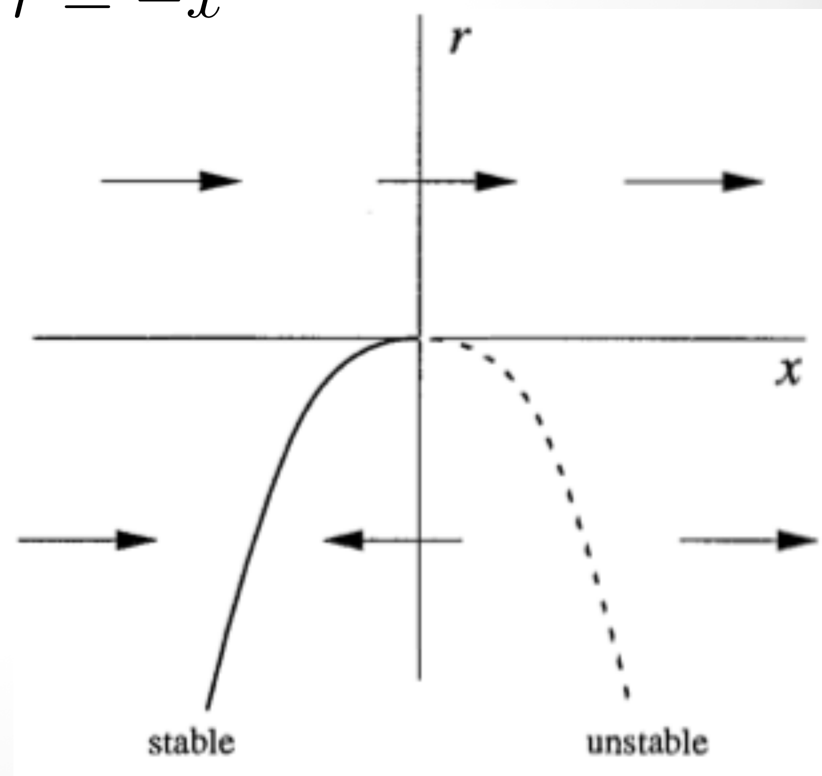
$x^*_{1,2} = \pm\sqrt{-r}$ $\quad x^* = 0 \quad$ No FPs.

Stacked vector fields



Fixed points as a function of r:

for $\dot{x} = 0, \; r = -x^2$

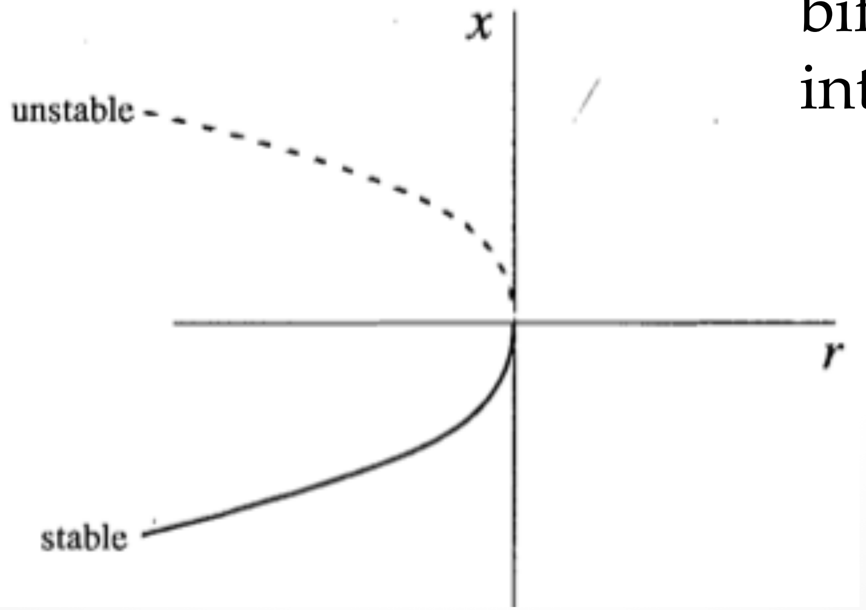


In the limit of a continuous stack of vector fields $\rightarrow$

Saddle-node bifurcations

$$\dot{x} = r + x^2$$

Conventional bifurcation diagram plotted as x vs r, since r is here viewed as the independent variable.



Again, the word bifurcation: splitting into two branches.

Example I

Do the linear stability analysis of the fixed points of

$$\dot{x} = f(x) = r - x^2$$

r > 0

$$x^*_{1,2} = \pm\sqrt{r} \quad \rightarrow \quad f'(x^*) = -2x^* \quad \rightarrow f'(\pm\sqrt{r}) = \mp 2\sqrt{r}$$

$x^*_1 = \sqrt{r}$ stable $\qquad$ $x^*_2 = -\sqrt{r}$ unstable

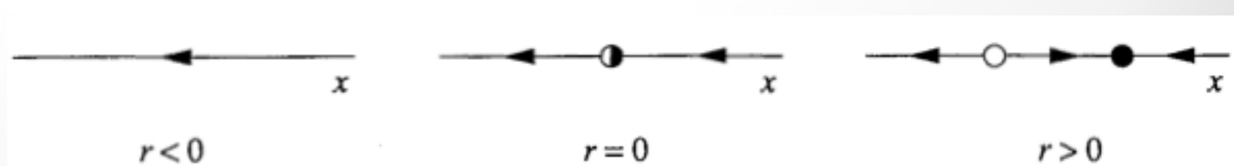
r = 0

$$x^* = 0 \quad \rightarrow \quad f'(x^*) = -2x^* \quad \rightarrow f'(0) = 0$$

Linearization vanishes when the points coalesce.

r < 0

No fixed points!



Example II

Show that the first-order system

$$\dot{x} = r - x - e^{-x}$$

undergoes a saddle-node bifurcation as *r* is varied.

Fixed points:

$$f(x^*) = 0 \quad \rightarrow \quad r - x^* - e^{-x^*} = 0$$

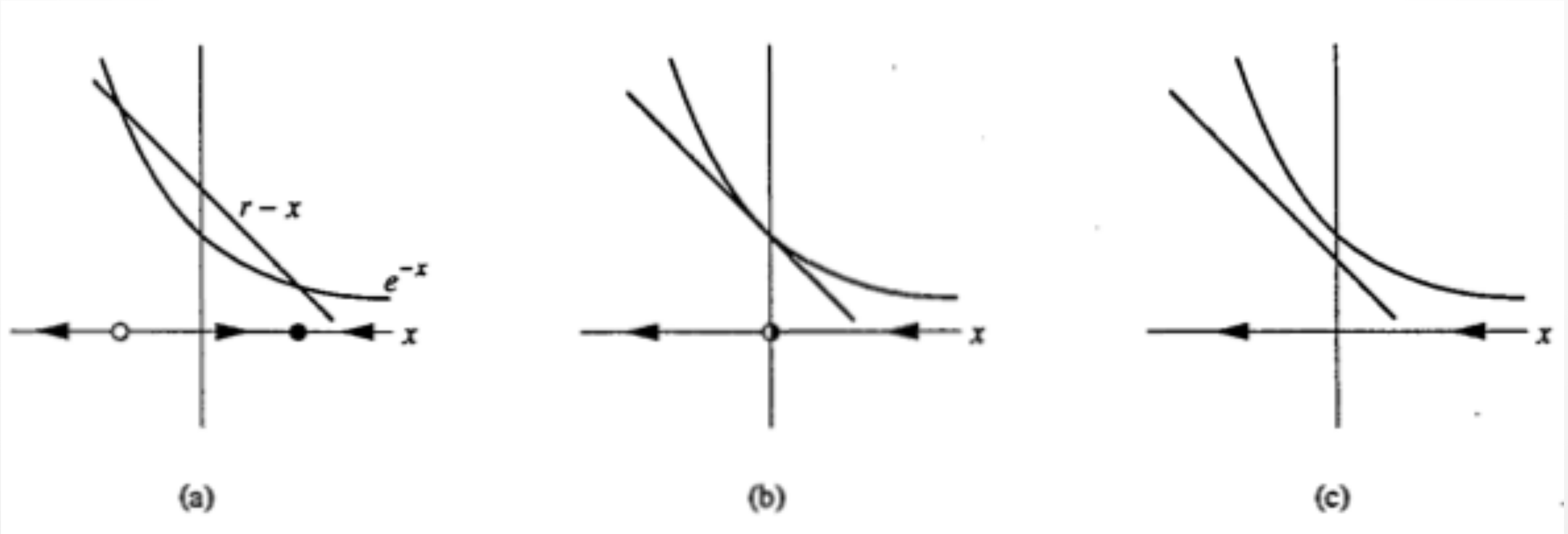
How to solve it? → Use the geometric method

$$\begin{cases} y & = & r - x \\ y & = & e^{-x} \end{cases}$$

Example II

$$\dot{x} = r - x - e^{-x}$$

$$\begin{cases} y &= r - x \\ y &= e^{-x} \end{cases}$$



Bifurcation point at the *r*-value for which the curves are tangent

Example II

$$\dot{x} = r - x - e^{-x}$$

$$\begin{cases} y = r - x \\ y = e^{-x} \end{cases}$$

Condition of tangential intersection: the curves must touch at that point and have equal derivative.

$$\begin{cases} e^{-x} = r - x \\ \frac{d}{dx} e^{-x} = \frac{d}{dx}(r - x) \end{cases} \Rightarrow \begin{cases} r = 1 \\ x = 0 \end{cases}$$

The bifurcation point is $r_c = 1$ and the bifurcation occurs at $x = 0$.

Normal forms

The equations $\dot{x} = r - x^2$ and $\dot{x} = r + x^2$

are prototypical in the sense that they are representative of all saddle-node bifurcations.

Consider, for example, $\dot{x} = r - x - e^{-x} \quad \rightarrow$

Normal forms

$$\dot{x} = r - x - e^{-x}$$

Taylor expansion about $x = 0$:

$$\begin{aligned} \dot{x} &= r - x - e^{-x} \\ &= r - x - \left[1 - x + \frac{x^2}{2!} + \cdots\right] \\ &= (r-1) - \frac{x^2}{2} + \cdots \end{aligned}$$

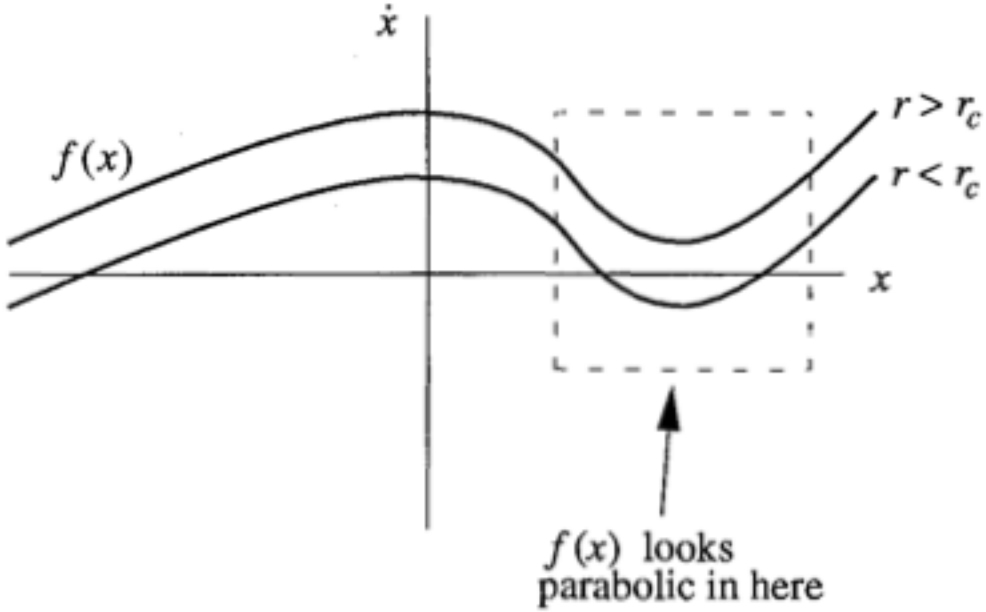
has the same algebraic form as

$$\dot{x} = r - x^2$$

(It can be made to agree exactly by $r \to r + 1$ and $x \to 2x$ in the Taylor expansion.)

Normal forms

Graphically: Two nearby roots of $f(x)$ are needed for a saddle-node bifurcation to occur. At the bifurcation point $f(x)$ is tangent to the x-axis.



Near the bifurcation $f(x)$ looks parabolic.

(Note that in the previous example the parabola actually opens downwards and there are no FPs when $r < r_c$ and two FPs when $r > r_c$.)

Normal forms

Taylor expansion about x^*, r_c

$$
\begin{aligned}
\dot{x} &= f(x,r) \\
&= f(x^*,r_c) + (x-x^*)\frac{\partial f}{\partial x}|_{(x^*,r_c)} \\
&+ (r-r_c)\frac{\partial f}{\partial r}|_{(x^*,r_c)} + \frac{1}{2}(x-x^*)^2\frac{\partial^2 f}{\partial^2 x}|_{(x^*,r_c)} + \cdots
\end{aligned}
$$

FP & tangency condition

$$
f(x^*,r_c)=0,\quad \frac{\partial f}{\partial x}|_{(x^*,r_c)}=0,\quad a=\partial f/\partial r|_{(x^*,r_c)},\quad b=\frac{1}{2}\partial^2 f/\partial x^2|_{(x^*,r_c)}
$$

$$
\boxed{\dot{x} = a(r-r_c) + b(x-x^*)^2 + \cdots}
$$

, which agrees with the prototypical forms.

$$
\dot{x} = r - x^2 \qquad \dot{x} = r + x^2
$$

= the **normal forms** of saddle-node bifurcation.

Transcritical bifurcation

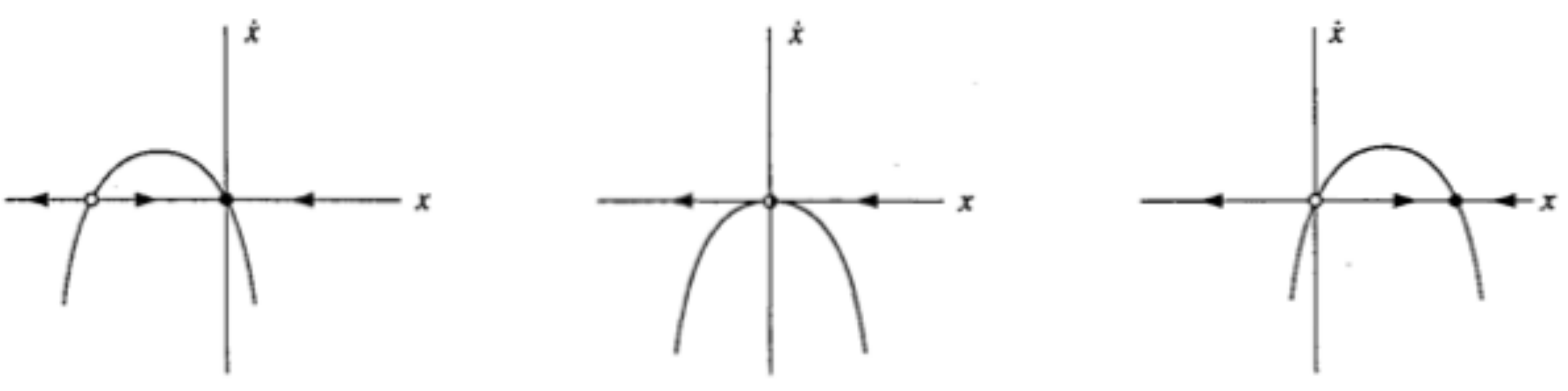
For a transcritical bifurcation *a fixed point always exists* (as e.g. in logistic eq. $\dot{N} = rN(1 - N/K)$), but it may change stability when a parameter varies.

The normal form for a transcritical bifurcation

$$\dot{x} = rx - x^2$$

(A logistic equation, where r can also have negative values.)

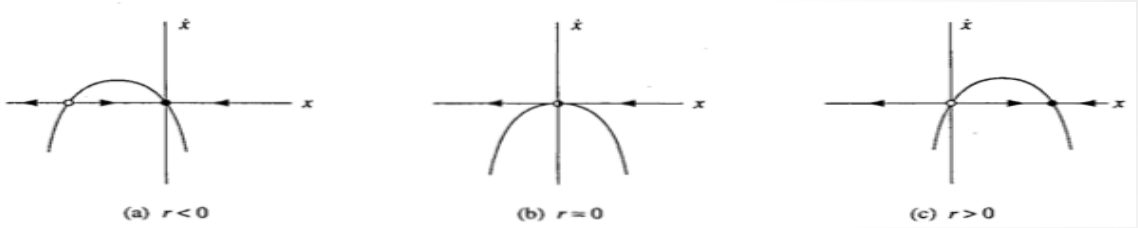
A fixed point at $x^* = 0$ for all values of r.



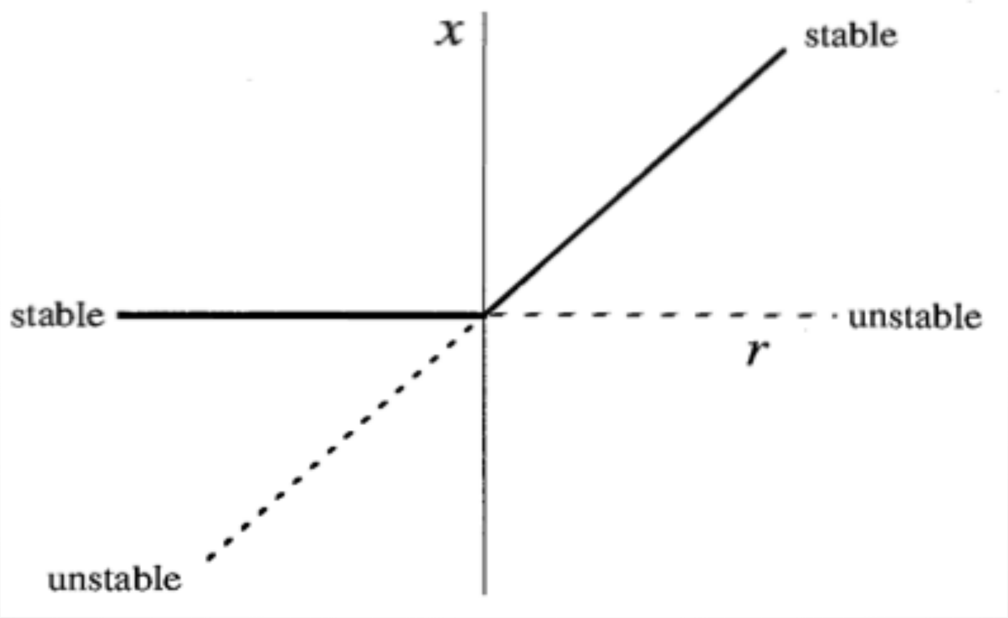
(a) $r<0$ (b) $r=0$ (c) $r>0$

Transcritical bifurcation

Unlike in the saddle-node bifurcation, in the transcritical bifurcation fixed points do not disappear – instead they just switch their stability.



Bifurcation diagram



Example I

Show that the first-order system

$$\dot{x} = x(1-x^2) - a(1-e^{-bx})$$

has a transcritical bifurcation at $x = 0$ when a and b satisfy a certain relation, to be determined. Then find an approximate formula for the FP that bifurcates from $x = 0$, assuming that the parameters are close to the bifurcation curve.

Clue for a transcritical bifurcation: $x^* = 0 \; \forall \; a, b$

Taylor expansion about $x^* = 0$

$$\begin{aligned} 1 - e^{-bx} &= 1 - [1 - bx + \tfrac{1}{2}b^2x^2 + O(x^3)] \\ &= bx - \tfrac{1}{2}b^2x^2 + O(x^3) \end{aligned}$$

$$\begin{aligned} \dot{x} &= x - a(bx - \tfrac{1}{2}b^2x^2) + O(x^3)] \\ &= (1-ab)x + (\tfrac{1}{2}ab^2)x^2 + O(x^3) \end{aligned}$$

$$\boxed{\dot{x} = rx - x^2}$$

Example I

$r = 0 \quad \to \quad ab = 1$ equation for **bifurcation curve** (on this curve bifurcation takes place in *a*, *b* –space).

Nonzero fixed point:

$$1 - ab + (\frac{1}{2}ab^2)x^* \approx 0 \quad \to \quad x^* \approx \frac{2(ab-1)}{ab^2}$$

Valid for small x $\Rightarrow ab \approx 1$

For $ab = 1$ (and small x): $\dot{x} = \dfrac{b}{2}x^2$

(So, parabolic near FP.)

Example II

Analyse the dynamics of

$$\dot{x} = r \ln x + x - 1$$

near $x = 1$, and show that the system undergoes a transcritical bifurcation at a certain value of r. Then find new variables such that the equation assumes $\dot{X} \approx RX - X^2$

A fixed point at $x^* = 1$ for all values of r. We are interested in the dynamics close to this FP, so introduce u:

$$u = x - 1 \quad \rightarrow \quad \begin{aligned} \dot{u} &= \dot{x} \\ &= r \ln(1+u) + u \\ &= r[u - \tfrac{1}{2}u^2 + O(u^3)] + u \\ &\approx (r+1)u - \tfrac{1}{2}ru^2 + O(u^3) \end{aligned}$$

Transcritical bifurcation for $r_c = -1$.

Example II

To get the normal form, coefficient of u^2 has to be 1.

$$u = av \quad \rightarrow \quad \dot{v} = (r+1)v - (\frac{1}{2}ra)v^2 + O(v^3)$$

$$\text{for } a = 2/r \quad \rightarrow \quad \dot{v} = (r+1)v - v^2 + O(v^3)$$

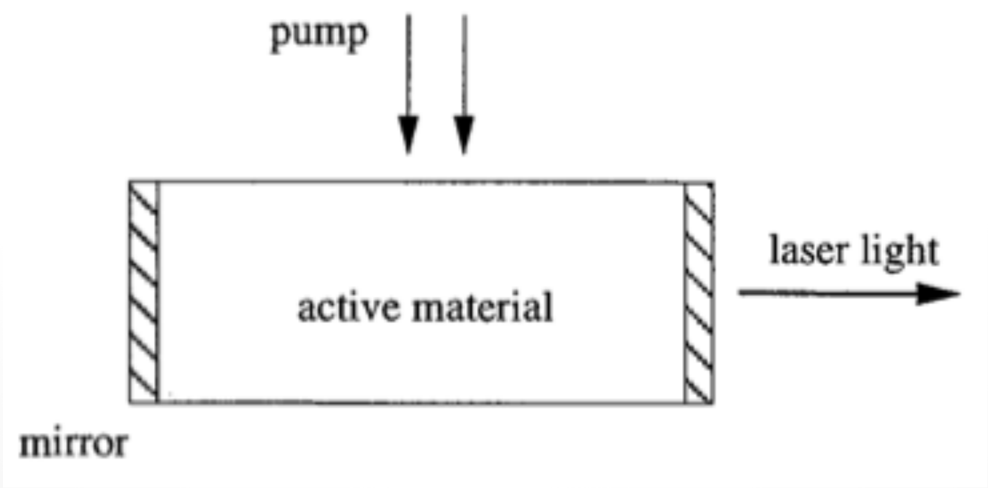
$$R = r+1,\ X = v \quad \rightarrow \quad \dot{X} \approx RX - X^2$$

In fact, the theory of normal forms ensures that one can put the system in normal form without neglecting higher order terms, so in the above case $\dot{X} \approx RX - X^2 \Rightarrow \dot{X} = RX - X^2$.

Laser threshold

Solid-state laser: collection of “laser-active” atoms embedded in a solid state matrix, bounded by partially reflecting mirrors at either end

External energy source is used to excite or “pump” the atoms out of their ground states



Laser threshold

When the pumping is relatively weak, laser acts as lamp: the excited atoms oscillate independently of one another and emit randomly light waves.

Above a certain threshold for the pumping, emitted photons from one atom triggers emission in others; atoms oscillate in phase → laser → the beam of radiation is much more coherent and intense than that produced below the laser threshold.

Atoms are being excited completely at random: where does coherence come from? From the cooperative interaction of stimulated emission among the atoms.

Laser threshold

Model

The dynamical variable: number of photons $n(t)$ in the laser field

$$\dot{n} = \text{gain} - \text{loss} = GnN - kn$$

Gain term from stimulated emission (photons stimulate excited atoms to emit other photons).

Rate of stimulated emission is proportional to the number of photons $n(t)$ and of excited atoms $N(t)$. $G > 0$ is the gain coefficient.

Loss term from photons escaping the laser, rate constant k.

Laser threshold

Model

Key idea: after an excited atom emits a photon it drops down to a lower energy and is no longer excited $\rightarrow N$ decreases due to emission of photons.

In the absence of laser action, the pump keeps the number of excited atoms fixed at $N_0 \rightarrow$ the actual number of excited atoms will be reduced by the laser process

$$N(t) = N_0 - \alpha\, n(t)$$

$$\begin{aligned} \dot{n} &= Gn\,(N_0 - \alpha\, n) - kn \\ &= (GN_0 - k)\, n - (\alpha\, G) n^2 \end{aligned}$$

(α is the rate at which atoms drop from stimulated to the ground state.)

Laser threshold

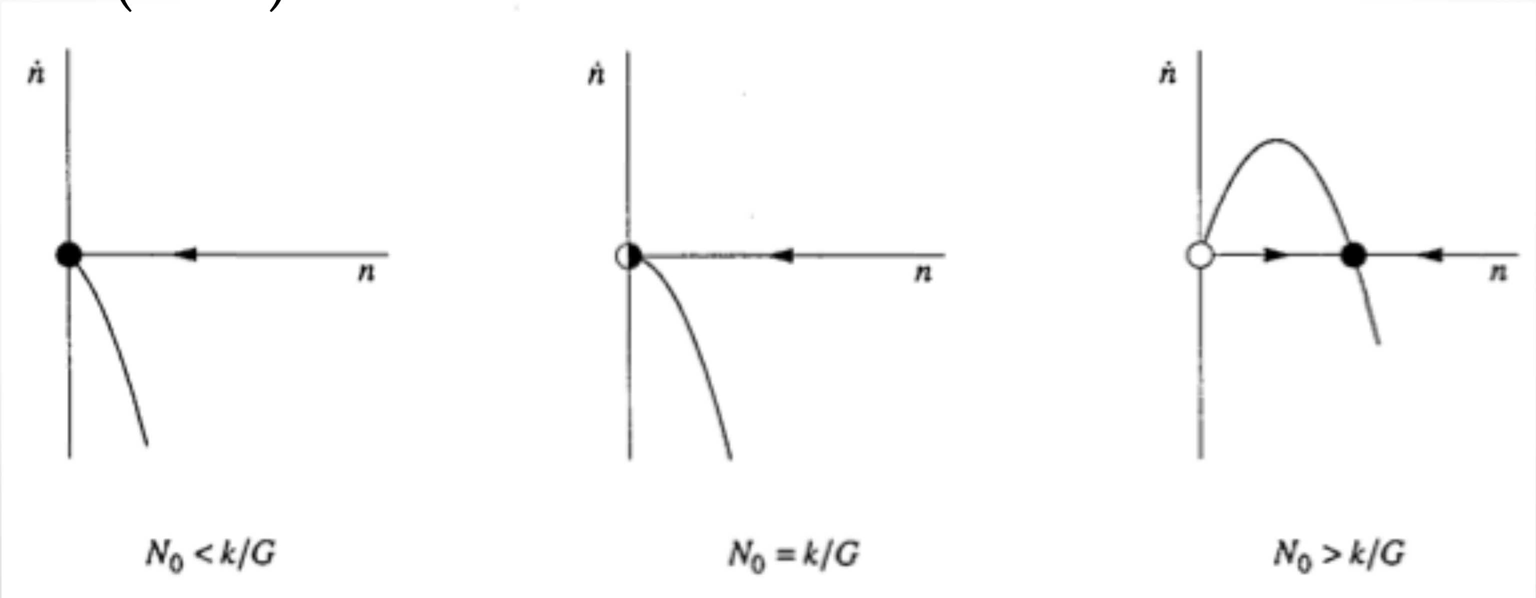
Model

$$\dot{n} = (GN_0 - k)\, n - (\alpha\, G)n^2$$

Fixed point $n^* = 0$ for all values of parameters.

For $N_0 < k/G \rightarrow n^* = 0$ is stable (no laser action).

For $N_0 > k/G \rightarrow n^* = 0$ is unstable and $n^* = (GN_0 - k)/\alpha G > 0$ is stable (laser).



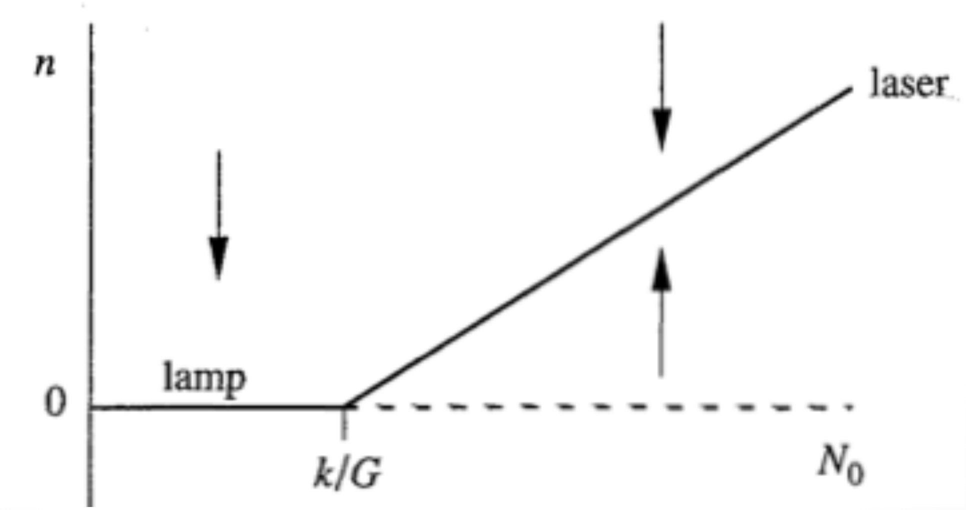
Laser threshold

Model

$$\dot{n} = (GN_0 - k)\, n - (\alpha\, G) n^2$$

$N_0 = k/G$ is the laser threshold.

Bifurcation diagram:



(This simplified model ignores dynamics of excited atoms, existence of spontaneous emissions, etc.)

Pitchfork bifurcation

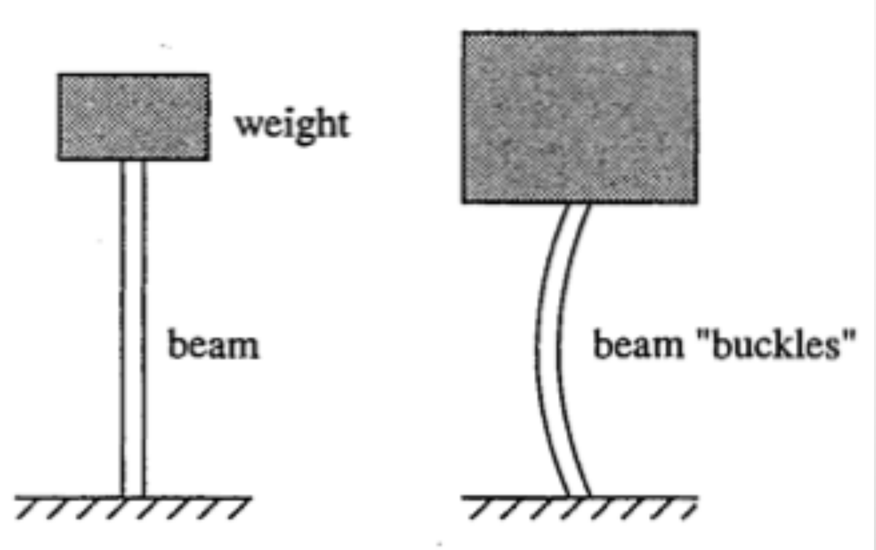
Common in physical problems having a symmetry.

For example in problems having a spatial symmetry between left and right, *fixed points tend to appear and disappear in symmetrical pairs*.

Beam buckling has a left-right symmetry:

Two types of pitchfork bifurcations:

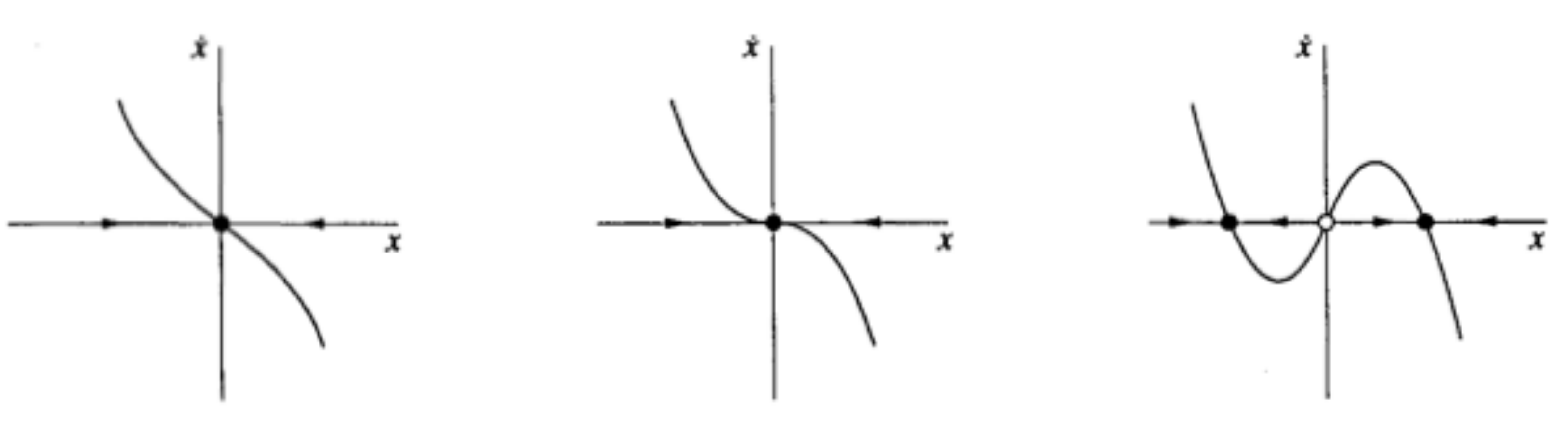
1) Supercritical (related to 2nd order phase transitions)
2) Subcritical (related to 1st order phase transitions)



Supercritical pitchfork bifurcation

The normal form: $\dot{x} = rx - x^3$

Left-right symmetry: The equation is invariant under the change of variables $x \to -x$ (left-right symmetry).

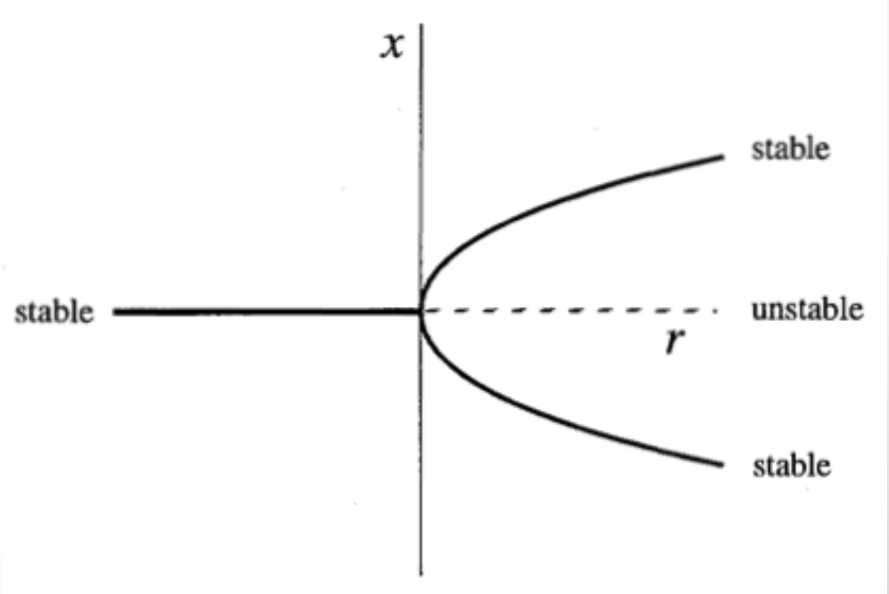


(a) $r<0$ (b) $r=0$ (c) $r>0$

Supercritical pitchfork bifurcation

1) $r < 0$: the origin is the only fixed point (stable).
2) $r = 0$: the origin is still stable, but solutions no longer decay exponentially fast (no linear term), but have algebraic (power-law) decay (critical slowing down).
3) $r > 0$: the origin becomes unstable, two new stable fixed points appear at $x^* = \pm\sqrt{r}$.

Non-zero FPs for $r > 0$: "supercritical"
Supercritical: bifurcating FPs are stable.



Example I

Show that the equation

$$\dot{x} = -x + \beta \tanh x$$

undergoes a pitchfork bifurcation as β is varied.

Fixed points

$$x^* = \beta \tanh x^*$$

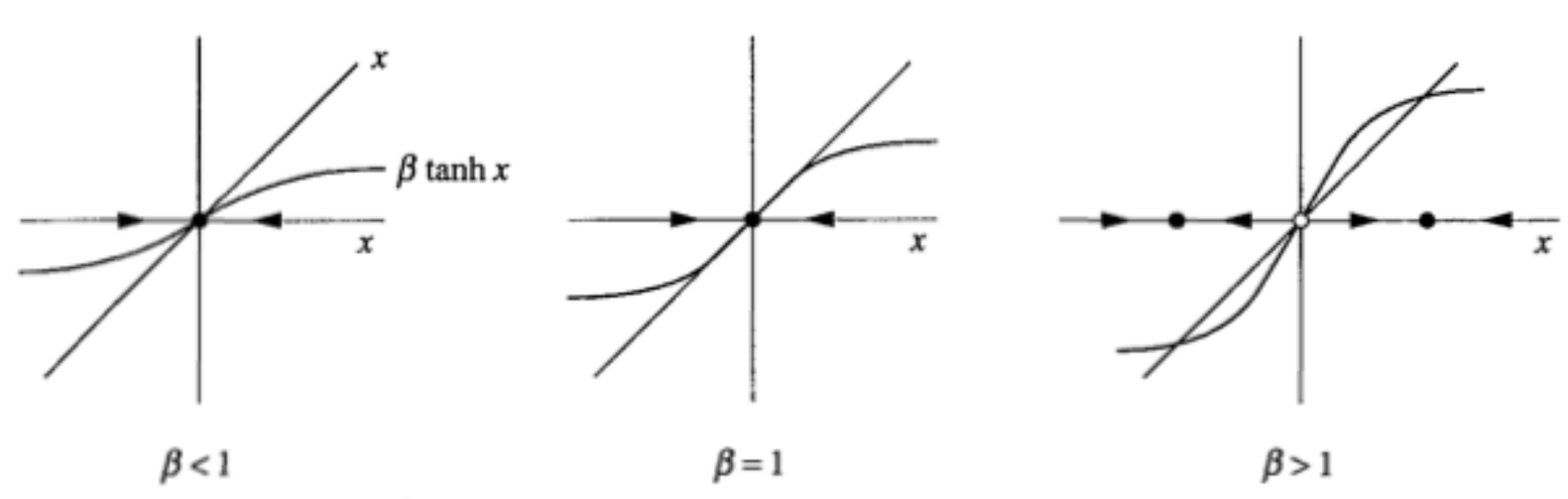
Geometric approach

$$\left\{ \begin{array}{ccc} y & = & x \\ y & = & \beta \tanh x \end{array} \right.$$

Example I

Geometric approach

$$\begin{cases} y = x \\ y = \beta \tanh x \end{cases}$$



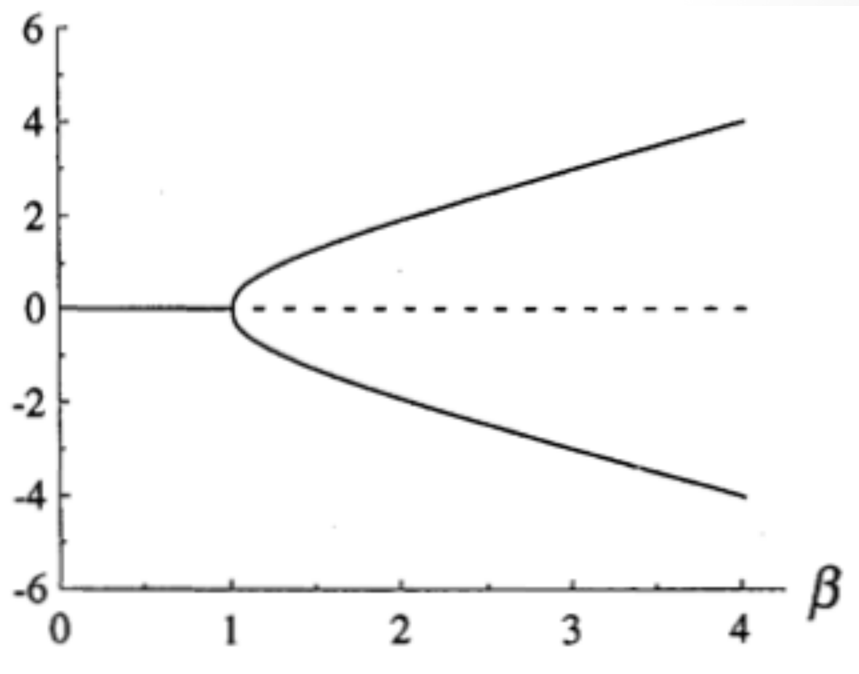
Pitchfork bifurcation at $\beta = 1$, $x^* = 0$.

Example I

It is easier to treat x as an independent variable and compute β as a function of x. Then plot the bifurcation diagram in the usual way:

$$x^* = \beta \tanh x^* \quad \rightarrow \quad \beta = \frac{x^*}{\tanh x^*}$$

This shortcut is based on $f(x, \beta) = -x + \beta \tanh(x)$ depending more simply on β than on x. Typically, the dependence on the control parameter is simpler than on x.

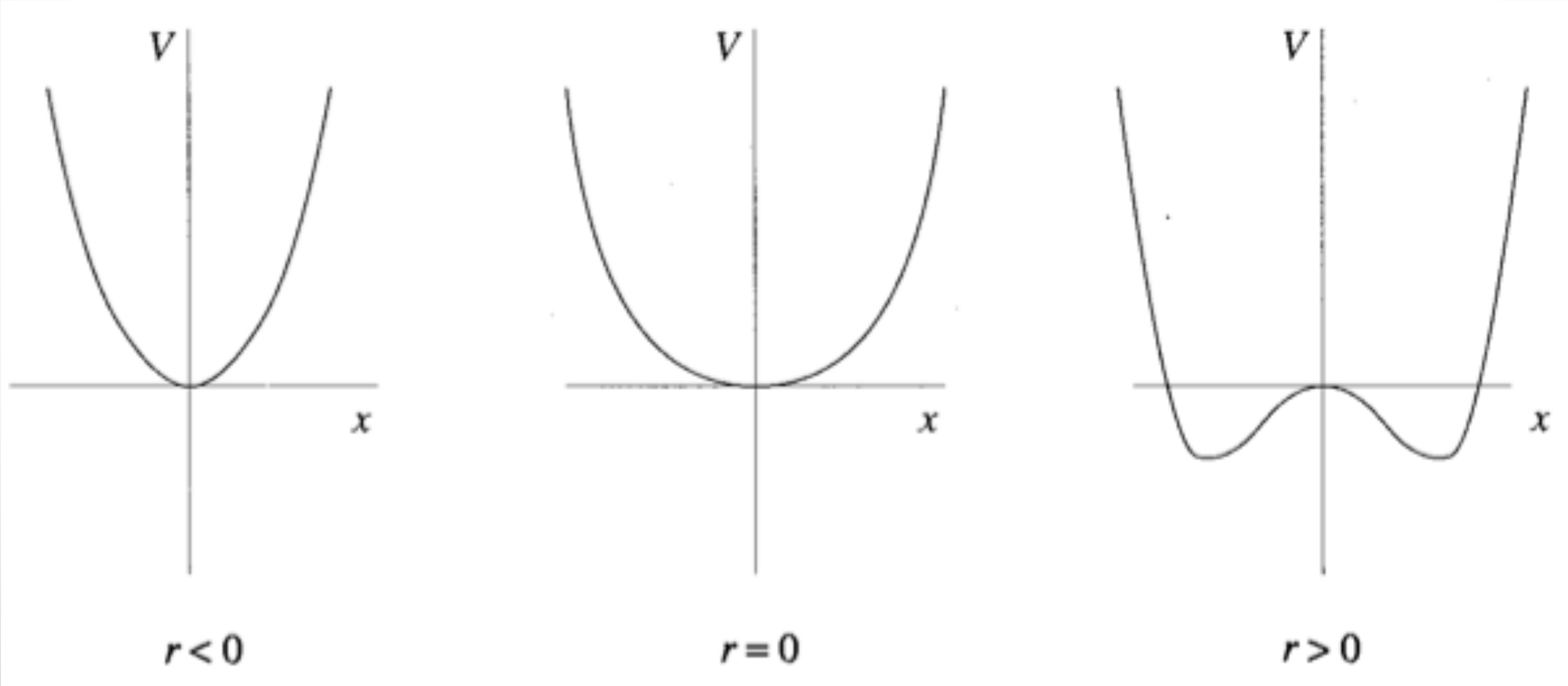


Example II

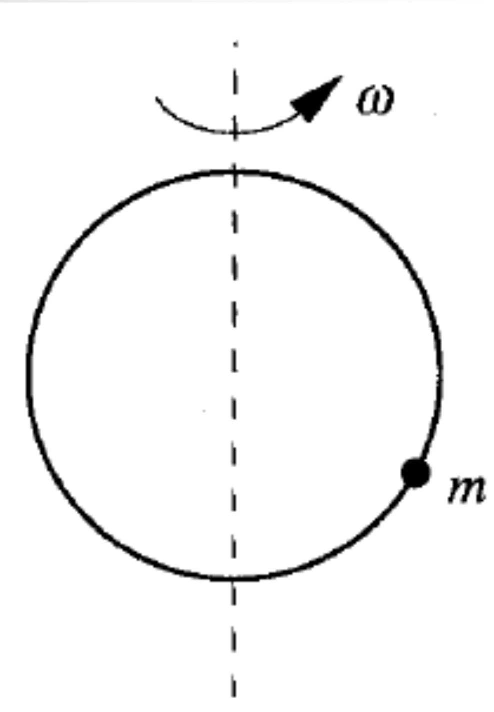
Plot the potential $V(x)$ for the cases $r < 0$, $r = 0$, $r > 0$ for the system

$$\dot{x} = rx - x^3$$

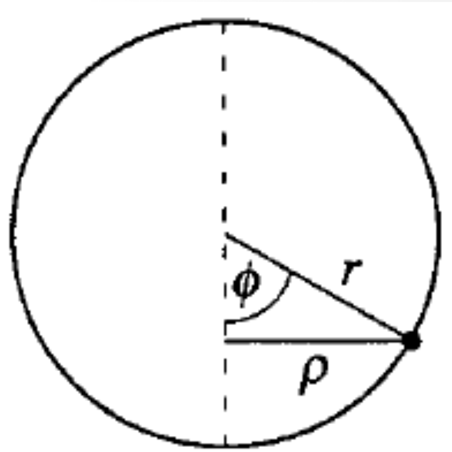
$$f(x) = -\frac{dV}{dx} \quad \rightarrow \quad -\frac{dV}{dx} = rx - x^3 \quad \rightarrow \quad V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4$$



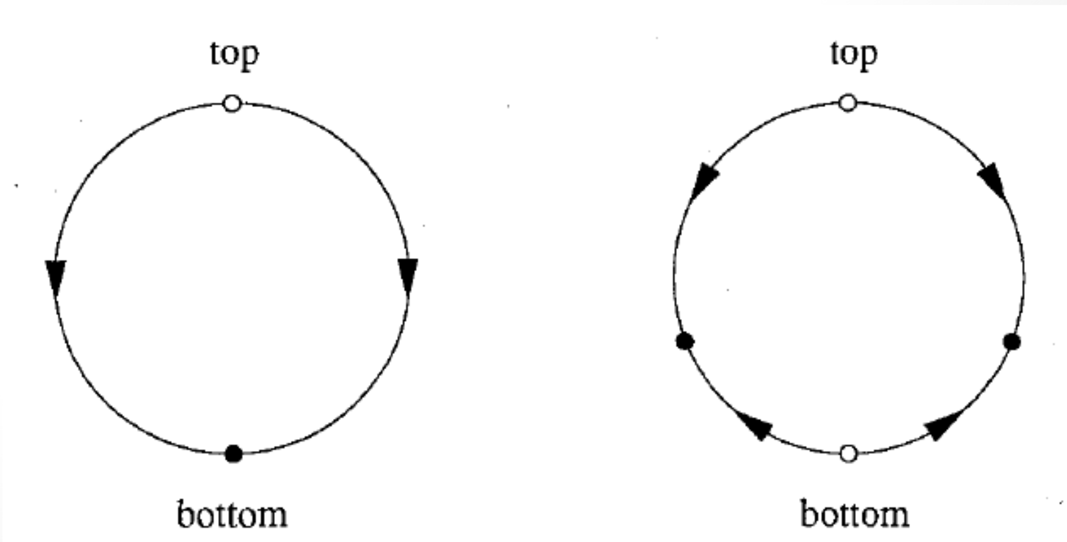
Example III: Overdamped bead on a rotating hoop



Coordinates:



Solutions for slow and fast rotation:

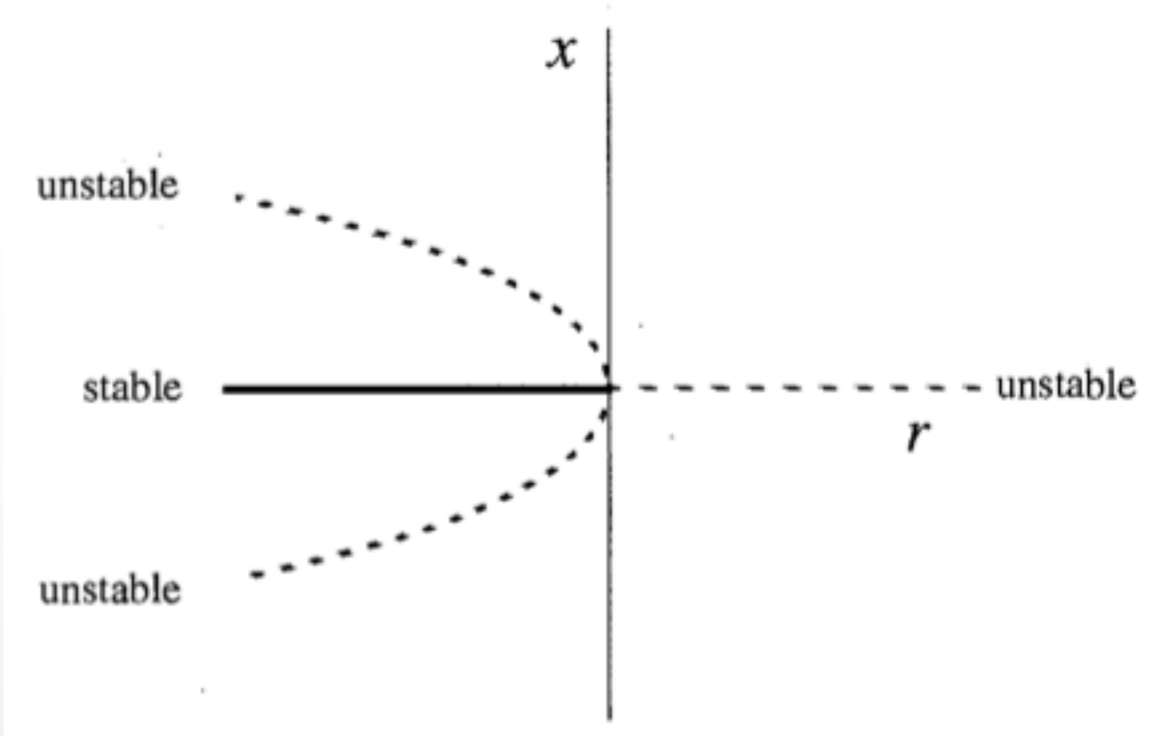


Supercritical pitchfork bifurcation.

Subcritical pitchfork bifurcation

The normal form: $\dot{x} = rx + x^3$

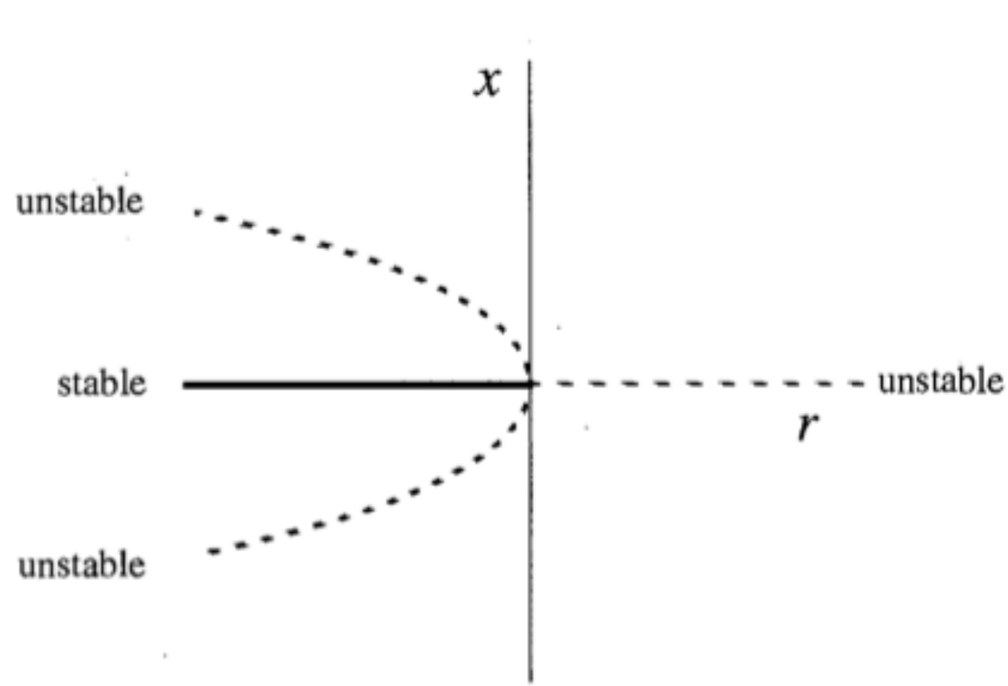
Bifurcation diagram



Subcritical pitchfork bifurcation

$$\dot{x} = rx + x^3 \qquad \left(V(x) = -\frac{r}{2}x^2 - \frac{1}{4}x^4\right)$$

The **destabilizing** term $+x^3$ makes a world of difference. Here the nonzero fixed points are unstable and exist only below the bifurcation $r < 0 \rightarrow$ "subcritical".



This cubic term also drives the trajectories starting from $x \neq 0$ to infinity in a finite time when $r > 0$ (blow-up).

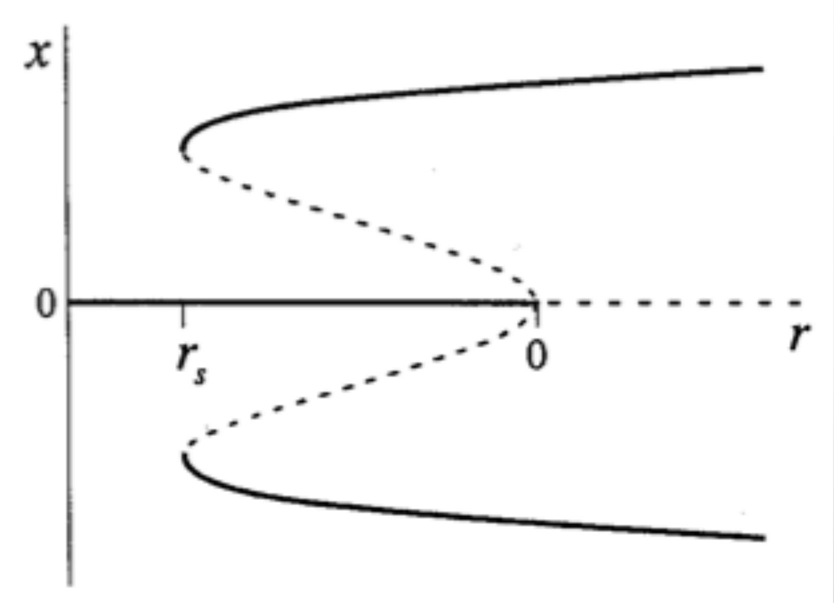
Subcritical pitchfork bifurcation

In real systems such an explosive instability is usually opposed by the stabilizing influence of higher-order terms

$$\dot{x} = rx + x^3 - x^5$$

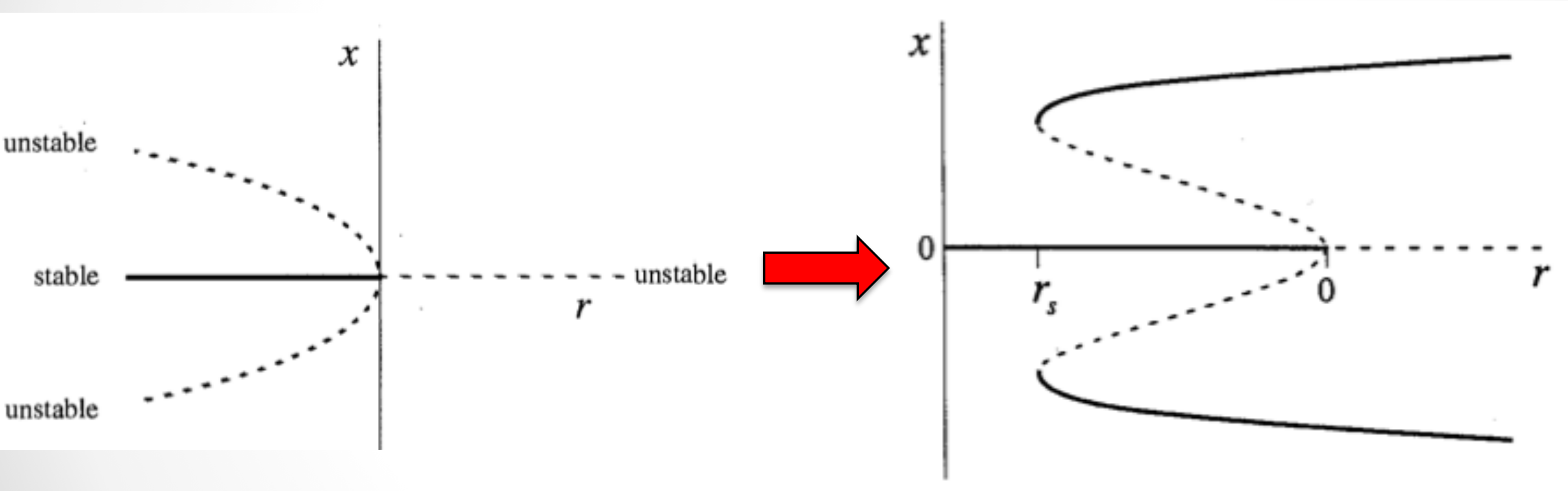
The first high-order power that maintains the left-right symmetry is x^5.

Bifurcation diagram.



Subcritical pitchfork bifurcation

$$\dot{x} = rx + x^3 - x^5$$

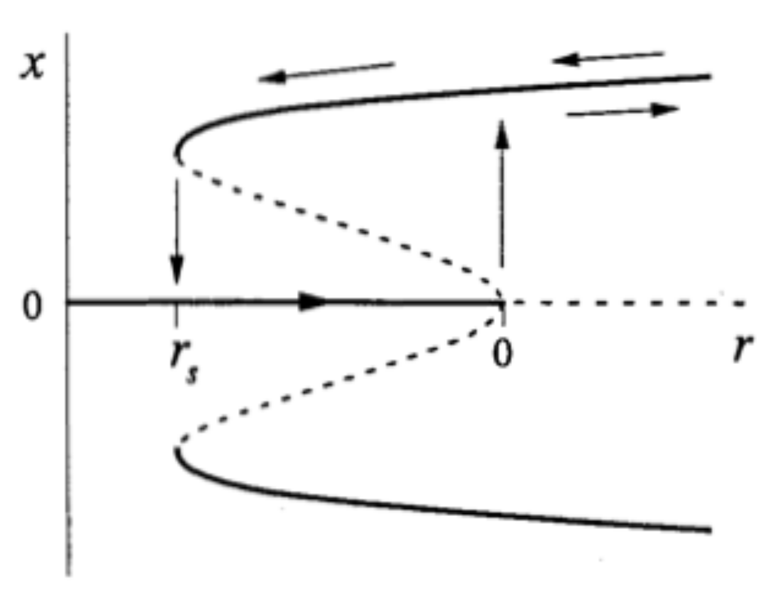


Unstable branches turn around and become stable for $r = r_s < 0$.

Subcritical pitchfork bifurcation

Remarks

1) In the range $r_s < r < 0$, the origin and the large-amplitude fixed points are stable. The origin is locally, but not globally stable.
2) Jumps and hysteresis are possible: if we start at the origin, and tune r from negative to positive values, the slightest perturbation makes the system jump to the large-amplitude fixed points, where it stays even if we bring r back to negative values, as long as $r > r_s$.
3) Saddle-node bifurcation at r_s.



Imperfect bifurcations

In many real-world circumstances the symmetry of the system is only approximate due to imperfections.

$$\dot{x} = h + rx - x^3$$

For $h = 0$: normal supercritical pitchfork bifurcation. For $h \neq 0$ *the left-right symmetry is broken*. h is the imperfection parameter.

Two independent parameters, h and r. Think of keeping r fixed and varying h.

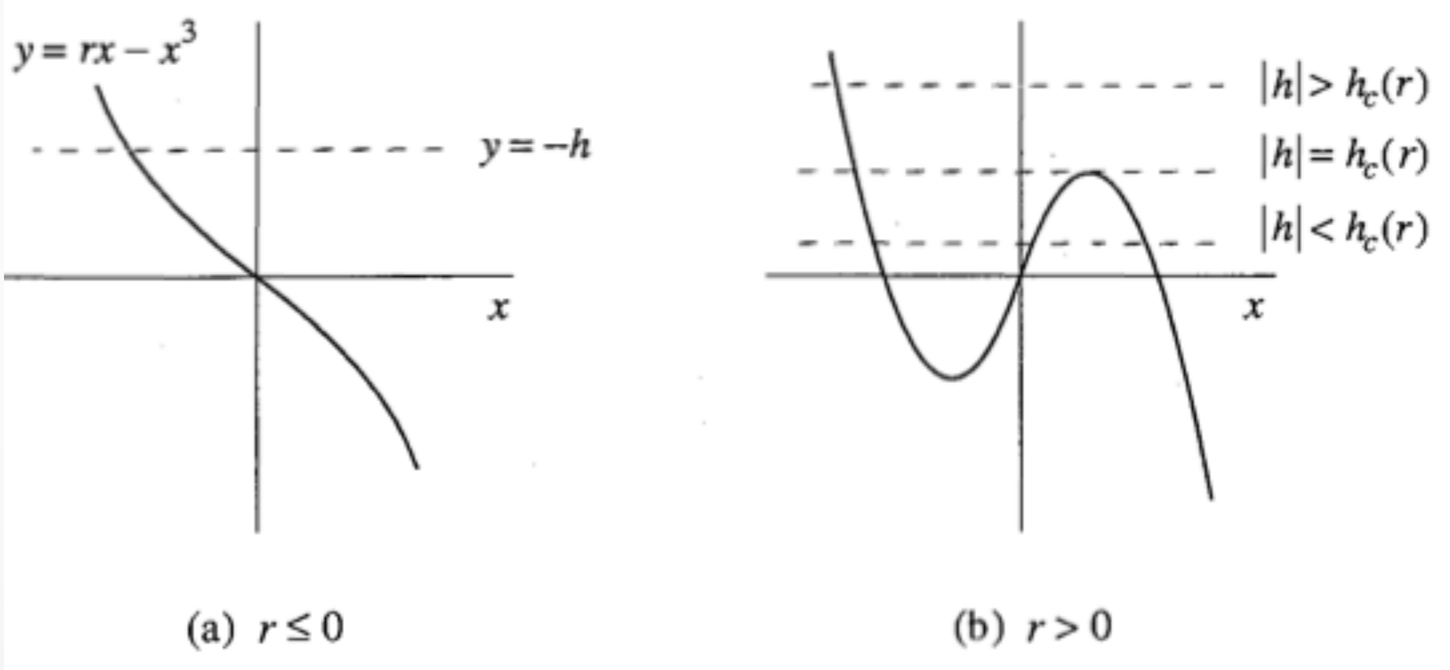
Solving fixed points exactly is messy → graphical approach.

$$h + rx - x^3 = 0 \quad \rightarrow \quad = \left\{ \begin{array}{lcl} y & = & -h \\ y & = & rx - x^3 \end{array} \right.$$

Imperfect bifurcations

$$\dot{x} = h + rx - x^3$$

FPs at intersections of $y = rx - x^3$ and $y = -h$.



Only one fixed point.

One, two, or three fixed points, depending on the value of *h*.

Imperfect bifurcations

$$\dot{x} = h + rx - x^3$$

Saddle-node bifurcation occurs when the horizontal line is tangent to the minimum or the maximum of the cubic.

The extrema of the cubic:

$$\frac{d}{dx}(rx - x^3) = r - 3x^2 = 0 \quad \rightarrow \quad x_{ext} = \pm\sqrt{\frac{r}{3}}$$

Values of the cubic at the extrema:

$$rx_{ext} - x_{ext}^3 = \pm r\sqrt{\frac{r}{3}} \mp \frac{r}{3}\sqrt{\frac{r}{3}} = \pm\frac{2r}{3}\sqrt{\frac{r}{3}}$$

Imperfect bifurcations

$$\dot{x} = h + rx - x^3$$

The condition for a saddle-node bifurcation to occur:

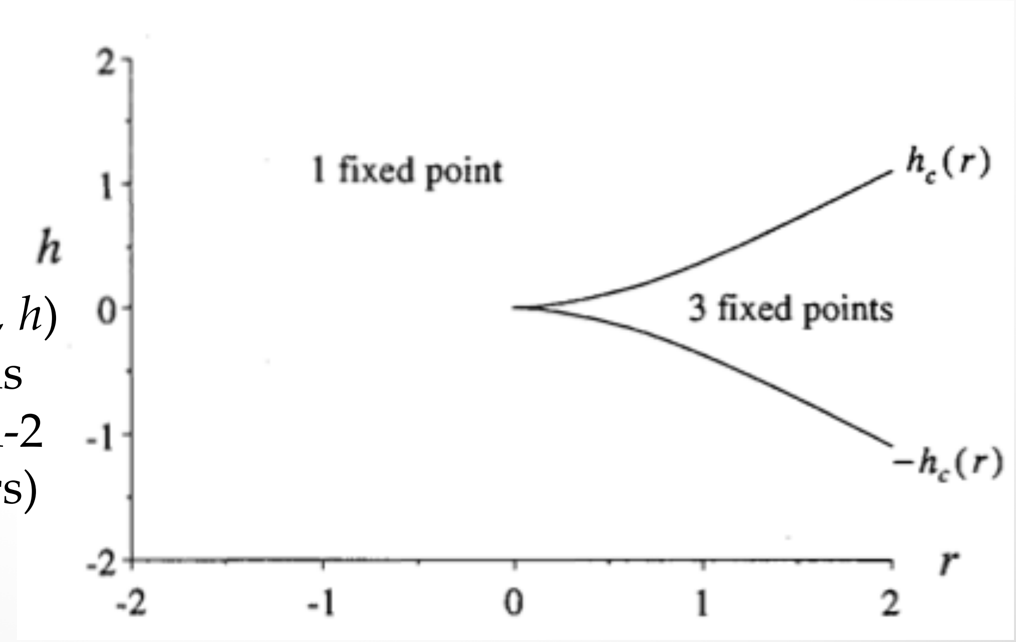
$$|h| = h_c = \frac{2r}{3}\sqrt{\frac{r}{3}}$$

Stability diagram

Three FPs for $|h| < h_c(r)$.

One FP for $|h| > h_c(r)$.

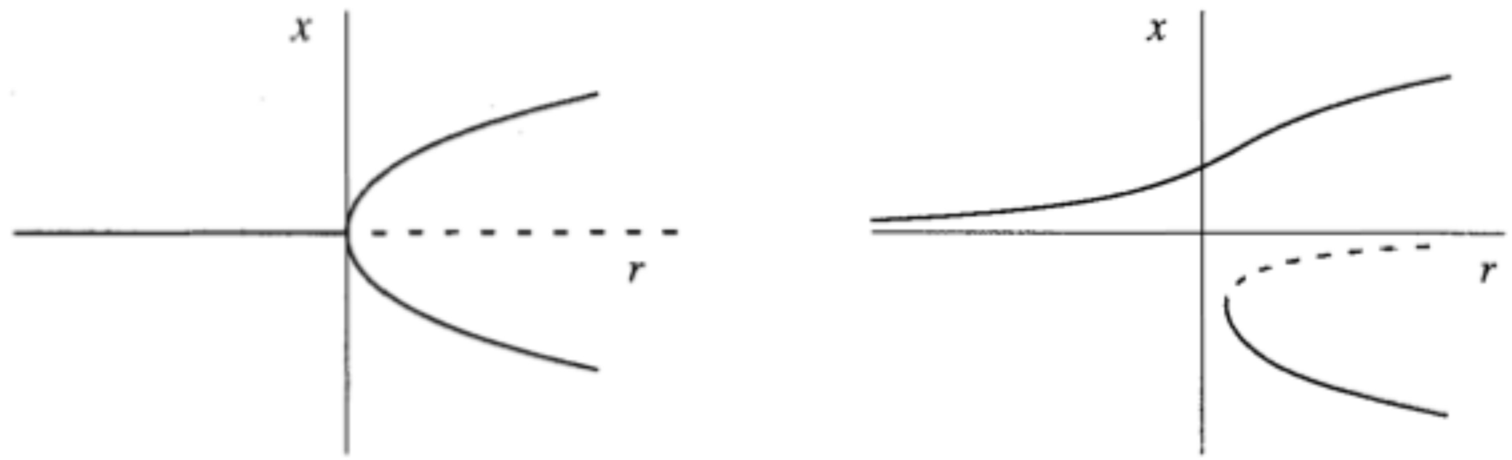
The two bifurcation curves meet tangentially at the **cusp point** $(r, h) = (0, 0)$. Saddle-node bifurcations along the curves. A codimension-2 bifurcation (2 tunable parameters) at the cusp point.



Imperfect bifurcations

$$\dot{x} = h + rx - x^3$$

Bifurcation diagram



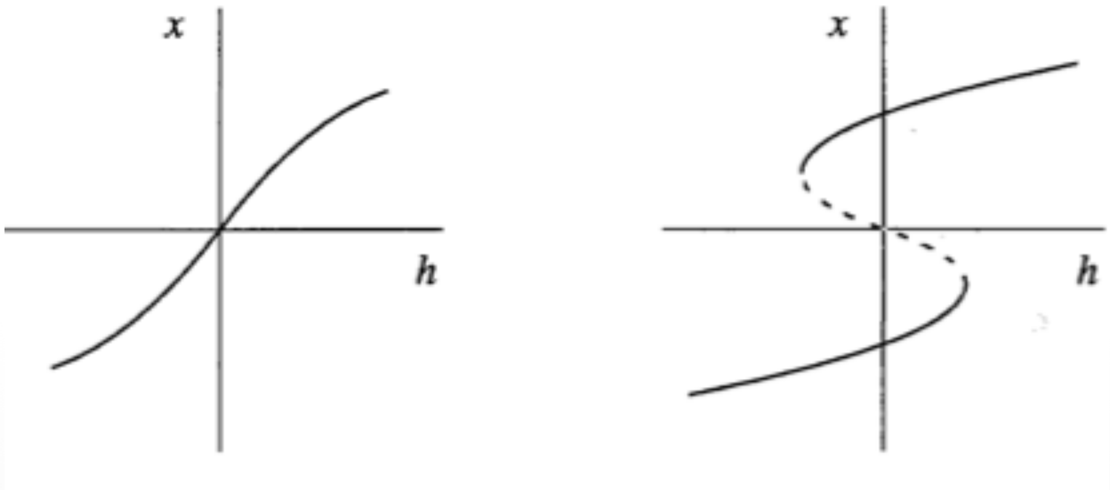
(a) $h = 0$ (b) $h \neq 0$

A supercritical pitchfork bifurcation for $h = 0$. For $h \neq 0$ the pitchfork disconnects into two pieces. Then increasing r from negative values makes the fixed point glide smoothly along the upper branch: the lower branch is not accessible unless a large disturbance is made.

Imperfect bifurcations

$$\dot{x} = h + rx - x^3$$

An alternative bifurcation diagram when we keep r fixed and vary h.

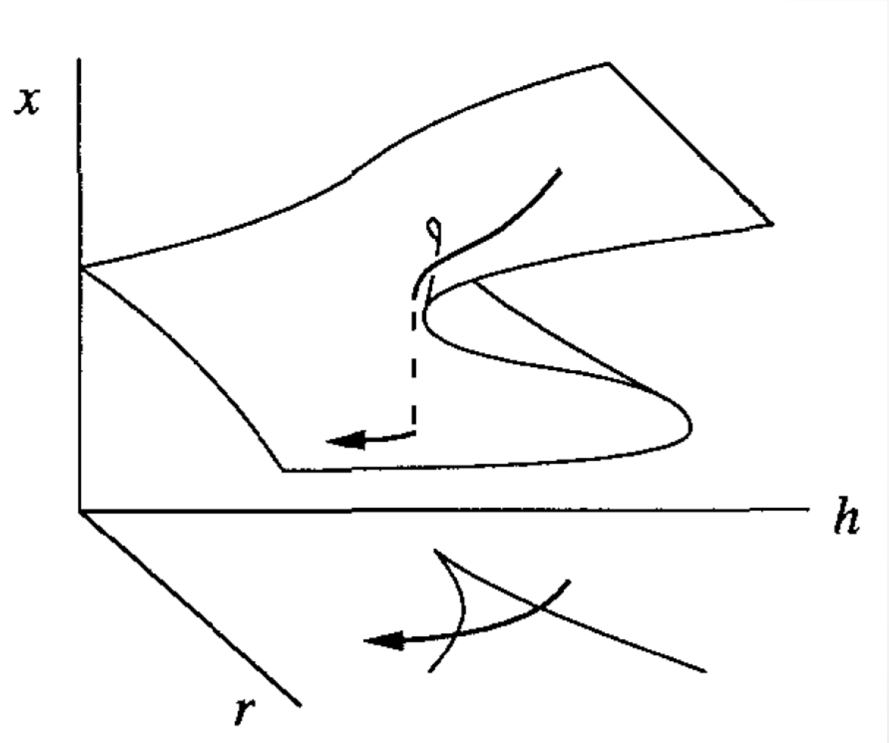


(a) $r \leq 0$ (b) $r > 0$

Three solutions for $|h| < h_c$: the middle one is unstable, the external ones are stable.

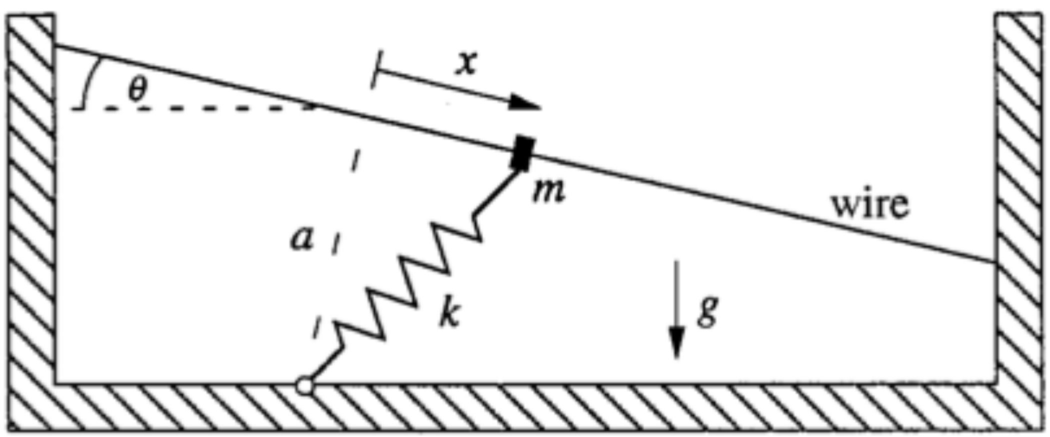
Catastrophe with bifurcations

Discontinuous jump between branches. Plot x^* above (r, h) plane: **cusp catastrophe** surface. This jump can be catastrophic e.g. for the equilibrium of a bridge or a building.



Example

Bead on a tilted wire

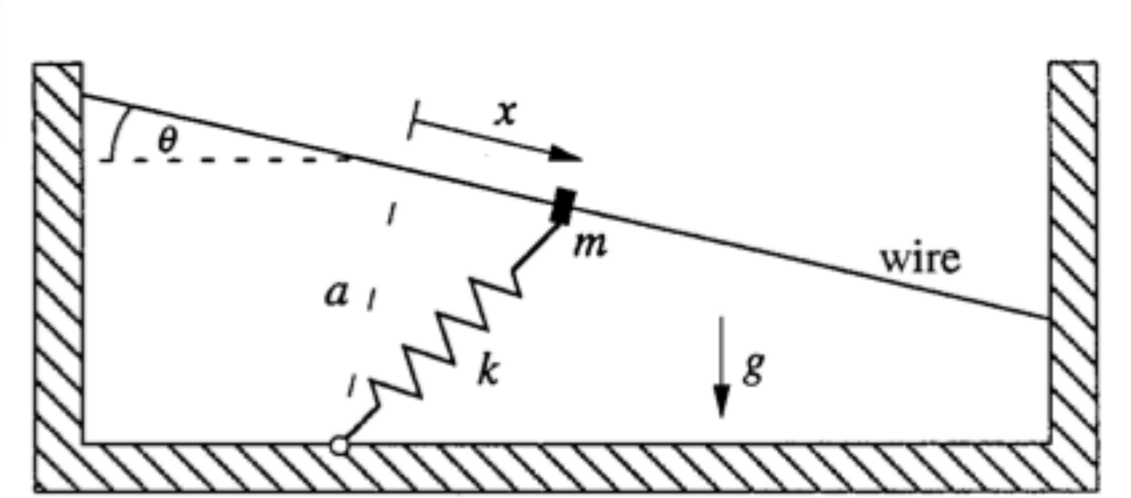


- A bead of mass m is constrained to glide along a straight wire inclined at an angle θ with respect to the horizontal
- The mass is attached to a spring of stiffness k and relaxed length L_0, and is also acted on by gravity

Choice of coordinates: $x = 0$ at the point closest to the support point of the spring

Example

Bead on a tilted wire

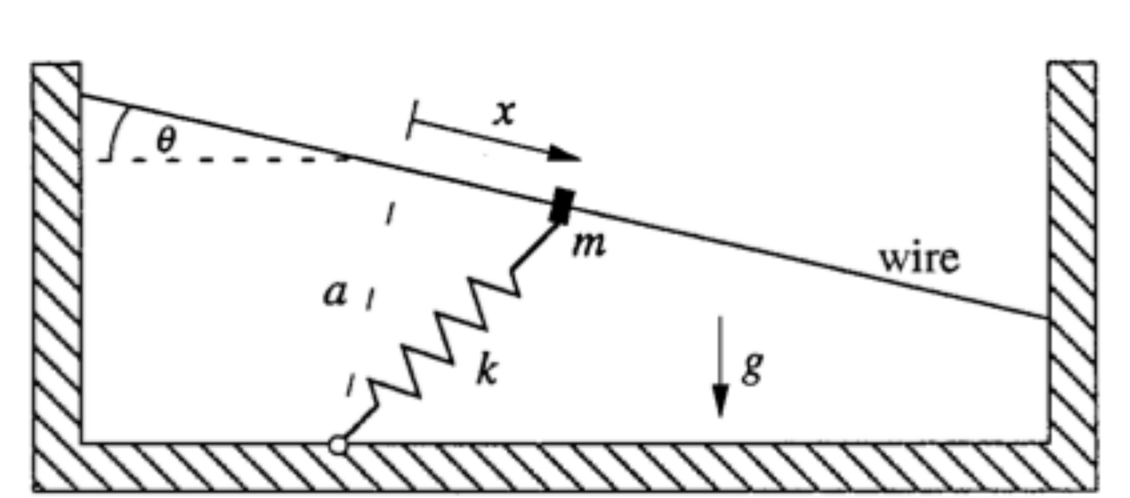


Horizontal wire ($\theta = 0$): $x = 0$ is the equilibrium position.

- $L_0 < a$: the spring is in tension and the equilibrium is stable
- $L_0 > a$: the spring is compressed and the equilibrium is unstable → two stable equilibria to either side of it

Example

Bead on a tilted wire



Tilted wire ($\theta \neq 0$)

- For small θ, there are still three equilibria (one unstable, two stable) if $L_0 > a$
- For θ not too small, uphill equilibrium might suddenly disappear and the bead would jump - **catastrophically** - to the downhill equilibrium!

Summary

Bifurcation changes the dynamics of the system qualitatively.

Three main classes of bifurcations:

1. Saddle-node
2. Transcritical
3. Pitchfork
   - supercritical
   - subcritical

Imperfect bifurcations take place in real life and may be catastrophic.